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FEJÉR ORTHOGONAL POLYNOMIALS AND RATIONAL MODIFICATION OF A MEASURE ON THE UNIT CIRCLE*

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Abstract. Relations between the monic orthogonal polynomials associated with a measure on the unit circle and the monic orthogonal polynomials associated with a rational modification of this measure are known. In this paper we deal with some generalization in order to give an explicit expression of the Fejér orthogonal polynomials on the unit circle. Furthermore we give a simple and efficient algorithm to compute the monic orthogonal polynomials associated with a rational modification of a measure.

Key words. Fejér kernel, orthogonal polynomials, rational modification of a measure

AMS subject classifications. 42C05

1. Introduction. Let μ be a finite and positive Borel measure on the unit circle $\mathcal{T} = \{z \in \mathcal{C},\ |z| = 1\}$. The measure μ provides an inner product

$$\langle f, g\rangle_\mu = \int_{\mathcal{T}} f(z)\overline{g(z)}d\mu(z),\ f, g \in \Pi$$

on the space Π of algebraic polynomials with complex coefficients.

Orthogonalising the basis $\{1, z, z^2, \dots\}$ by the Gram-Schmidt process with respect to this inner product we obtain a sequence of algebraic orthogonal polynomials known as *Szegö polynomials*; see, e.g., [13, 14]. We denote by $\{\varrho_n(z;\mu)\}_{n=0}^{\infty}$ the sequence of monic Szegö polynomials orthogonal with respect to μ, and by $\{\varrho_n^*(z;\mu)\}_{n=0}^{\infty}$, where

$$\varrho_n^*(z;\mu) = z^n\overline{\varrho_n(1/\overline{z};\mu)},\ n = 0, 1, 2, \dots,$$

the sequence of *reversed polynomials*. Let

$$\gamma_k(\mu) = \int_{\mathcal{T}} z^{-k}d\mu(z),\ k = 0, \pm1, \pm2, \dots$$

be the *moments* corresponding to the measure μ. Let us consider the Hermitian (take into account that $\overline{\gamma_k(\mu)} = \gamma_{-k}(\mu)$) Toeplitz matrices, called moment matrices, given by

$$M_n = \begin{bmatrix} \gamma_0(\mu) & \gamma_{-1}(\mu) & \dots & \gamma_{-n}(\mu) \\ \gamma_1(\mu) & \gamma_0(\mu) & \dots & \gamma_{-n+1}(\mu) \\ \vdots & \vdots & & \vdots \\ \gamma_n(\mu) & \gamma_{n-1}(\mu) & \dots & \gamma_0(\mu) \end{bmatrix},\ n = 0, 1, 2, \dots.$$

Let $\Delta_n = \det(M_n),\ n = 0, 1, 2, \dots (\Delta_{-1} = 1)$. Since for any nonvanishing column vector $v = (v_0, v_1, \dots, v_n)^t$, it holds that

$$\overline{v^t}M_n v = \langle p, p\rangle_\mu = \int_{\mathcal{T}} |p(z)|^2 d\mu(z) > 0,$$

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where $p(z) = v_0 + v_1 z + \cdots + v_n z^n$, we conclude that M_n is positive definite and hence $\Delta_n > 0,\ n = 1, 2, \ldots$. It is immediate to show (based on linear dependence properties of rows in a determinant) that the monic Szegö polynomials admit the expression $\varrho_0(z;\mu) = 1$ and

$$\varrho_n(z;\mu) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \gamma_0(\mu) & \gamma_{-1}(\mu) & \ldots & \gamma_{-n}(\mu) \\ \gamma_1(\mu) & \gamma_0(\mu) & \ldots & \gamma_{-n+1}(\mu) \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1}(\mu) & \gamma_{n-2}(\mu) & \ldots & \gamma_{-1}(\mu) \\ 1 & z & \ldots & z^n \end{vmatrix},\ n = 1, 2, 3, \ldots. \tag{1.1}$$

Since $\overline{\gamma_k(\mu)} = \gamma_{-k}(\mu)$ and $\Delta_n > 0$, it follows that the reversed polynomials admit the expression $\varrho_0^*(z;\mu) = 1$ and

$$\varrho_n^*(z;\mu) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \gamma_0(\mu) & \gamma_1(\mu) & \ldots & \gamma_n(\mu) \\ \gamma_{-1}(\mu) & \gamma_0(\mu) & \ldots & \gamma_{n-1}(\mu) \\ \vdots & \vdots & & \vdots \\ \gamma_{-n+1}(\mu) & \gamma_{-n+2}(\mu) & \ldots & \gamma_1(\mu) \\ z^n & z^{n-1} & \ldots & 1 \end{vmatrix},\ n = 1, 2, 3, \ldots. \tag{1.2}$$

From (1.1) and (1.2) is deduced that the polynomials $\varrho_n(z;\mu)$ and $\varrho_n^*(z;\mu)$ satisfy the orthogonality conditions

$$\langle \varrho_n(z;\mu), z^m \rangle_\mu = \begin{cases} 0, & 0 \le m \le n-1, \\ \Delta_n/\Delta_{n-1}, & m = n, \end{cases} \tag{1.3}$$

and

$$\langle \varrho_n^*(z;\mu), z^m \rangle_\mu = \begin{cases} \Delta_n/\Delta_{n-1}, & m = 0, \\ 0, & 1 \le m \le n. \end{cases} \tag{1.4}$$

The polynomials $\varrho_n(z;\mu)$ satisfy the forward recurrence relations (see, e.g., [9])

$$\begin{aligned} &\varrho_0(z;\mu) = 1, \\ &\varrho_n(z;\mu) = z\varrho_{n-1}(z;\mu) + \delta_n \varrho_{n-1}^*(z;\mu),\ n = 1, 2, 3, \ldots, \end{aligned} \tag{1.5}$$

Thus

$$\begin{aligned} &\varrho_0^*(z;\mu) = 1, \\ &\varrho_n^*(z;\mu) = \overline{\delta}_n z\varrho_{n-1}(z;\mu) + \varrho_{n-1}^*(z;\mu),\ n = 1, 2, 3, \ldots. \end{aligned}$$

The coefficients δ_n are called *Verblunsky coefficients*. Observe from (1.5) that $\delta_n = \varrho_n(0;\mu)$. They can be computed from (1.5), taking into account (1.3)-(1.4), in terms of the moments $\gamma_k(\mu)$ by the following procedure known as *Levinson's algorithm* (see [10]),

$$\delta_n = -\frac{\langle z\varrho_{n-1}(z;\mu), 1\rangle}{\langle \varrho_{n-1}^*(z;\mu), 1\rangle} = -\frac{\displaystyle\sum_{j=0}^{n-1} q_j^{(n-1)}\gamma_{-j-1}(\mu)}{\displaystyle\sum_{j=0}^{n} q_j^{(n-1)}\gamma_{j+1-n}(\mu)}\ ,\quad \varrho_n(z;\mu) = \sum_{j=0}^{n} q_j^{(n)} z^j.$$

