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EXTREMAL INTERPOLATORY PROBLEM OF FEJÉR TYPE FOR ALL CLASSICAL WEIGHT FUNCTIONS*

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Abstract. Several constructive solutions of interpolating problems of Fejér, Egerváry and Turán, connected with the optimal, most economical and stable interpolation are known for Jacobi, Hermite and Laguerre orthogonal polynomials. In this paper we solve the interpolatory weighted problem of Fejér type for all positive solutions of the Pearson differential equation, which generate finite or infinite sequences of the classical orthogonal polynomials. More precisely, we establish that the Fejér problem is generic in this class of polynomials and present an elementary unified proof of this fact. Next, these results are used to establish a complete solution of the Egerváry and Turán interpolatory problem.

Key words. Fejér extremal problem, Pearson differential equation, Sturm-Liouville differential equation, classical orthogonal polynomials, Lagrange optimal interpolation, weighted stability, most economical systems.

AMS subject classifications. 41A05, 42C05, 65D05, 49K35.

1. Introduction and preliminaries. Let $w(x)$ be a positive weight function on a finite or infinite interval (a,b). This weight function is said to be classical, if it satisfies the Pearson differential equation

$$\frac{d}{dx}\left[A(x)\,w(x)\right] = B(x)\,w(x)\,,\ a<x<b,$$

and boundary conditions of the form

$$\lim_{x\downarrow a} A(x)\,w(x) = \lim_{x\uparrow b} A(x)\,w(x) = 0,$$

where the polynomials

$$A(x) = a_0 + a_1 x + a_2 x^2 \text{ and } B(x) = b_0 + b_1 x$$

are such that $A(x) > 0$ on (a,b) and $b_1 \neq 0$.

Moreover, let $q_n(x)$, $n = 0,1,\ldots$, denote the sequence of polynomials of degree n, orthogonal with respect to the inner product

$$(f,g)_w = \int_a^b f(x)\,g(x)\,w(x)\,dx$$

in the Hilbert space $L_w^2(a,b)$, where $w(x)$ is a classical weight function. Additionally, if there exists a finite or infinite n_w such that the orthogonal polynomials $q_n(x)$ $(0 \le n < n_w)$ are solutions of the following Sturm-Liouville differential equation

$$\frac{d}{dx}\left[A(x)\,w(x)\,\frac{d}{dx}q_n(x)\right] = \lambda_n w(x)\,q_n(x)\,,\ a<x<b, \tag{1.1}$$

with the coefficients λ_n equal to

$$\lambda_n = n\left[(n-1)\,a_2 + b_1\right],$$

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then the sequence $q_n(x)$ $(0 \leq n < n_w)$ is called classical; cf. Agarwal and Milovanović [1], Al-Salam [2], Koekoek et al. [22], Lesky [31], Maroni [34], Mastroianni and Milovanović [35], Nikiforov and Uvarov [36] and Suetin [40]. Conversely, linearly independent polynomial $L^2_w(a,b)$-solutions of the equation (1.1) are orthogonal with respect to the classical weight function $w(x)$; cf. Bochner [6], Lesky [30] and Koepf and Masjed-Jamei [23]. We note that the classical orthogonal polynomials were called continuous classical orthogonal polynomials and extensively studied by Koekoek, Lesky, and Swarttouw in their recent monograph [22].

It is interesting and important that the sequence of derivatives $q_n^{(k)}(x)$ $(k<n<n_w)$ of the classical orthogonal polynomials $q_n(x)$ $(0 < n < n_w)$ is also the sequence of classical polynomials, orthogonal with respect to the classical weight function

$$w_k(x) = A^k(x)\, w(x)\,,\ a < x < b.$$

In this case the weight function $w_k(x)$ satisfies the Pearson differential equation of the form

$$\frac{d}{dx}\left[A(x)\, w_k(x)\right] = \left[B(x) + kA'(x)\right] w_k(x)\,,\ a < x < b, \tag{1.2}$$

where the coefficient $b_1 + 2ka_2$ at x on the right-hand side should be distinct from zero for $k = 0, 1, \cdots, n-1$. Moreover the derivatives $q_n^{(k)}(x)$ $(k < n < n_w)$ satisfy the following Sturm-Liouville differential equation

$$\frac{d}{dx}\left[A(x)\, w_k(x)\, \frac{d}{dx} q_n^{(k)}(x)\right] = \lambda_{n,k} w_k(x)\, q_n^{(k)}(x)\,,\ a < x < b,$$

with coefficients

$$\lambda_{n,k} = (n-k)\left[(n+k-1)\, a_2 + b_1\right];$$

cf. Hahn [15], Krall [25, 26, 27], Agarwal and Milovanović [1] and Mastroianni and Milovanović [35].

