Electronic Transactions on Numerical Analysis.
Volume 40, pp. 407-413, 2013.

ISSN 1068-9613.

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ON COMPUTING STABILIZABILITY RADII OF LINEAR TIME-INVARIANT CONTINUOUS SYSTEMS*

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Abstract. In this paper we focus on a non-convex and non-smooth singular value optimization problem. Our framework encompasses the distance to stabilizability of a linear system (A, B) when both A and B or only one of them are perturbed. We propose a trisection algorithm for the numerical solution of the singular value optimization problem. This method requires $O(n^4)$ operations on average, where n is the order of the system. Numerical experiments indicate that the method is reliable in practice.

Key words. stabilizability radius, optimization, trisection algorithm, linear time-invariant continuous system

AMS subject classifications. 65F15, 93D15, 65K10

1. Introduction. Consider the linear time-invariant continuous dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geq 0, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$, with state vector $x(t) \in \mathbb{C}^n$ and control vector $u(t) \in \mathbb{C}^m$ for all $t \geq 0$. The system (1.1) is called *stabilizable* if there exists a feedback $u(t) = Fx(t)$, where F is a fixed matrix, making the closed-loop system $\dot{x}(t) = (A + BF)x(t)$ asymptotically stable. It is well-known that the system (1.1) is stabilizable if and only if the condition $\operatorname{rank}[A-\lambda I\ B]=n$ holds for all $\lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \Re\lambda \geq 0\}$; see, for example, [10].

One of the most effective and flexible approaches towards problems of robustness of stabilizability is based on the concept of the *stabilizability radius*—the norm of the smallest perturbation that makes a given system unstabilizable, introduced in [5, 6] as

$$\begin{aligned}\tau_0(A, B) := \inf \{\|[\Delta_A,\ \Delta_B]\| : &\text{ the perturbed system} \\ &\dot{x} = (A + \Delta_A)x + (B + \Delta_B)u \text{ is unstabilizable}\},\end{aligned}$$

where $\|.\|$ denotes the spectral norm. This definition is inspired from the definition of the controllability radius in [14]. The stabilizability radius is expressed in [5, 6] as

$$\tau_0(A, B) = \min_{\lambda\in\mathbb{C}_+} \sigma_{\min}([A - \lambda I\ B]), \tag{1.2}$$

where $\sigma_{\min}(.)$ denotes the smallest singular value of its matrix argument. Recently in [12], we considered stabilizability radii when only one of the system matrices, A or B, is perturbed

$$\tau_1(A) := \inf \{\|\Delta_A\| : \text{the perturbed system } \dot{x} = (A + \Delta_A)x + Bu \text{ is unstabilizable}\},$$

$$\tau_2(B) := \inf \{\|\Delta_B\| : \text{the perturbed system } \dot{x} = Ax + (B + \Delta_B)u \text{ is unstabilizable}\},$$

and the formulas for these values are given by

$$\tau_1(A) = \min_{\lambda\in\mathbb{C}_+} \sigma_{\min}([\textbf{null}\ (B^*)]^*(A - \lambda I)), \tag{1.3}$$

*Received January 13, 2009. Accepted August 30, 2013. Published online on December 9, 2013. Recommended by P. Van Dooren. This work was supported by the Vietnam National Foundation for Science and Technology Development Grant No. 101.01-2012.12.
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$$\tau_2(B) = \min_{\lambda\in\sigma(A)\cap\mathbb{C}_+} \sigma_{\min}(B^*\textbf{null}\,(A-\lambda I)), \tag{1.4}$$

where **null** $(.)$ is the matrix whose columns form an orthonormal basis of the null space of its matrix argument.

Problem (1.4) can be computed easily. But problems (1.2) and (1.3) are non-smooth optimization problems in two real variables α and β, the real and imaginary parts of λ. Moreover, the objective functions $\sigma_{\min}\left([A-\lambda I\ B]\right)$ or $\sigma_{\min}\left([\textbf{null}\,(B^*)]^*(A-\lambda I)\right)$ are not convex and may have many local minima, so standard optimization methods, which usually are only guaranteed to converge to a local minimum, will not yield reliable results in general.