The Verblunsky coefficients play an important role in the construction of Szegö *quadrature formulas on the unit circle*; see [6]. Szegö quadrature formulas (see, e.g., [5]) are used for the approximation of integrals on the unit circle of the form

$$\int_{\mathcal{T}} f(z) d\mu(z).$$

Let us now consider the absolutely continuous measure μ on the unit circle given by $d\mu(z) = \mu'(z)|dz| = K_N(t)dt$, where

$$\begin{aligned} K_N(t) &= \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) z^j \\ &= \frac{1}{N+1} \left| \frac{z^{N+1}-1}{z-1} \right|^2, \; z = e^{it}, \; -\pi \le t \le \pi, \; N = 0, 1, 2, \ldots, \end{aligned} \tag{1.6}$$

is the Fejér kernel. Hence,

$$\int_{\mathcal{T}} f(z) d\mu(z) = \int_{\mathcal{T}} f(z) \mu'(z)|dz| = \int_{-\pi}^{\pi} f(e^{it}) K_N(t) dt.$$

The Fejér kernel is one of the most important summability kernel in Fourier series. The importance comes from the Fejér's theorem; see, e.g., [15, Chapter 3]. This classical theorem states that for 2π-periodic continuous functions $f(x)$ the sequence of Cesàro means $\{\sigma_N\}$ of the partial sums of the Fourier series of $f(x)$ converges uniformly to $f(x)$ on $[-\pi, \pi]$. It holds that

$$\sigma_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) K_N(t) dt. \tag{1.7}$$

Observe that for 2π-periodic continuous functions of the form $f(x) = g(e^{ix})$, the Cesàro means admits a representation in terms of an integral on the unit circle with respect to the Fejér kernel. Given the connections between Szegö quadrature formulas and Szegö orthogonal polynomials on the unit circle (see, e.g., [9]) they motivates us to study in view of (1.7) the Szegö orthogonal polynomials with respect to the Fejér kernel and relative to the inner product

$$\langle f, g \rangle_{K_N} = \int_{-\pi}^{\pi} f(e^{it}) \overline{g(e^{it})} K_N(t) dt.$$

They are called *Fejér orthogonal polynomials on the unit circle* or, briefly, Fejér orthogonal polynomials. It is known that they are related to orthogonal polynomials on the real line associated with certain generalization of the Jacobi weight function; see [12]. We clarify that the Fejér orthogonal polynomials studied in this paper are different from the family introduced by Fejér, called *Fejér polynomials*, and studied in [7] in view of its applications in the study of Taylor series. On the other hand, we comment that the study of polynomials related with the Fejér kernel and the study of modifications of the Fejér summability method are currently active research areas; see [2] and [11], respectively.

For $N = 0$, one has $K_0(t) = 1, \; t \in [-\pi, \pi]$. In this case, the monic orthogonal polynomials $\varrho_n(z; K_0)$ are well known, $\varrho_n(z; K_0) = z^n, \; n = 0, 1, 2, \ldots$; see [14, pp. 289-290]. For $N = 1$, the monic orthogonal polynomials $\varrho_n(z; K_1)$ are given by

$$\varrho_n(z; K_1) = \sum_{k=0}^{n} (-1)^{n-k} \frac{k+1}{n+1} z^k, \; n = 0, 1, 2, \ldots;$$

see [3]. Furthermore, it is known (see [12]) that for the set of values $0 \leq n \leq N+1$, $N = 0, 1, 2, \ldots$, the monic orthogonal polynomials associated with the Fejér kernel are given by

$$\begin{aligned} &\varrho_0(z; K_N) = 1, \\ &\varrho_n(z; K_N) = \frac{1}{2N-n+3} - \frac{2N-n+2}{2N-n+3} z^{n-1} + z^n. \end{aligned} \tag{1.8}$$

In the literature, as far as we know, an explicit expression of the monic orthogonal polynomials for the set of values $n = N+2, N+3, \ldots$ for $N = 2, 3, 4, \ldots$, is not given. Such an explicit expression is given in Section 2 and it constitutes a first goal of our contribution. For this first goal, it will be relevant, in view of (1.6), to recall the following theorem relative to orthogonal polynomials with respect to a rational modification of a measure. Orthogonal polynomials on the unit circle with respect to a rational modification have been studied in [3, 4, 8]. (In [1], orthogonal rational functions with respect to a rational modification of a Borel measure on $\mathcal{T}$ are studied.)

Let μ be a finite positive Borel measure on $\mathcal{T}$. Let us consider the rational modification

$$d\mu_2 = \frac{1}{|z-\alpha|^2} d\mu, \ \alpha \notin \mathcal{T}.$$

Let $\{\Phi_n(z;\mu)\}$ and $\{\Phi_n(z;\mu_2)\}$ be the monic orthogonal polynomial sequences on $\mathcal{T}$ associated to μ and μ_2, respectively, and denote by $\Phi_n^*(z;\mu) = z^n \overline{\Phi_n(1/\overline{z})}$, the reversed polynomials.

THEOREM 1.1. *(See [4, Proposition 6]) The monic orthogonal polynomials with respect to μ_2 on the unit circle $\mathcal{T}$ satisfy*

$$\Phi_{n+1}(z;\mu_2) = (z - A_n(\alpha))\Phi_n(z;\mu) + B_n(\alpha)\Phi_n^*(z;\mu), \ \forall n \geq 1$$

and

$$\Phi_0(z;\mu_2) = 1, \ \ \Phi_1(z;\mu_2) = z - \alpha + \frac{\overline{Q_0(\alpha)}}{\|\mu_2\|},$$

where

$$A_n(\alpha) = \alpha \frac{e_{n+1}(\mu_2)}{e_n(\mu)} = \alpha \left[\frac{\|\mu_2\| - \sum_{j=0}^{n} |q_j(\alpha)|^2}{\|\mu_2\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2} \right]$$

and

$$B_n(\alpha) = \frac{1}{\alpha^{n+1}} \left[\frac{\overline{q_n(\alpha)}\,\overline{q_n(\frac{1}{\overline{\alpha}})}}{\|\mu_2\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2} \right]$$

with

$$q_k(t) = \frac{1}{\sqrt{e_k(\mu)}} \int_{\mathcal{T}} \frac{\overline{\Phi_k(z;\mu)}}{t-z} d\mu(z), \ Q_0(\alpha) = \int_{\mathcal{T}} \frac{d\mu(z)}{\alpha - z} \ \text{and} \ \|\mu_2\| = \int_{\mathcal{T}} d\mu_2.$$

Notice that from Theorem 1.1 we cannot implement an algorithm to compute the monic orthogonal polynomials $\Phi_n(z;\mu_2)$ or, equivalently, the values $A_n(\alpha)$ and $B_n(\alpha)$, in terms of orthogonal polynomials $\Phi_n(z;\mu)$. In fact, a second goal of our contribution is to implement such a useful and simple algorithm. This is done in Section 3.

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2. Explicit expression of the Fejér orthogonal polynomials. The pole α of the rational modification considered in Theorem 1.1 is supposed $\alpha \notin \mathcal{T}$. From (1.6) we need to consider the case $\alpha \in \mathcal{T}$.