REMARK 1.1. In view of the six statements (a)-(f) from the proof of Lemma 2.6 presented in Section 2, it is clear that the conditions $b_1 + 2ka_2 \neq 0$ $(0 \leq k < n < n_w)$ are satisfied, whenever the classical weight $w(x)$ admits the existence of classical orthogonal polynomials of degree $n < n_w$.

It is important to note that there exist exactly six classes, up to a linear change of variable, of the classical orthogonal polynomials [22, page 93]. They can be expressed by the following Rodrigues formula

$$q_n(x) = \frac{\kappa_n}{w(x)} \frac{d^n}{dx^n}\left[w(x)\, A^n(x)\right],\ 0 \leq n < n_w,$$

where $\kappa_n \neq 0$ are arbitrary constants. For the simplicity, we will use below the notation $w_0(x) = w(x)$. We will also use the floor function $\lfloor x \rfloor$ which is the largest integer less than or equal to x. Moreover we note that $n_w = +\infty$ in the cases (i), (ii) and (iii) given below.

(i) *Hermite classical orthogonal polynomials* with $(a,b) = (-\infty, +\infty)$, $A(x) = 1$, $B(x) = -2x$ and

$$w_k(x) = e^{-x^2}.$$

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(ii) *Jacobi classical orthogonal polynomials*, whenever $(a,b) = (-1,1)$, $A(x) = 1 - x^2$, $B(x) = \beta - \alpha - (\alpha + \beta + 2)x$, $\alpha > -1$, $\beta > -1$ and

$$w_k(x) = (1-x)^{\alpha+k}(1+x)^{\beta+k}.$$

(iii) *Laguerre classical orthogonal polynomials* correspond to $(a,b) = (0,+\infty)$, $A(x) = x$, $B(x) = \alpha + 1 - x$, $\alpha > -1$ and

$$w_k(x) = x^{\alpha+k}e^{-x}.$$

(iv) *The generalized Bessel classical orthogonal polynomials* $q_n(x)$ $(0 \le n < n_w)$ with $(a,b) = (0,+\infty)$, $A(x) = x^2$, $B(x) = \alpha x + \beta$, $\alpha < -1$, $\alpha \notin \{-2,-3,\ldots\}$, $\beta > 0$ and

$$w_k(x) = x^{\alpha+2k-2}e^{-\frac{\beta}{x}}, \; n_w = \left\lfloor \frac{1-\alpha}{2} \right\rfloor.$$

(v) *Jacobi classical orthogonal polynomials* $q_n(x)$ $(0 \le n < n_w)$ *on* $(0,+\infty)$, whenever $(a,b) = (0,+\infty)$, $A(x) = x^2 + x$, $B(x) = (2-\alpha)x + \beta + 1$, $\alpha \ne 2$, $\beta > -1$ and

$$w_k(x) = \frac{x^{\beta+k}}{(1+x)^{\alpha+\beta-k}}, \; n_w = \left\lfloor \frac{\alpha-1}{2} \right\rfloor.$$