To the best of our knowledge, the problem of computing stabilizability radii has not been studied in the literature even though several algorithms have been designed to compute controllability radii; see [3, 4, 7, 8, 9, 11, 13]. In the papers [4, 7, 11], two-dimensional grid techniques are used that are too costly for high accuracy. Gu [8] proposed a bisection method which can correctly estimate $\tau_0(A,B)$ within a factor of two in polynomial time in n. Burke et al. [3] suggested a trisection variant to retrieve the distance to uncontrollability to any desired accuracy with $O(n^6)$ complexity. Gu et al. [9] and Mengi [13] reduced the average running time to $O(n^4)$ by employing inverse iterations and the shift-and-invert preconditioned Arnoldi method. It is the main purpose of this paper to describe a numerical method for computing both problems (1.2) and (1.3).

Indeed, we consider the following non-convex and non-smooth general optimization problem

$$\tau(P,Q) = \min_{\lambda\in\mathbb{C}_+} \sigma_{\min}\left(P-\lambda Q\right), \tag{1.5}$$

where P, Q are some given matrices in $\mathbb{C}^{p\times q}$ such that $p \le q$ and $\operatorname{rank}(Q) = p$.

We can check that if $P := [A\ B], Q := [I\ 0]$, then problem (1.5) reduces to problem (1.2), and if $P := [\textbf{null}(B^*)]^*A$, $Q := [\textbf{null}\ (B^*)]^*$, then problem (1.5) reduces to problem (1.3). In this paper, based on the idea of the trisection algorithm introduced in [3], we present a method to solve (1.5).

The structure of this paper is as follows. In the next section, we give a modified version of Gu's result [8, Theorem 3.1] that is applicable to (1.5). Then, we apply the obtained results to state a method for solving this problem in Section 3. Finally, the reliability of the algorithm is demonstrated by numerical examples in Section 4.

2. Modified version of Gu's theorem.

The methods for computing the controllability radius in [8, 9, 13] are based on a simultaneous comparison of two bounds $\delta_1 > \delta_2$ with $\tau_0(A,B)$, i.e., one of the following inequalities is verified

$$\tau_0(A,B) \le \delta_1 \quad \text{or} \quad \tau_0(A,B) > \delta_2.$$

This so-called Gu's test based on [8, Theorem 3.1] returns some information about only one of the inequalities even if both of them may be satisfied.

It is remarkable that Gu's test in [8] cannot be applied to solve the general problem (1.5). The search space of (1.5) is just the closed right half of the complex plane, which naturally leads to the idea of a *vertical search* as in the following theorem.

THEOREM 2.1. *Assume that* $\delta > \tau(P,Q)$. *Given a number* $\eta \in (0, \frac{2(\delta-\tau(P,Q))}{\|Q\|_2}]$, *then there exists a pair* (α,β) *of a non-negative number* α *and a real number* β *such that*

$$\sigma_{\min}\left[P-(\alpha+\beta i)Q\right] = \sigma_{\min}\left[P-(\alpha+\eta i+\beta i)Q\right] = \delta. \tag{2.1}$$

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We omit the proof of this theorem, since it is similar to that of [8, Theorem 3.1] but with horizontal search replaced by vertical search. If we rewrite problem (1.5) as

$$\tau(P,Q) = \min_{\lambda\in\mathbb{C},\Re\lambda\geq 0} \sigma_{\min}\left(iP-\lambda Q\right),$$

the horizontal search can also be applied to solve (1.5). For the numerical verification, we only need a relation implied by (2.1).

COROLLARY 2.2. *Assume that* $\delta > \tau(P,Q)$. *Given a number* $\eta \in (0, \frac{2(\delta-\tau(P,Q))}{\|Q\|_2}]$, *then there exists a pair* (α,β) *of a non-negative number* α *and a real number* β *such that*

$$\delta \in \sigma\left[P-(\alpha+\beta i)Q\right] \cap \sigma\left[P-(\alpha+\eta i+\beta i)Q\right], \tag{2.2}$$

where $\sigma(.)$ *is the set of all singular values of its matrix argument.*

For $\delta_1 > \delta_2 > 0$, set $\delta = \delta_1$ and $\eta = \frac{2(\delta_1-\delta_2)}{\|Q\|_2}$. This corollary implies that when no pair (α,β) satisfying (2.2) exists, the inequality $\eta > \frac{2(\delta-\tau(P,Q))}{\|Q\|_2}$ is valid, so condition

$$\tau(P,Q) > \delta_2 \tag{2.3}$$

holds. On the other hand, when a pair exists, then by definition we can conclude that

$$\tau(P,Q) \leq \delta_1. \tag{2.4}$$

This is called the *modified Gu test for the stabilizability radius*.