Let $w(t)$ be a weight function on the unit circle

$$w(t) = |q(e^{it})|^2, \ t \in [-\pi, \pi], \tag{2.1}$$

where $q(z)$ is an algebraic polynomial not identically equal to zero. Let $\{\varrho_n(z; w)\}$ be the monic orthogonal polynomial sequence with respect to $w(t)$. We assume that the polynomials $\varrho_n(z; w)$ are known. On the other hand, consider the rational modification $\tilde{w}(t)$ given by

$$\tilde{w}(t) = \left| \frac{q(e^{it})}{e^{it} - \alpha} \right|^2, \ t \in [-\pi, \pi], \ \alpha \in \mathcal{C}. \tag{2.2}$$

If $\alpha \in \mathcal{T}$ then it is assumed that $q(\alpha) = 0$. We denote by $\{\varrho_n(z; \tilde{w})\}$ the monic orthogonal polynomial sequence with respect to $\tilde{w}(t)$.

THEOREM 2.1. *Let $w(t)$ be the weight function (2.1) and $\tilde{w}(t)$ be a rational modification $\tilde{w}(t) = \tilde{w}(t; \alpha_0)$ of the form (2.2) with $\alpha = \alpha_0 \in \mathcal{T}$. Then the monic orthogonal polynomials with respect to $\tilde{w}(t)$ satisfy*

$$\varrho_{n+1}(z; \tilde{w}) = (z - A_n(\alpha_0))\varrho_n(z; w) + B_n(\alpha_0)\varrho_n^*(z; w), \forall n \geq 1,$$

where $A_n(\alpha_0)$ and $B_n(\alpha_0)$ are given as in Theorem 1.1.

Proof. The proof is based on a continuity argument on the parameter α. Denote by $\mathcal{E} = \{z \in \mathcal{C}, \ |z| > 1\}$ and by $\mathcal{D} = \{z \in \mathcal{C}, \ |z| < 1\}$ the exterior and the interior of the unit disc of the complex plane, respectively. Let us consider the absolutely continuous measures μ and μ_2 on the unit circle given by $d\mu(z) = \mu'(z)|dz| = w(s)ds$ and $d\mu_2(z) = \frac{1}{|z-\alpha|^2} d\mu(z) = \tilde{w}(s)ds, \ \alpha \notin \mathcal{T}, \ z = e^{is}, \ s \in [-\pi, \pi]$. For these measures we apply Theorem 1.1. The corresponding functions $q_k(t)$ take the form

$$q_k(t) = \frac{1}{\sqrt{e_k(w)}} \int_{-\pi}^{\pi} \frac{\overline{\varrho_k(z; w)}}{t - z} |q(z)|^2 ds, \ z = e^{is}, \ t \in \mathcal{E}.$$

Observe that the functions $q_k(t)$ are *Laurent polynomials* in the variable t, that is, functions of the form $\sum_{k=m}^{n} c_k z^k, \ c_k \in \mathcal{C}, \ -\infty < m \leq k \leq n < \infty$. (Specifically, the functions $q_k(t)$ are Laurent polynomials that vanish at infinity.) Hence, the functions $q_k(t)$ are continuous functions in $\mathcal{E}$. Furthermore, for $t = \alpha_0 \in \mathcal{T}$ the value $q_k(\alpha_0)$ exists since for $\alpha_0 \in \mathcal{T}$ is assumed that $q(\alpha_0) = 0$.

On the other hand, according to [4, Proposition 1] we get

$$\frac{\Delta_n(\tilde{w}(\cdot; \alpha))}{\Delta_{n-1}(w)} = \|\tilde{w}(\cdot; \alpha)\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2 > 0, \ \alpha \in \mathcal{E}.$$

Thus, by continuity

$$\begin{aligned} \lim_{\alpha \to \alpha_0, \alpha \in \mathcal{E}} \frac{\Delta_n(\tilde{w}(\cdot; \alpha))}{\Delta_{n-1}(w)} &= \lim_{\alpha \to \alpha_0, \alpha \in \mathcal{E}} \|\tilde{w}(\cdot; \alpha)\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2 \\ &= \|\tilde{w}(\cdot; \alpha_0)\| - \sum_{j=0}^{n-1} |q_j(\alpha_0)|^2. \end{aligned}$$

Since the moments $\gamma_k(\tilde{w}) = \gamma_k(\tilde{w};\alpha) = \int_{-\pi}^{\pi} e^{-ikt}\tilde{w}(t;\alpha)dt$ are continuous functions of α, the determinant $\Delta_n(w(\cdot;\alpha))$ is also a continuous function of α and one can write

$$\lim_{\alpha\to\alpha_0,\alpha\in\mathcal{E}} \frac{\Delta_n(\tilde{w}(\cdot;\alpha))}{\Delta_{n-1}(w)} = \frac{\Delta_n(\tilde{w}(\cdot;\alpha_0))}{\Delta_{n-1}(w)}.$$

Furthermore, $\Delta_n(\tilde{w}(\cdot;\alpha_0))$ and $\Delta_{n-1}(w)$ are positive values since they are the determinants of positive definite matrices. We have obtained

$$\frac{\Delta_n(\tilde{w}(\cdot;\alpha_0))}{\Delta_{n-1}(w)} = \|\tilde{w}(\cdot;\alpha_0)\| - \sum_{j=0}^{n-1} |q_j(\alpha_0)|^2 > 0. \tag{2.3}$$

From the continuity of the functions $q_k(t)$ in $\mathcal{E}$ (for $t \in \mathcal{D}$ the functions $q_k(t)$ are also continuous since they are algebraic polynomials), we obtain that

$$A_n(\alpha) = \alpha \left[\frac{\|\tilde{w}(\cdot;\alpha)\| - \sum_{j=0}^{n} |q_j(\alpha)|^2}{\|\tilde{w}(\cdot;\alpha)\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2} \right]$$

and

$$B_n(\alpha) = \frac{1}{\alpha^{n+1}} \left[\frac{\overline{q_n(\alpha)}\,\overline{q_n(\frac{1}{\alpha})}}{\|\tilde{w}(\cdot;\alpha)\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2} \right]$$

are continuous functions of α in $\mathcal{E}$. Additionally, the values $A_n(\alpha_0)$ and $B_n(\alpha_0)$ are well defined by virtue of (2.3).

By virtue of Theorem 1.1 it holds that

$$\varrho_{n+1}(z;\tilde{w}(\cdot;\alpha)) = (z - A_n(\alpha))\varrho_n(z;w) + B_n(\alpha)\varrho_n^*(z;w),\ \alpha \in \mathcal{E}. \tag{2.4}$$

Taking limits,

$$\lim_{\alpha\to\alpha_0,\alpha\in\mathcal{E}} \varrho_{n+1}(z;\tilde{w}(\cdot;\alpha)) = \varrho_{n+1}(z;\tilde{w}(\cdot;\alpha_0))$$

since the moments $\gamma_k(\tilde{w}) = \gamma_k(\tilde{w};\alpha)$ are continuous functions of α and the orthogonal polynomials depend continuously on the moments in view of (1.1). The right-hand side of (2.4) tends to $(z - A_n(\alpha_0))\varrho_n(z;w) + B_n(\alpha_0)\varrho_n^*(z;w)$ by the continuity property of the functions $A_n(\alpha)$ and $B_n(\alpha)$. The statement of the theorem follows. □