(vi) *Pseudo-Jacobi classical orthogonal polynomials* $q_n(x)$ $(0 \le n < n_w)$ correspond to $(a,b) = (-\infty,+\infty)$, $\alpha \ne 1$, $\beta \in \mathbb{R}$, $n_w = \lfloor \alpha - \frac{1}{2} \rfloor$ and

$$A(x) = x^2 + 2\frac{\mathcal{AB}+\mathcal{CD}}{\mathcal{A}^2+\mathcal{C}^2}x + \frac{\mathcal{B}^2+\mathcal{D}^2}{\mathcal{A}^2+\mathcal{C}^2},$$

$$B(x) = 2(1-\alpha)x + \frac{\beta(\mathcal{AD}-\mathcal{BC}) + 2(1-\alpha)(\mathcal{AB}+\mathcal{CD})}{\mathcal{A}^2+\mathcal{C}^2},$$

$$w_k(x) = \left[\frac{(\mathcal{A}x+\mathcal{B})^2 + (\mathcal{C}x+\mathcal{D})^2}{(\mathcal{A}^2+\mathcal{C}^2)}\right]^{-\alpha+k} e^{\beta \arctan \frac{\mathcal{A}x+\mathcal{B}}{\mathcal{C}x+\mathcal{D}}},$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are real parameters such that

$$\mathcal{AD} - \mathcal{BC} > 0 \text{ and } \mathcal{A}^2 + \mathcal{C}^2 > 0.$$

In Section 2 of this paper we state our main results, which are connected with the interpolatory problems originated by Fejér [11] and Egerváry and Turán [9, 10]. These problems can be formulated for each classical weight function $w(x)$ on a finite or inifinte interval (a,b). For this purpose let $w_1(x) = A(x)w(x)$ and consider the weighted norm

$$\Lambda_{w_1}(x_1,\cdots,x_n) = \sup_{a<x<b} w_1(x)\lambda_{w_1}(x;x_1,\cdots,x_n), \; a < x_1 < \cdots < x_n < b,$$

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of the weighted Lebesgue type function

$$\lambda_{w_1}(x; x_1, \cdots, x_n) = \sum_{k=1}^{n} \frac{l_k^2(x)}{w_1(x_k)},$$

of the interpolating Hermite operator

$$(H_n f)(x) = w_1(x) \sum_{k=1}^{n} \frac{f(x_k)\, l_k^2(x)}{w_1(x_k)},$$

where the fundamental Lagrange interpolating polynomials $l_k(x)$ are equal to

$$l_k(x) = \frac{p_n(x)}{(x - x_k)\, p_n'(x_k)}$$

with

$$p_n(x) = \prod_{i=1}^{n} (x - x_i), \; p_n'(x_k) = \prod_{i=1,\, i\neq k}^{n} (x_k - x_i).$$

Note that this operator satisfies the interpolating conditions of the form

$$(H_n f)(x_i) = f(x_i), \; (H_n f)'(x_i) = 0, \; i = 1, \cdots, n,$$

whenever $p_n(x)$ satisfies the Sturm-Liouville differential equation (1.1) in (a, b).

The Fejér problem is equivalent to finding points $x_1, \ldots, x_n$ for which the weighted w_1-norm $\Lambda_{w_1}(x_1, \cdots, x_n)$ attains its minimal possible value equal to 1. It was a great discovery of Fejér [11], that the unique solution of this problem gives the roots $x_1, \cdots, x_n$ of the classical Legendre orthogonal polynomials corresponding to $(a, b) = (-1, 1)$, $w(x) = 1$ and $w_1(x) = 1 - x^2$. After that Karlin and Studden [21] used the von Neumann's Minimax Theorem to solve the Fejér problem for Hermite, Jacobi and Laguerre polynomials. Other more elementary proofs of their results were given by Balázs [3] and Lau and Studden [28, 29], together with some new results for non-classical weights. The last two references provide also a good summary of the Fejér problem. It should be noticed that several interesting modifications of the Fejér problem were studied recently by Lubinsky [32], Szabó [41] and Horváth [17, 18]. These papers include also extensive references on the subject.