3. Trisection algorithm for computing stabilizability radii. The bisection algorithm of Gu [8] keeps only an upper bound on the distance to uncontrollability. It refines the upper bound until condition (2.3) is satisfied, and at termination the controllability radius lies within a factor of 2 of δ_1, i.e., $\delta_1/2 < \tau(P,Q) \leq 2\delta_1$. To obtain the distance to uncontrollability with better accuracy, Burke et al. [3] proposed a trisection variant. Since the derivation of the details of the algorithm presented in this section follows [9] step by step, we just give brief results. Using the modified Gu test for the stabilizability radius instead of Gu's test, the trisection algorithm yields bounds of $\tau(P,Q)$ in form of an interval $[l,u]$ and reduces the length of this interval by a factor of $\frac{2}{3}$ at each iteration.

Trisection algorithm for computing (1.5)

Input: $P,Q\in\mathbb{C}^{k\times n}$ with $k\leq n$ and $\mathrm{rank}(Q)=k$ and a tolerance $\varepsilon>0$.
Output: Scalars l and u satisfying $l<\tau(P,Q)\leq u$ and $u-l<\varepsilon$.

```
Initialize the lower bound as l = 0 and the upper bound as u = σ_min(P).
   repeat
      δ1 = l + (2/3)(u − l)
      δ2 = l + (1/3)(u − l)
      Apply the modified Gu test for stabilizability radius.
      if (2.4) is verified then
         u ← δ1.
      else
         % Otherwise (2.3) is verified.
         l ← δ2.
      end if
   until u − l < ε
Return l and u.
```

3.1. Verification scheme. Let the singular value decomposition of Q be

$$Q = U\,[\Sigma\ 0]\,V^*,$$

and define

$$[P_1\ P_2] := U^* P V,$$

where $P_1 \in \mathbb{C}^{p\times p}$ and $P_2 \in \mathbb{C}^{p\times n-p}$. It is remarkable that $\delta \in \sigma[P-(\alpha+\beta i)Q]$ if and only if α is an eigenvalue of the following matrix

$$H(\beta,\delta) := \begin{bmatrix} \Sigma^{-1}P_1^* + i\beta I & -\delta\Sigma^{-1} \\ \Sigma^{-1}\hat{P}_2 & \Sigma^{-1}P_1 - i\beta I \end{bmatrix},$$

where $\hat{P}_2 := \frac{P_2P_2^*}{\delta} - \delta I$. Then, condition (2.2) is equivalent to the fact that the following Sylvester equation

$$H(\beta,\delta)X + XH(\beta+\eta,\delta) = 0$$

has a nontrivial solution. By the Kroneckerization of this Sylvester equation, for the verification of a pair (α,β) satisfying (2.2), we search at first for an imaginary eigenvalue $i\beta$ of the following matrix

$$\begin{aligned} \mathcal{A} := \frac{1}{2}\Bigg(&\begin{bmatrix} -\delta C_2 & \delta C_1 \\ B_1 & -B_2 \end{bmatrix}\begin{bmatrix} A_3 - A_4 & 0 \\ 0 & A_2 - A_1 \end{bmatrix}^{-1}\begin{bmatrix} B_2 & \delta C_1 \\ B_1 & \delta C_2 \end{bmatrix} \\ &+ \begin{bmatrix} A_2 - A_3 & 0 \\ 0 & A_1 - A_4 \end{bmatrix}\Bigg), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} A_1 &:= I \otimes \left(\Sigma^{-1}P_1\right), & A_2 &:= \left(\Sigma^{-1}P_1 - i\eta I\right)^T \otimes I, \\ A_3 &:= I \otimes \left(\Sigma^{-1}P_1^*\right), & A_4 &:= \left(\Sigma^{-1}P_1^* + i\eta I\right)^T \otimes I, \\ B_1 &:= I \otimes \left(\Sigma^{-1}\hat{P}_2\right), & B_2 &:= \left(\Sigma^{-1}\hat{P}_2\right)^T \otimes I, \\ C_1 &:= I \otimes (\Sigma^{-1}), & C_2 &:= (\Sigma^{-1})^T \otimes I. \end{aligned}$$

Next, if there exists an imaginary eigenvalue $i\beta$ of (3.1) such that the matrices $H(\beta,\delta)$ and $H(\beta+\eta,\delta)$ share a common non-negative eigenvalue α, then the verification succeeds.