Consider the weight function $w_N(t) = |e^{iNt} - 1|^2,\ N = 1, 2, \ldots$. Observe that $K_N(t) = \tilde{w}_N(t) = \dfrac{1}{N+1}\dfrac{w_{N+1}(t)}{|e^{it}-1|^2},\ N \geq 0$, is the Fejér kernel, see (1.6). With the help of some computational experiments based on (1.2) we state the following

THEOREM 2.2. *The reversed polynomials $\varrho_n^*(z;w_N) = z^n\overline{\varrho_n(1/\overline{z};w_N)}$ of the monic orthogonal polynomials $\varrho_n(z;w_N)$ with respect to the weight function $w_N(t) = |e^{iNt} - 1|^2,\ N \geq 1$, are given by*

$$\varrho_n^*(z;w_N) = \begin{cases} 1, & n = 0, 1, \ldots, N-1, \\ \dfrac{1}{\lfloor \frac{n}{N} \rfloor + 1} \displaystyle\sum_{k=1}^{\lfloor \frac{n}{N} \rfloor + 1} k z^{N(\lfloor \frac{n}{N} \rfloor + 1 - k)}, & n = N, N+1, \ldots, \end{cases} \tag{2.5}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Proof. Let $N \geq 1$ be fixed. The moments, $\gamma_\ell(w_N),\ \ell = 0, \pm 1, \pm 2, \ldots$, for the weight function $w_N(t)$ are given by $\gamma_0(w_N) = 4\pi,\ \gamma_{-N}(w_N) = \gamma_N(w_N) = -2\pi$, and $\gamma_\ell(w_N) = 0$, otherwise. We have to show that $\langle \varrho_n^*(z; w_N), 1\rangle_{w_N} \neq 0$ and $\langle \varrho_n^*(z; w_N), z^\ell\rangle_{w_N} = 0$, $\ell = 1, 2, \ldots, n$. This is clearly fulfilled if $0 \leq n \leq N-1$. Consider now $n \geq N$. It holds, $\langle \varrho_n^*(z; w_N), 1\rangle_{w_N} = \frac{2\pi}{\lfloor \frac{n}{N} \rfloor + 1}\left(\lfloor \frac{n}{N} \rfloor + 2\right) \neq 0$. Note that all the exponents of the variable z in the polynomial $\varrho_n^*(z; w_N)$ are multiples of N. Hence, if $\ell = 1, 2, \ldots, n$, is not a multiple of N, then $\langle \varrho_n^*(z; w_N), z^\ell\rangle_{w_N} = 0$. If $\ell = 1, 2, \ldots, n$ is a multiple of N, then $\ell = sN$, $s = 1, 2, \ldots, \lfloor \frac{n}{N} \rfloor$. In this case,

$$\begin{aligned}\langle \varrho_n^*(z; w_N), z^\ell\rangle_{w_N} &= \int_{-\pi}^{\pi} \varrho_n^*(z; w_N) \frac{1}{z^{sN}} \left(2 - z^N - \frac{1}{z^N}\right) dt \\ &= 4\pi \left(2 \frac{\lfloor \frac{n}{N} \rfloor - s + 1}{\lfloor \frac{n}{N} \rfloor + 1} - \frac{\lfloor \frac{n}{N} \rfloor - s + 2}{\lfloor \frac{n}{N} \rfloor + 1} - \frac{\lfloor \frac{n}{N} \rfloor - s}{\lfloor \frac{n}{N} \rfloor + 1}\right) = 0.\end{aligned}$$

This completes the proof. □

Thus, we have determined the monic orthogonal polynomials $\varrho_n(z; w_N)$ by virtue of the relation $\varrho_n(z; w_N) = z^n \overline{\varrho_n^*(1/\overline{z}; w_N)}$, obtaining

$$\varrho_n(z; w_N) = \begin{cases} z^n, & n = 0, 1, \ldots, N-1, \\ \dfrac{1}{\lfloor \frac{n}{N} \rfloor + 1} \displaystyle\sum_{k=1}^{\lfloor \frac{n}{N} \rfloor + 1} k z^{n - N(\lfloor \frac{n}{N} \rfloor + 1 - k)}, & n = N, N+1, \ldots. \end{cases} \tag{2.6}$$

Taking into account that

$$K_N(t) = \tilde{w}_N(t) = \frac{1}{N+1} \frac{w_{N+1}(t)}{|e^{it} - 1|^2} \ ,$$

Theorem 2.1 gives a way to find an explicit expression of the Fejér orthogonal polynomials on the unit circle. Nevertheless, this results in a large amount of tedious calculations. For this reason we proceed alternatively as follows.

THEOREM 2.3. *Let k and ℓ, $0 \leq k, \ell \leq n$, be constants such that*

$$D_n = \begin{vmatrix} \langle \varrho_n(z; w), z^k\rangle_{\tilde{w}} & \langle \varrho_n^*(z; w), z^k\rangle_{\tilde{w}} \\ \langle \varrho_n(z; w), z^\ell\rangle_{\tilde{w}} & \langle \varrho_n^*(z; w), z^\ell\rangle_{\tilde{w}} \end{vmatrix} \neq 0.$$

Then

$$\varrho_{n+1}(z; \tilde{w}) = (z - A_n)\, \varrho_n(z; w) + B_n \varrho_n^*(z; w),$$

where

$$A_n = \begin{vmatrix} \langle z\varrho_n(z; w), z^k\rangle_{\tilde{w}} & \langle \varrho_n^*(z; w), z^k\rangle_{\tilde{w}} \\ \langle z\varrho_n(z; w), z^\ell\rangle_{\tilde{w}} & \langle \varrho_n^*(z; w), z^\ell\rangle_{\tilde{w}} \end{vmatrix} / D_n$$

and

$$B_n = -\begin{vmatrix} \langle \varrho_n(z; w), z^k\rangle_{\tilde{w}} & \langle z\varrho_n(z; w), z^k\rangle_{\tilde{w}} \\ \langle \varrho_n(z; w), z^\ell\rangle_{\tilde{w}} & \langle z\varrho_n(z; w), z^\ell\rangle_{\tilde{w}} \end{vmatrix} / D_n\ .$$

(There are values of k and ℓ such that $D_n \neq 0$ as is shown in Theorem 2.4, below.)

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Proof. By virtue of Theorem 2.1 there exist constants A_n and B_n such that $\varrho_{n+1}(z;\tilde{w}) = (z - A_n)\,\varrho_n(z;w) + B_n\varrho_n^*(z;w)$. For any $\nu,\ 0 \le \nu \le n$, it holds that

$$\langle \varrho_{n+1}(z;\tilde{w}), z^\nu\rangle_{\tilde{w}} = 0 = \langle z\varrho_n(z;w), z^\nu\rangle_{\tilde{w}} - A_n\langle \varrho_n(z;w), z^\nu\rangle_{\tilde{w}} + B_n\langle \varrho_n^*(z;w), z^\nu\rangle_{\tilde{w}}.$$

The proof follows from the classical Cramer's rule. □

The moments $\gamma_k(\tilde{w}_N),\ k = 0, \pm 1, \pm 2, \dots,\ N = 0, 1, 2, \dots,$ for the Fejér kernel are given by (see [12])

$$\gamma_k(\tilde{w}_N) = \int_{-\pi}^{\pi} e^{-ikt}\tilde{w}_N(t)dt = \begin{cases} 2\pi\left(1 - \frac{|k|}{N+1}\right), & \text{if } |k| \le N, \\ 0, & \text{if } |k| > N. \end{cases}$$

We need the following observation

$$\langle \varrho_n^*(z;w_{N+1}), z^{n-k}\rangle_{\tilde{w}_N} = \overline{\langle \varrho_n^*(z;w_{N+1}), z^{n-k}\rangle}_{\tilde{w}_N} = \langle \varrho_n(z;w_{N+1}), z^k\rangle_{\tilde{w}_N},\ 0 \le k \le n,$$

for the next theorem that gives the explicit expression of the monic Fejér orthogonal polynomials $\varrho_n(z;K_N)$ for $n = N+1, N+2, \dots,$ and $N = 1, 2, \dots$.