Note that each Lagrange fundamental polynomial $l_k(x)$ is a unique polynomial of degree $n - 1$ which satisfies interpolating conditions

$$l_k(x_i) = \delta_{ki} \;\; (i = 1, \cdots, n), \tag{1.3}$$

where δ_{ki} denotes the Kronecker delta. These conditions do not guarantee the uniqueness of $l_k(x)$ without the additional assumption that $l_k(x)$ should be a polynomial of degree $n - 1$. Below we denote by $\widehat{l}_k(x)$ any polynomial of arbitrary degree $\deg\left(\widehat{l}_k(x)\right) \geq n - 1$ for which the conditions (1.3) hold. Then the polynomial

$$Q_{w_1}(x) = \sum_{k=1}^{n} y_k \frac{\widehat{l}_k(x)}{w_1(x_k)}, \; a < x_1 < \cdots < x_n < b,$$

satisfies the interpolating conditions

$$Q_{w_1}(x_i) = \frac{y_i}{w_1(x_i)}, \; i = 1, \cdots, n.$$

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Further, following [9, 10, 21], the interpolatory system of polynomials $\widehat{l}_k(x)$, $k = 1, \cdots, n$, is said to be w_1-stable if the inequality

$$0 \le w_1(x) \sum_{k=1}^{n} y_k \frac{\widehat{l}_k(x)}{w_1(x_k)} \le \max_{1\le k\le n} y_k, \; a < x < b,$$

holds for all nonnegative real numbers $y_1, \cdots, y_n$. Additionally, if the sum

$$\sum_{k=1}^{n} \deg\left(\widehat{l}_k(x)\right)$$

of degrees of polynomials $\widehat{l}_k(x)$ is minimal, then the w_1-stable interpolatory system $\widehat{l}_k(x)$, $k = 1, \cdots, n$, is called the most economical [4, 9, 10, 19, 20, 21]. It is interesting that the Fejér problem can be easily applied to solve the problem of finding the w_1-stable and the most economical interpolatory systems. This problem was originated by Egerváry and Turán, who established it for Legendre and Hermite polynomials in [9, 10]. Next, this problem was investigated in a series of papers of Balázs [4] and Joó [19, 20] for the Jacobi and Laguerre polynomials. It should be noticed that the interpolating problems of Fejér, Egerváry and Turán have found several applications in the area of polynomial approximation and interpolation of functions and in numerical analysis; cf. e.g., [8, 21, 35, 40, 42].

Finally, in Section 3 we present a new elementary unified proof of a theorem, which completes the solution of the Fejér problem [11] for all classical orthogonal polynomials. It is inspired by the papers of Balázs [3], Joó [19, 20] and our recent papers on the univariate and multivariate inequalities of Chernoff type [38, 39] and on the electrostatic equilibrium problem [37] in the class of all classical orthogonal polynomials. Next, following Joó [19, 20] we apply this result to complete the solution of the Egerváry and Turán problem for all classical weight functions, which satisfy the Pearson equation.

2. Main results and auxiliary lemmas.

The first of our theorems provides a complete solution of the Fejér problem in the class of all classical orthogonal polynomials. In the case of orthogonal polynomials of Hermite, Jacobi and Laguerre mentioned in (i), (ii) and (iii) of Section 1, the sufficiency part of this theorem reduces to Theorems 4.4, 4.1, 4.3 from the monograph of Karlin and Studden [21, chapter X]. It is new for the remaining classical orthogonal polynomials (iv), (v) and (vi) of Section 1. Moreover, the necessity part of the theorem is also new.

THEOREM 2.1. *Let $q_n(x)$ be the classical polynomial, orthogonal with respect to a classical weight function $w(x)$ on (a, b). Then we have*

$$\inf_{a<z_1<\cdots<z_n<b} \Lambda_{w_1}(z_1, \cdots, z_n) = \Lambda_{w_1}(x_1, \cdots, x_n) = 1$$

for some $x_1 < \cdots < x_n$ in (a, b) if and only if $x_1 < \cdots < x_n$ are the roots of $q_n(x)$ in the interval (a, b).

In view of formula (1.2), Remark 1.1 and Theorem 2.1, we directly derive the following corollary for the derivatives $q_n^{(k)}(x)$ of polynomials $q_n(x)$, which seems to be of independent interest.

COROLLARY 2.2. *Let $q_n(x)$ be the classical polynomial, orthogonal with respect to a classical weight function $w(x)$ on (a, b). Then we have*

$$\inf_{a<z_1<\cdots<z_{n-k}<b} \Lambda_{w_{k+1}}(z_1, \cdots, z_{n-k}) = \Lambda_{w_{k+1}}(x_1, \cdots, x_{n-k}) = 1, \; k = 1, \cdots, n-1,$$

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for some $x_1 < \cdots < x_{n-k}$ *in* (a,b) *if and only if points* $x_1 < \cdots < x_{n-k}$ *are the roots of* $q_n^{(k)}(x)$ *in the interval* (a,b).