3.2. Eigenvalue search. We can observe that searching for an imaginary eigenvalue $i\beta$ of $\mathcal{A} \in \mathbb{C}^{2p^2\times 2p^2}$ is the key operation of the modified Gu test for the stabilizability radius. In [3], the Matlab function *eig* can solve the search problem at a cost of $O(k^6)$. In [9], an algorithm for searching real eigenvalues of a matrix in $\mathbb{C}^{2p^2\times 2p^2}$ is given at a cost of $O(p^4)$ on average by employing a shift-and-invert Arnoldi method of the matrix at a cost of $O(p^3)$. Clearly, we can search for a real eigenvalue β of $-i\mathcal{A}$ instead of searching for an imaginary eigenvalue $i\beta$ of $\mathcal{A}$. Moreover, the linear system $(-i\mathcal{A} - \nu I)v = u$ can be solved at a cost of $\mathcal{O}(p^3)$ for any $\nu \in \mathbb{R}$, by means of a Sylvester equation solver (such as the LAPACK routine *dtrsyl* [1]) as indicated by the following lemma.

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LEMMA 3.1. *Let* $u \in \mathbb{C}^{2p^2}$ *and suppose that* $v \in \mathbb{C}^{2p^2}$ *is the solution of the linear system* $(-i\mathcal{A} - \nu I)v = iu$. *Then the following Sylvester equation*

$$\begin{bmatrix} \Sigma^{-1}P_1^* + i\nu I - \delta\Sigma^{-1} \\ \Sigma^{-1}\hat{P}_2\Sigma^{-1}P_1 - i\nu I \end{bmatrix} Z - Z \begin{bmatrix} \Sigma^{-1}P_1^* + i(\eta+\nu)I - \delta\Sigma^{-1} \\ \Sigma^{-1}\hat{P}_2\Sigma^{-1}P_1 - i(\eta+\nu)I \end{bmatrix} = 2\begin{bmatrix} 0 & -U_1 \\ U_2 & 0 \end{bmatrix} \tag{3.2}$$

has a unique solution $Z = \begin{bmatrix} W_1 & V_1 \\ V_2 & W_2 \end{bmatrix}$, *where* $u = \begin{bmatrix} \mathbf{vec}(U_1) \\ \mathbf{vec}(U_2) \end{bmatrix}$ *and* $v = \begin{bmatrix} \mathbf{vec}(V_1) \\ \mathbf{vec}(V_2) \end{bmatrix}$.

4. Numerical experiments.

Setting $P := A$ and $Q := I$, we observe that $\tau(P, Q)$ is the stability radius of the system (1.1). Hence, we can compare the accuracy of our method given in this paper and the method for computing the stability radius given in [2, 13]. Table 4.1 presents this comparison for a variety of examples chosen from EigTool [15] with a tolerance of 10^{-4}. All the tests are run using Matlab 6.5 under Linux on a PC.

TABLE 4.1
Stability radii with tolerance = 10^{-4}.

Example	New method	Method in [2]
Airy(5)	(0.00370, 0.00380]	0.0038
Airy(10)	(0.01245, 0.01254]	0.0125
Convection Diffusion(5)	(0.60395, 0.60403]	0.6040
Convection Diffusion(10)	(0.75310, 0.75317]	0.7532
Transient(5)	(0.02935, 0.02942]	0.0294
Transient(10)	(0.02025, 0.02032]	0.0203

For the next results, we again choose matrices A from EigTool [15] and let B be matrices with normally distributed entries. Table 4.2 displays the results of $\tau_0(A, B)$ and $\tau_1(A)$ in the second and the third columns, respectively, when we apply our method for computing problem (1.5). These results illustrate the inequality given in [12] that $\tau_0(A, B) \leq \tau_1(A)$. In particular, the result for the Godunov and Skew Laplacian matrices show the importance of computing the stabilizability radius when only the system matrix A is perturbed.