THEOREM 2.4. *Let* $n = m(N+1) + s,\ m \ge 1,\ 0 \le s \le N$. *Then*

$$\varrho_{n+1}(z;\tilde{w}_N) = \left(z - \frac{mN + 2N + m + 1 - s}{mN + 2N + m + 2 - s}\right)\varrho_n(z;w_{N+1}) + \frac{\varrho_n^*(z;w_{N+1})}{mN + 2N + m + 2 - s}\,,$$

where $\varrho_n^*(z;w_{N+1})$ *and* $\varrho_n(z;w_{N+1}),\ (n \ge N+1)$, *are given by (2.5) and (2.6), respectively.*

Proof. Consider the values $k = n$ and $\ell = 0$ in Theorem 2.3. It holds that

$$\begin{aligned}
\langle \varrho_n(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} &= \langle \varrho_n^*(z;w_{N+1}), 1\rangle_{\tilde{w}_N} = 2\pi, \\
\langle z\varrho_n(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} &= \langle \varrho_n(z;w_{N+1}), z^{n-1}\rangle_{\tilde{w}_N} \\
&= \langle \varrho_n^*(z;w_{N+1}), z\rangle_{\tilde{w}_N} = \frac{2\pi}{N+1}\left(N + \frac{\lfloor \frac{n}{N+1}\rfloor}{1 + \lfloor \frac{n}{N+1}\rfloor}\right), \\
\langle \varrho_n^*(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} &= \langle \varrho_n(z;w_{N+1}), 1\rangle_{\tilde{w}_N} = \frac{2\pi}{1 + \lfloor \frac{n}{N+1}\rfloor}\left(1 - \frac{s}{N+1}\right), \\
\langle z\varrho_n(z;w_{N+1}), 1\rangle_{\tilde{w}_N} &= \frac{2\pi}{1 + \lfloor \frac{n}{N+1}\rfloor}\left(1 - \frac{s+1}{N+1}\right).
\end{aligned}$$

Hence, after some computations we get

$$\begin{aligned}
D_n &= \begin{vmatrix} \langle \varrho_n(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} & \langle \varrho_n^*(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} \\ \langle \varrho_n(z;w_{N+1}), 1\rangle_{\tilde{w}_N} & \langle \varrho_n^*(z;w_{N+1}), 1\rangle_{\tilde{w}_N} \end{vmatrix} \\
&= \frac{4\pi^2(m^2N^2 + 2m^2N + m^2 + 2mN^2 + 4mN + 2m + 2Ns + 2s - s^2)}{(m+1)^2(N+1)^2} \neq 0,
\end{aligned}$$

where $m = \lfloor \frac{n}{N+1}\rfloor$. Taking into account $2mN^2 > s^2$, from Theorem 2.3 we deduce

$$\varrho_{n+1}(z;\tilde{w}_N) = (z - A_n)\,\varrho_n(z;w_{N+1}) + B_n\varrho_n^*(z;w_{N+1})$$

where

$$\begin{aligned}
A_n &= \begin{vmatrix} \langle z\varrho_n(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} & \langle \varrho_n^*(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} \\ \langle z\varrho_n(z;w_{N+1}), 1\rangle_{\tilde{w}_N} & \langle \varrho_n^*(z;w_{N+1}), 1\rangle_{\tilde{w}_N} \end{vmatrix} / D_n \\
&= \frac{mN + 2N + m + 1 - s}{mN + 2N + m + 2 - s}
\end{aligned}$$

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and

$$B_n = -\begin{vmatrix} \langle \varrho_n(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} & \langle z\varrho_n(z;w_{N+1}), z^n\rangle_{\tilde{w}_N} \\ \langle \varrho_n(z;w_{N+1}), 1\rangle_{\tilde{w}_N} & \langle z\varrho_n(z;w_{N+1}), 1\rangle_{\tilde{w}_N} \end{vmatrix} / D_n$$
$$= \frac{1}{mN+2N+m+2-s}.$$

The proof is complete. □

In the following theorem we give the whole sequence of monic Fejér orthogonal polynomials.

THEOREM 2.5. *Let $N \geq 1$ be given. The monic Fejér orthogonal polynomials $\varrho_n(z;K_N)$ reproduce*

$$\varrho_0(z;K_N) = 1$$

and

$$\varrho_{n+1}(z;K_N) = \left(z - \frac{mN+2N+m+1-s}{mN+2N+m+2-s}\right)\varrho_n(z;w_{N+1}) + \frac{\varrho_n^*(z;w_{N+1})}{mN+2N+m+2-s},$$

where $n = m(N+1)+s \geq 0$, $m = \lfloor \frac{n}{N+1} \rfloor \geq 0$, $0 \leq s \leq N$, and where $\varrho_n^*(z;w_N)$ and $\varrho_n(z;w_N)$ are given by (2.5) and (2.6), respectively.

Proof. For $m \geq 1$, and hence $n \geq N+1$, we deal with the monic Fejér orthogonal polynomials $\varrho_n(z;K_N)$ given in Theorem 2.4. For $m=0$, is easy to check that they reproduce the monic Fejér orthogonal polynomials for $0 \leq n \leq N$ given in (1.8). □

COROLLARY 2.6. *Let $N \geq 1$ be given. The Verblunsky coefficients $\delta_n = \delta_n(K_N)$, $n = 0,1,2,\dots$, corresponding to the Fejér kernel $K_N(t)$ are given by $\delta_0 = 1$ and*

$$\delta_{n+1} = \begin{cases} -\dfrac{N}{(m+1)(N+1)}, & \text{if } n = m(N+1), m = 0,1,2,\dots, \\ \dfrac{1}{(m+2)(N+1)-s}, & \text{if } n = m(N+1)+s, 1 \leq s \leq N, m = 0,1,2,\dots. \end{cases}$$

Proof. The above expression for the Verblunsky coefficients $\delta_n(K_N) = \varrho_n(0;K_N)$ follows from Theorem 2.5 and Eq. (2.6). □