In the next theorem we complete the solution of the most economical interpolatory problem, which was originated by Egerváry and Turán [9, 10]; cf. also Karlin and Studden [21]. This problem was also solved in Theorems I, II and III in the excellent paper of Joó [20], in the case of Jacobi, Laguerre and Hermite weight functions. In our paper we present a unified approach to solve the most economical interpolatory problem in the class of all classical weights, which include the generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi weights.

THEOREM 2.3. *Let* $q_n(x)$ *be the classical monic polynomial, orthogonal with respect to a classical weight function* $w(x)$ *on* (a,b). *Then the interpolatory system* $\widehat{l}_k(x)$, $k = 1,\cdots,n$, *is* w_1*-stable and most economical if and only if*

$$\widehat{l}_k(x) = \left[\frac{q_n(x)}{(x-x_k)\,q_n'(x_k)}\right]^2, \; k = 1,\cdots,n,$$

where $x_1,\cdots,x_n \in (a,b)$ *are roots of* $q_n(x)$.

For the proof of Theorem 2.1 we need the inequality

$$\left(\frac{1}{w_1(x)}\right)^{(2n)} > 0, \; a < x < b, \; 0 < n < n_w,$$

for any classical weight function $w(x)$. In view of Joó [20], such results are due to R. Askey in the case of Jacobi and Laguerre weight functions. Their proofs presented in [20] use deep theorems on the distribution of zeros of Jacobi and Laguerre polynomials [42]. Other proofs were suggested by Balázs [3] and Bogmér [7] in the case of Jacobi and Laguerre polynomials. We note that Balázs's approach to the proofs was based on some suitable explicit integral formulae for the function $1/w_1(x)$ which should be easy to differentiate. More precisely, by applying three times the formula

$$\int_0^{+\infty} t^{\nu-1} e^{-\mu t} dt = \Gamma(\nu)\,\mu^{-\nu} \;\; (\mu > 0, \; \nu > 0)$$

given in [14, Equation 3.381.4], we conclude that

$$\frac{1}{w_1(x)} = \frac{1}{\Gamma(\alpha+1)} \int_0^{+\infty} t^\alpha e^{(1-t)x} dt$$

for the Laguerre weight $w_1(x) = x^{\alpha+1}e^{-x}$ on $(0,+\infty)$, and

$$\frac{1}{w_1(x)} = \frac{1}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} \int_0^{+\infty}\int_0^{+\infty} t^\alpha s^\beta e^{-(t+s)+(t-s)x} dtds$$

for the Jacobi weight $w_1(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}$ on $(-1,1)$. Hence the derivatives

$$\left(\frac{1}{w_1(x)}\right)^{(2n)} = \frac{1}{\Gamma(\alpha+1)} \int_0^{+\infty} (1-t)^{2n}\, t^\alpha e^{(1-t)x} dt, \; 0 < x < +\infty,$$

and

$$\left(\frac{1}{w_1(x)}\right)^{(2n)} = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{+\infty}\int_0^{+\infty} (t-s)^{2n}\, t^\alpha s^\beta e^{-(t+s)+(t-s)x} dtds, \; -1<x<1,$$

are evidently positive. In the case of the Hermite weight

$$w_1(x) = e^{-x^2} \quad (-\infty < x < +\infty)$$

the proof is even simpler. Indeed, we have

$$\left(\frac{1}{w_1(x)}\right)^{(2n)} = \sum_{k=2n}^{+\infty} \frac{(2k)(2k-1)\cdot\dots\cdot(2k-2n+1)}{k!} x^{2k-2n} > 0$$

for all $x \in (-\infty, +\infty)$.

Unfortunately, it seems that these ideas can not be applied to the remaining classical weights given in (iv), (v) and (vi) of Section 1. On the other hand, we have found an elementary unified way to prove the required inequality for all classical weights, which generate nontrivial sequences of classical orthogonal polynomials. For this purpose, we need the following three lemmas, which are of independent interest.