Acknowledgment. The Matlab software used for computing the general problem (1.5) in the numerical experiments is based on software provided by Emre Mengi. The authors thank the referees for their careful reading of the manuscript and insightful comments.

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TABLE 4.2
Stabilizability radii with tolerance = 10^{-4}.

Example	$\tau_0(A,B)$	$\tau_1(A)$
Airy(5,2)	(0.03777, 0.03786]	(0.04286, 0.04296]
Airy(10,4)	(0.16442, 0.16449]	(0.16822, 0.16829]
Basor Morrison(5,2)	(0.72733, 0.72740]	(1.45174, 1.45181]
Basor Morrison(10,4)	(0.76689, 0.76696]	(1.41940, 1.41948]
Chebyshev(5,2)	(0.75035, 0.75044]	(1.61063, 1.61073]
Chebyshev(10,4)	(0.82703, 0.82711]	(2.50581, 2.50591]
Companion(5,2)	(0.42431, 0.42438]	(0.71518, 0.71525]
Companion(10,4)	(0.65862, 0.65868]	(0.75474, 0.75481]
Convection Diffusion(5,2)	(0.76432, 0.76439]	(1.12964, 1.12971]
Convection Diffusion(10,4)	(2.08213, 2.08222]	(3.10039, 3.10048]
Davies(5,2)	(0.23170, 0.23176]	(5.51655, 5.51663]
Davies(10,4)	(0.70003, 0.70012]	(6.19431, 6.19440]
Demmel(5,2)	(0.10152, 0.10159]	(0.10152, 0.10159]
Demmel(10,4)	(0.27415, 0.27424]	(0.27417, 0.27425]
Frank(5,2)	(0.45907, 0.45916]	(0.80848, 0.80858]
Frank(10,4)	(0.69384, 0.69393]	(0.85247, 0.85255]
Gallery(5,2)	(0.50544, 0.50551]	(0.43033, 0.43042]
Gauss Seidel(5,2)	(0.06383, 0.06392]	(0.06323, 0.06332]
Gauss Seidel(10,4)	(0.07038, 0.07046]	(0.07039, 0.07046]
Godunov(7,3)	(1.39651, 1.39659]	(425.81455, 425.81464]
Grcar(5,2)	(0.49571, 0.49579]	(0.57320, 0.57328]
Grcar(10,4)	(0.44178, 0.44185]	(0.51903, 0.51911]
Hatano(5,2)	(1.31802, 1.31809]	(0.48167, 0.48175]
Hatano(10,4)	(0.44865, 0.44872]	(0.67699, 0.67708]
Kahan(5,2)	(0.18594, 0.18601]	(0.18710, 0.18717]
Kahan(10,4)	(0.05592, 0.05599]	(0.05651, 0.05658]
Landau(5,2)	(0.41766, 0.41773]	(1.04564, 1.04572]
Landau(10,4)	(0.36615, 0.36623]	(0.34016, 0.34023]
Markov Chain(5,2)	(0.04355, 0.04365]	(0.04470, 0.04477]
Markov Chain(10,4)	(0.20548, 0.20557]	(0.20852, 0.20861]
Orr Sommerfield(5,2)	(0.04834, 0.04841]	(0.05296, 0.05305]
Orr Sommerfield(10,4)	(0.08401, 0.08408]	(0.09904, 0.09911]
Skew Laplacian(8,3)	(27.86320, 27.86330]	(32.94072, 32.94079]
Supg(4,2)	(0.06546, 0.06554]	(0.06612, 0.06619]
Supg(9,4)	(0.03810, 0.03817]	(0.03638, 0.03644]
Transient(5,2)	(0.38264, 0.38273]	(0.39787, 0.39796]
Transient(10,4)	(0.34967, 0.34974]	(0.35715, 0.35722]
Twisted(5,2)	(0.14929, 0.14936]	(0.20265, 0.20273]
Twisted(10,4)	(0.77197, 0.77204]	(1.01131, 1.01140]

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