EXAMPLE 2.7. We consider $N = 3$. Then we deal with the Fejér kernel $K_3(t) = \tilde{w}_3(t) = \dfrac{1}{4}\dfrac{w_4(t)}{|e^{it}-1|^2}$. Consider, for example, $n = 0,1,2,\dots,10$. Then, the corresponding monic orthogonal polynomials $\varrho_n(z;K_3)$ are, according to Theorem 2.5,

$$\begin{aligned} \varrho_0(z;K_3) &= 1, \\ \varrho_1(z;K_3) &= (z - \frac{7}{8})\varrho_0(z;w_4) + \frac{1}{8}\varrho_0^*(z;w_4) \\ &= z - \frac{3}{4}, \\ \varrho_2(z;K_3) &= (z - \frac{6}{7})\varrho_1(z;w_4) + \frac{1}{7}\varrho_1^*(z;w_4) \\ &= z^2 - \frac{6}{7}z + \frac{1}{7}, \end{aligned}$$

$$\begin{aligned}
\varrho_3(z;K_3) &= (z-\frac{5}{6})\varrho_2(z;w_4)+\frac{1}{6}\varrho_2^*(z;w_4)\\
&= z^3-\frac{5}{6}z^2+\frac{1}{6},\\
\varrho_4(z;K_3) &= (z-\frac{4}{5})\varrho_3(z;w_4)+\frac{1}{5}\varrho_3^*(z;w_4)\\
&= z^4-\frac{4}{5}z^3+\frac{1}{5},\\
\varrho_5(z;K_3) &= (z-\frac{11}{12})\varrho_4(z;w_4)+\frac{1}{12}\varrho_4^*(z;w_4)\\
&= z^5-\frac{7}{8}z^4+\frac{1}{2}z-\frac{3}{8},\\
\varrho_6(z;K_3) &= (z-\frac{10}{11})\varrho_5(z;w_4)+\frac{1}{11}\varrho_5^*(z;w_4)\\
&= z^6-\frac{10}{11}z^5+\frac{1}{22}z^4+\frac{1}{2}z^2-\frac{5}{11}z+\frac{1}{11},\\
\varrho_7(z;K_3) &= (z-\frac{9}{10})\varrho_6(z;w_4)+\frac{1}{10}\varrho_6^*(z;w_4)\\
&= z^7-\frac{9}{10}z^6+\frac{1}{20}z^4+\frac{1}{2}z^3-\frac{9}{20}z^2+\frac{1}{10},\\
\varrho_8(z;K_3) &= (z-\frac{8}{9})\varrho_7(z;w_4)+\frac{1}{9}\varrho_7^*(z;w_4)\\
&= z^8-\frac{8}{9}z^7+\frac{5}{9}z^4-\frac{4}{9}z^3+\frac{1}{9},\\
\varrho_9(z;K_3) &= (z-\frac{15}{16})\varrho_8(z;w_4)+\frac{1}{16}\varrho_8^*(z;w_4)\\
&= z^9-\frac{11}{12}z^8+\frac{2}{3}z^5-\frac{7}{12}z^4+\frac{1}{3}z-\frac{1}{4},\\
\varrho_{10}(z;K_3) &= (z-\frac{14}{15})\varrho_9(z;w_4)+\frac{1}{15}\varrho_9^*(z;w_4)\\
&= z^{10}-\frac{14}{15}z^9+\frac{1}{45}z^8+\frac{2}{3}z^6-\frac{28}{45}z^5+\frac{2}{45}z^4+\frac{1}{3}z^2-\frac{14}{45}z+\frac{1}{15}.
\end{aligned}$$

We point out that these polynomials were computed as an example in [12], although there they were calculated using a different method.

3. Computation of the monic orthogonal polynomials associated with a rational modification of a measure. Let μ be a finite positive Borel measure on $\mathcal{T}$. Consider the rational modification

$$d\tilde{\mu} = \frac{1}{|z-\alpha|^2}d\mu,\ \alpha \notin \mathcal{T}.$$

By virtue of Theorem 1.1, there are constants A_n and B_n such that

$$\varrho_{n+1}(z;\tilde{\mu}) = (z-A_n)\varrho_n(z;\mu)+B_n\varrho_n^*(z;\mu).$$

The formulas for these constants given in Theorem 1.1 are not appropriate for computation. In this section we are interested in obtaining a simple and efficient algorithm to compute the orthogonal polynomials $\varrho_n(z;\tilde{\mu})$ in terms of the orthogonal polynomials $\varrho_n(z;\mu)$. With this goal and as a starting point we give alternative expressions for the constants A_n and B_n.

We remark that the results in this section also hold for weight functions $w(t)$ of the form (2.1) and a rational modification $\tilde{w}(t)$ of the form (2.2). In this case, the existence of the constants A_n and B_n is proved in Theorem 2.1.

THEOREM 3.1. *Let $n \geq 1$ be given. Then*

$$\varrho_{n+1}(z;\tilde{\mu}) = (z - A_n)\varrho_n(z;\mu) + B_n\varrho_n^*(z;\mu) \tag{3.1}$$

where

$$A_n = \alpha \frac{\langle \varrho_{n+1}(z;\tilde{\mu}), \varrho_{n+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}}{\langle \varrho_n(z;\mu), \varrho_n(z;\mu)\rangle_\mu}$$

and

$$B_n = -\frac{\langle z\varrho_n(z;\mu), 1\rangle_\mu + \overline{\alpha}\langle z\varrho_{n+1}(z;\tilde{\mu}), 1\rangle_{\tilde{\mu}}}{\langle \varrho_n^*(z;\mu), 1\rangle_\mu}.$$

Proof. From (3.1) and taking into account that $\langle \varrho_n(z;\mu), 1\rangle_\mu = 0$, we can write

$$\langle \varrho_{n+1}(z;\tilde{\mu}), 1\rangle_\mu = \langle z\varrho_n(z;\mu), 1\rangle_\mu + B_n\langle \varrho_n^*(z;\mu), 1\rangle_\mu.$$

Note that

$$\begin{aligned}\langle \varrho_{n+1}(z;\tilde{\mu}), 1\rangle_\mu &= \int_{-\pi}^{\pi} \varrho_{n+1}(e^{it};\tilde{\mu}) \left(1 + |\alpha|^2 - \overline{\alpha}e^{it} - \alpha e^{-it}\right) \frac{d\mu}{|e^{it} - \alpha|^2} \\ &= -\overline{\alpha}\langle z\varrho_{n+1}(z;\tilde{\mu}), 1\rangle_{\tilde{\mu}}.\end{aligned}$$

The value of B_n follows. From (3.1) we can write

$$\begin{aligned}\langle \varrho_{n+1}(z;\tilde{\mu}), \varrho_n(z;\mu)\rangle_\mu = \langle (z - A_n)\varrho_n(z;\mu), \varrho_n(z;\mu)\rangle_\mu + \\ B_n\langle \varrho_n^*(z;\mu), \varrho_n(z;\mu)\rangle_\mu.\end{aligned} \tag{3.2}$$

The following relations hold

$$\begin{aligned}\langle \varrho_{n+1}(z;\tilde{\mu}), \varrho_n(z;\mu)\rangle_\mu &= \int_{-\pi}^{\pi} \varrho_{n+1}(e^{it};\tilde{\mu})\overline{\varrho_n(e^{it};\mu)} \left(1 + |\alpha|^2 - \alpha e^{-it} - \overline{\alpha}e^{it}\right) d\tilde{\mu} \\ &= -\alpha\langle \varrho_{n+1}(z;\tilde{\mu}), \varrho_{n+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}} - \overline{\alpha}\overline{\delta_n(\mu)}\langle z\varrho_{n+1}(z;\tilde{\mu}), 1\rangle_{\tilde{\mu}}, \\ \langle z\varrho_n(z;\mu), \varrho_n(z;\mu)\rangle_\mu &= \overline{\delta_n(\mu)}\langle z\varrho_n(z;\mu), 1\rangle_\mu, \\ \langle \varrho_n^*(z;\mu), \varrho_n(z;\mu)\rangle_\mu &= \overline{\delta_n(\mu)}\langle \varrho_n^*(z;\mu), 1\rangle_\mu.\end{aligned}$$

Replacing these relations and the obtained value of B_n in (3.2), we get A_n. □

The unknown values $\langle \varrho_{n+1}(z;\tilde{\mu}), \varrho_{n+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}$ and $\langle z\varrho_{n+1}(z;\tilde{\mu}), 1\rangle_{\tilde{\mu}}$ appearing in the obtained expression of A_n and B_n are iteratively computed within the next algorithm.