LEMMA 2.4. *Let $w(x)$ be a classical weight on (a, b) and let $s_n(x)$ be defined by*

$$\left(\frac{1}{w_1(x)}\right)^{(n)} = \frac{(-1)^n}{w_{n+1}(x)} s_n(x). \tag{2.1}$$

Then $s_n(x)$ is a polynomial of degree n of the form

$$s_n(x) = c_n + \sum_{k=1}^{n} \left(\binom{n}{k} c_{n-k} \prod_{i=1}^{k} [b_1 + (n-i) a_2] \right) x^k \tag{2.2}$$

with coefficients c_n satisfying recurrent relations

$$\begin{gathered} c_0 = 1,\ c_1 = b_0, \\ c_n = -(n-1)[b_1 + (n-2) a_2] a_0 c_{n-2} + [b_0 + (n-1) a_1] c_{n-1}, \end{gathered} \tag{2.3}$$

whenever $2 \le n < n_w$. Moreover, the polynomial $s_n(x)$ satisfies the recurrent formula

$$s_n(x) = -s'_{n-1}(x) A(x) + s_{n-1}(x) [B(x) + (n-1) A'(x)], \ n = 1, 2, \cdots. \tag{2.4}$$

Proof. Let $s_n(x)$ be as in formula (2.1). Then the formulae (2.1) and (2.4) with the initial condition $s_0(x) = 1$ are equivalent. Indeed, we can rewrite (2.1) in the equivalent form

$$\left(\frac{1}{w_1(x)}\right)^{(n)} = (-1)^{n-1} \frac{s'_{n-1}(x) w_n(x) - s_{n-1}(x) w'_n(x)}{w_n^2(x)}.$$

Since the Pearson differential equation (1.2) is equivalent to

$$\frac{w'_k(x)}{w_k(x)} = \frac{B(x) + (k-1) A'(x)}{A(x)},$$

the proof of equivalence is completed.

Hence it remains to prove formulae (2.2) and (2.3) by induction. Since we have $s_0(x) = 1$ they are trivial for $n = 0$. Moreover (2.2), (2.3) and (2.4) yield

$$\left(\frac{1}{w_1(x)}\right)' = -\frac{b_0 + b_1 x}{w_2(x)} \text{ and } s_1(x) = b_0 + b_1 x.$$

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Hence it is clear that formulae (2.2) and (2.3) hold also for $n=1$. Now suppose that these formulae are true for $0,1,\cdots,n-1$ $(n>1)$. Then one can insert $A(x)=a_0+a_1x+a_2x^2$ and $B(x)=b_0+b_1x$ into the right-hand side of (2.4) and use the induction hypothesis to get

$$\begin{aligned}
&-s_{n-1}'(x)A(x)+s_{n-1}(x)\left[B(x)+(n-1)A'(x)\right]\\
&=\left\{-(n-1)\left[b_1+(n-2)a_2\right]a_0c_{n-2}+\left[b_0+(n-1)a_1\right]c_{n-1}\right\}x^0\\
&\quad+nc_{n-1}\left[b_1+(n-1)a_2\right]x^1\\
&\quad+\sum_{k=2}^{n-2}\left(\binom{n}{k}c_{n-k}\prod_{i=1}^{k}\left[b_1+(n-i)a_2\right]\right)x^k\\
&\quad+\left(nc_1\prod_{i=1}^{n-1}\left[b_1+(n-i)a_2\right]\right)x^{n-1}\\
&\quad+\left(c_0\prod_{i=1}^{n}\left[b_1+(n-i)a_2\right]\right)x^n\\
&=c_n+\sum_{k=1}^{n}\left(\binom{n}{k}c_{n-k}\prod_{i=1}^{k}\left[b_1+(n-i)a_2\right]\right)x^k\\
&=s_n(x).
\end{aligned}$$