THEOREM 3.2. *For $n \geq 1$ it holds that*

$$\delta_{n+2}(\tilde{\mu}) = \frac{\alpha}{\overline{\alpha}}\delta_n(\mu) + \frac{1}{\overline{\alpha}} \frac{\langle \varrho_n(z;\mu), \varrho_n(z;\mu)\rangle_\mu}{\langle \varrho_{n+1}(z;\tilde{\mu}), \varrho_{n+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}}(\delta_{n+1}(\tilde{\mu}) - \delta_{n+1}(\mu)).$$

Proof. By setting $z = 0$ in (3.1) we obtain

$$\begin{aligned}\delta_{n+1}(\tilde{\mu}) = -\alpha \frac{\langle \varrho_{n+1}(z;\tilde{\mu}), \varrho_{n+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}}{\langle \varrho_n(z;\mu), \varrho_n(z;\mu)\rangle_\mu}\delta_n(\mu) \\ - \frac{\langle z\varrho_n(z;\mu), 1\rangle_\mu + \overline{\alpha}\langle z\varrho_{n+1}(z;\tilde{\mu}), 1\rangle_{\tilde{\mu}}}{\langle \varrho_n^*(z;\mu), 1\rangle_\mu}.\end{aligned} \tag{3.3}$$

Using the recurrence relation (1.5) for $\tilde{\mu}$ and $n+1$, we get

$$(3.4)\qquad \begin{aligned} \langle z\varrho_{n+1}(z;\tilde{\mu}),1\rangle_{\tilde{\mu}} &= -\delta_{n+2}(\tilde{\mu})\langle \varrho^*_{n+1}(z;\tilde{\mu}),1\rangle_{\tilde{\mu}} \\ &= -\delta_{n+2}(\tilde{\mu})\langle \varrho_{n+1}(z;\tilde{\mu}),\varrho_{n+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}} \end{aligned}$$

and

$$(3.5)\quad \langle z\varrho_n(z;\mu),1\rangle_{\mu} = -\delta_{n+1}(\mu)\langle \varrho^*_n(z;\mu),1\rangle_{\mu} = -\delta_{n+1}(\mu)\langle \varrho_n(z;\mu),\varrho_n(z;\mu)\rangle_{\mu}.$$

The result follows substituting (3.4) and (3.5) into (3.3) and solving for $\delta_{n+2}(\tilde{\mu})$. □

The proposed algorithm to compute the constants A_n and B_n is given below. We will use the relation

$$\langle \varrho_n(z;\tilde{\mu}),\varrho_n(z;\tilde{\mu})\rangle_{\tilde{\mu}} = \langle \varrho_{n-1}(z;\tilde{\mu}),\varrho_{n-1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}(1-|\delta_n(\tilde{\mu})|^2),\, n=1,2,\dots,$$

which is a consequence of the recurrence (1.5).

ALGORITHM 3.3. Algorithm to compute the constants A_n and B_n

Input: $\alpha,\ \gamma_0(\tilde{\mu}),\ \gamma_{-1}(\tilde{\mu}),\ \gamma_{-2}(\tilde{\mu}),\ n,\ \langle \varrho_\ell(z;\mu),\varrho_\ell(z;\mu)\rangle_\mu,\ \ell=1,2,\dots,n$, and $\delta_\ell(\mu),\ \ell=1,2,\dots,n+1$.

Output: $A_\ell,\ B_\ell,\ \ell=1,2,\dots,n.$

Computation of the initial values, $\delta_1(\tilde{\mu}),\ \langle z\varrho_1(z;\tilde{\mu}),1\rangle_{\tilde{\mu}},\ \langle \varrho_1(z;\tilde{\mu}),\varrho_1(z;\tilde{\mu})\rangle_{\tilde{\mu}}$ and $\delta_2(\tilde{\mu})$.

Compute $\delta_1(\tilde{\mu})$ from $\langle \varrho_1(z;\tilde{\mu}),1\rangle_{\tilde{\mu}} = \langle z+\delta_1(\tilde{\mu}),1\rangle_{\tilde{\mu}} = 0.$

Step 1. $\delta_1(\tilde{\mu}) \leftarrow -\dfrac{\gamma_{-1}(\tilde{\mu})}{\gamma_0(\tilde{\mu})}$;

Compute $\langle z\varrho_1(z;\tilde{\mu}),1\rangle_{\tilde{\mu}} = \langle z(z+\delta_1(\tilde{\mu})),1\rangle_{\tilde{\mu}} = \gamma_{-2}(\tilde{\mu})+\delta_1(\tilde{\mu})\gamma_{-1}(\tilde{\mu}).$

Step 2. $\langle z\varrho_1(z;\tilde{\mu}),1\rangle_{\tilde{\mu}} \leftarrow \gamma_{-2}(\tilde{\mu})+\delta_1(\tilde{\mu})\gamma_{-1}(\tilde{\mu})$;

Compute $\langle \varrho_1(z;\tilde{\mu}),\varrho_1(z;\tilde{\mu})\rangle_{\tilde{\mu}} = \langle \varrho_0(z;\tilde{\mu}),\varrho_0(z;\tilde{\mu})\rangle_{\tilde{\mu}}(1-|\delta_1(\tilde{\mu})|^2) = \gamma_0(\tilde{\mu})\left(1-\left|\dfrac{\gamma_{-1}(\tilde{\mu})}{\gamma_0(\tilde{\mu})}\right|^2\right).$

Step 3. $\langle \varrho_1(z;\tilde{\mu}),\varrho_1(z;\tilde{\mu})\rangle_{\tilde{\mu}} \leftarrow \gamma_0(\tilde{\mu})\left(1-\left|\dfrac{\gamma_{-1}(\tilde{\mu})}{\gamma_0(\tilde{\mu})}\right|^2\right)$;

Step 4. $\delta_2(\tilde{\mu}) \leftarrow -\dfrac{\langle z\varrho_1(z;\tilde{\mu}),1\rangle_{\tilde{\mu}}}{\langle \varrho_1(z;\tilde{\mu}),\varrho_1(z;\tilde{\mu})\rangle_{\tilde{\mu}}}$;

Step 5. for $\ell=1,2,\dots,n$ **do**

$\langle \varrho_{\ell+1}(z;\tilde{\mu}),\varrho_{\ell+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}} \leftarrow \langle \varrho_\ell(z;\tilde{\mu}),\varrho_\ell(z;\tilde{\mu})\rangle_{\tilde{\mu}}(1-|\delta_{\ell+1}(\tilde{\mu})|^2)$;

$A_\ell \leftarrow \alpha\dfrac{\langle \varrho_{\ell+1}(z;\tilde{\mu}),\varrho_{\ell+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}}{\langle \varrho_\ell(z;\mu),\varrho_\ell(z;\mu)\rangle_\mu}$;

$\delta_{\ell+2}(\tilde{\mu}) \leftarrow \dfrac{\alpha}{\overline{\alpha}}\delta_\ell(\mu)+\dfrac{1}{\overline{\alpha}}\dfrac{\langle \varrho_\ell(z;\mu),\varrho_\ell(z;\mu)\rangle_\mu}{\langle \varrho_{\ell+1}(z;\tilde{\mu}),\varrho_{\ell+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}}(\delta_{\ell+1}(\tilde{\mu})-\delta_{\ell+1}(\mu))$;

$\langle z\varrho_{\ell+1}(z;\tilde{\mu}),1\rangle_{\tilde{\mu}} \leftarrow -\delta_{\ell+2}(\tilde{\mu})\langle \varrho_{\ell+1}(z;\tilde{\mu}),\varrho_{\ell+1}(z;\tilde{\mu})\rangle_{\tilde{\mu}}$;

Compute B_ℓ by Theorem 3.1 and (3.5).

$B_\ell \leftarrow \dfrac{\delta_{\ell+1}(\mu)\langle \varrho_\ell(z;\mu),\varrho_\ell(z;\mu)\rangle_\mu - \overline{\alpha}\langle z\varrho_{\ell+1}(z;\tilde{\mu}),1\rangle_{\tilde{\mu}}}{\langle \varrho_\ell(z;\mu),\varrho_\ell(z;\mu)\rangle_\mu}$;

end

Observe that the number of operations needed to compute A_k and $B_k,\ k=1,\dots,n$, is $O(n)$. Once these constants are computed, the orthogonal polynomials $\varrho_k(z;\tilde{\mu})$ are obtained from the relation $\varrho_{k+1}(z;\tilde{\mu}) = (z-A_k)\varrho_k(z;\mu)+B_k\varrho^*_k(z;\mu)$. On the other hand, Levinson's algorithm [10] computes the Verblunsky coefficients $\{\delta_k(\tilde{\mu})\}_{k=1}^n$ with a number of operations of order $O(n^2)$. Then the orthogonal polynomials $\varrho_k(z;\tilde{\mu})$ can be obtained from the forward recurrence relation, $\varrho_k(z;\tilde{\mu}) = z\varrho_{k-1}(z;\tilde{\mu}) + \delta_k(\tilde{\mu})\varrho^*_{k-1}(z;\tilde{\mu})$. This reduction in the number of operations in our method is natural. One expects a reduction in the number of operations when computing the orthogonal polynomials associated with a rational modification of a measure if one uses the orthogonal polynomials of the initial measure.

EXAMPLE 3.4. We have implemented a Maple procedure to illustrate the proposed algorithm. We executed the procedure for the previously considered weight function $w_4(t) = |e^{i4t} - 1|^2$ and its rational modification $\tilde{w}_3(t) = K_3(t) = \dfrac{1}{4}\dfrac{w_4(t)}{|e^{it}-1|^2}$. We need the initial data $\alpha = 1,\ \gamma_0(\tilde{w}_3) = 2\pi,\ \gamma_{-1}(\tilde{w}_3) = \frac{3\pi}{2}$ and $\gamma_{-2}(\tilde{w}_3) = \pi$. Furthermore, we need that $\langle \varrho_n(z;w_4), \varrho_n(z;w_4)\rangle_{w_4} = \pi\frac{\lfloor \frac{n}{4}\rfloor + 2}{2(\lfloor \frac{n}{4}\rfloor + 1)},\ n \geq 0$, and $\delta_n(w_4) = \frac{1}{m+1}$ if $n = 4m,\ m \geq 0$ and $\delta_n(w_4) = 0$, otherwise. The obtained values of A_n and B_n are the ones obtained by the corresponding formula for A_n and B_n given in Theorem 2.5.

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REFERENCES

[1] K. DECKERS AND A. BULTHEEL, *Orthogonal rational functions and rational modifications of a measure on the unit circle*, J. Approx. Theory, 157 (2009), pp. 1–18.
[2] M. GALAZ-LARIOS, R. GARCÍA-OLIVO, AND J. LÓPEZ-BONILLA, *Fejér and Dirichlet kernels: Their associated polynomials*, J. Sci. Res., 1 (2) (2009), pp. 281–284.
[3] E. GODOY AND F. MARCELLÁN, *An analog of the Christoffel formula for polynomial modification of a measure on the unit circle*, Boll. Un. Mat. Ital., 5-A (7) (1991), pp. 1–12.
[4] ———, *Orthogonal polynomials and rational modifications of measures*, Canad. J. Math., 45 (5) (1993), pp. 930–943.
[5] P. GONZÁLEZ-VERA, O. NJÅSTAD, AND J. C. SANTOS-LEÓN, *Some results about numerical quadrature on the unit circle*, Adv. Comput. Math., 5 (1996), pp. 297–328.
[6] W. B. GRAGG, *Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle* (in Russian), in E. S. Nicholaev, ed., Numerical Methods in Linear Algebra, pp. 16–32, Moscow University Press, Moscow, 1982. Published in English in slightly modified form in J. Comput. Appl. Math., 46 (1993), pp. 183–189.
[7] F. HERZOG AND G. PIRANIAN, *Some properties of the Fejér polynomials*, Proc. Amer. Math. Soc., 7 (3) (1956), pp. 379–386.
[8] M. E. H. ISMAIL AND R. W. RUEDEMANN, *Relation between polynomials orthogonal on the unit circle with respect to different weights*, J. Approx. Theory, 71 (1992), pp. 39–60.
[9] W. B. JONES, O. NJÅSTAD, AND W. J. THRON, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc., 21 (1989), pp. 113–152.
[10] N. LEVINSON, *The Wiener RMS (root mean square), error criterion in filter design and prediction*, J. Math. and Physics, 25 (1947), pp. 261–278.
[11] J. OBERMAIER, *A modified Fejér and Jackson summability method with respect to orthogonal polynomials*, J. Approx. Theory, 163 (2011), pp. 554–567.
[12] J.-C. SANTOS-LEÓN, *Szegö polynomials and Szegö quadrature for the Fejér kernel*, J. Comput. Appl. Math., 179 (2005), pp. 327–341.
[13] B. SIMON, *Orthogonal Polynomials on the Unit Circle*, Amer. Math. Soc. Colloq. Publ. 54, Amer. Math. Soc., Providence, RI, 2005. Two volumes
[14] G. SZEGÖ, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, New York, 1939, Fourth ed., 1975.
[15] A. ZYGMUND, *Trigonometric Series*, Vol. I and II, Cambridge University Press, Cambridge, 1977.