Thus the proof is completed. □

LEMMA 2.5. *The derivative of the polynomial $s_n(x)$ from Lemma 2.4 satisfies the following recurrent formula*

$$s_n'(x)=n\left[b_1+(n-1)a_2\right]s_{n-1}(x).$$

Proof. By the formula (2.2) we easily get

$$\begin{aligned}
s_n'(x)&=\sum_{k=1}^{n}\left(\binom{n}{k}kc_{n-k}\prod_{i=1}^{k}\left[b_1+(n-i)a_2\right]\right)x^{k-1}\\
&=\sum_{k=0}^{n-1}\left(\binom{n}{k+1}(k+1)c_{n-k-1}\prod_{i=1}^{k+1}\left[b_1+(n-i)a_2\right]\right)x^k\\
&=n\left[b_1+(n-1)a_2\right]c_{n-1}\\
&\quad+n\left[b_1+(n-1)a_2\right]\sum_{k=1}^{n-1}\left(\binom{n-1}{k}c_{n-k-1}\prod_{i=1}^{k}\left[b_1+(n-i-1)a_2\right]\right)x^k\\
&=n\left[b_1+(n-1)a_2\right]s_{n-1}(x),
\end{aligned}$$

which finishes the proof. □

LEMMA 2.6. *The polynomials $s_{2n}(x)$, occurring in Lemma 2.4, are convex and positive, whenever $0\le n<n_w$.*

Proof. First we restrict our attention to six classes of polynomials defined in (i)-(vi) of Section 1. Since we have $s_0(x)=1$, it follows that the proof is obvious for $n=0$. Suppose that it is true for $n\ge 1$, i.e., that we have

$$s_{2n-2}(x)>0 \text{ and } s_{2n-2}''(x)\ge 0$$

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on $\mathbb{R}$. Then it follows from Lemma 2.5 that

$$\begin{aligned} s''_{2n}(x) &= 2n\left[b_1 + (2n-1)\,a_2\right] s'_{2n-1}(x) \\ &= 2n\left[b_1 + (2n-1)\,a_2\right](2n-1)\left[b_1 + (2n-2)\,a_2\right] s_{2n-2}(x)\,. \end{aligned}$$

Hence the inequality $s''_{2n}(x) > 0$ ($a < x < b$), will be established, whenever we show that

$$\delta_{2n} = \left[b_1 + (2n-1)\,a_2\right]\left[b_1 + (2n-2)\,a_2\right] > 0$$

for all classical weights $w(x)$ given in (i)-(vi). For this purpose we note that:

(a) Hermite case. Since $a_2 = 0$ and $b_1 = -2$, we have $\delta_{2n} = (-2)(-2) > 0$.

(b) Jacobi case. Since $a_2 = -1$, $b_1 = -(\alpha + \beta + 2) < 0$, $\alpha > -1$ and $\beta > -1$, it follows that $\delta_{2n} = (-\alpha - \beta - 2n - 1)(-\alpha - \beta - 2n) > 0$.

(c) Laguerre case. Since $a_2 = 0$ and $b_1 = -1$, then $\delta_{2n} = (-1)(-1) > 0$.

(d) Generalized Bessel case. Since $a_2 = 1$, $b_1 = \alpha$ and $n < \left\lfloor \frac{1-\alpha}{2} \right\rfloor \leq \frac{1-\alpha}{2}$, we have $\delta_{2n} = (\alpha + 2n - 1)(\alpha + 2n - 2) > 0$.

(e) Jacobi on $(0, +\infty)$ case. Since $a_2 = 1$, $b_1 = 2 - \alpha$ and $n < \left\lfloor \frac{\alpha-1}{2} \right\rfloor \leq \frac{\alpha-1}{2}$, then $\delta_{2n} = (-\alpha + 2n + 1)(-\alpha + 2n) > 0$.

(f) Pseudo-Jacobi case. Since $a_2 = 1$, $b_1 = 2(1-\alpha)$ and $n < \left\lfloor \alpha - \frac{1}{2} \right\rfloor \leq \alpha - \frac{1}{2}$, it follows that $\delta_{2n} = (-2\alpha + 2n + 1)(-2\alpha + 2n) > 0$.

Hence $s_{2n}(x)$ is convex on $\mathbb{R}$. Additionally, $s_{2n}(x)$ is a polynomial of degree $2n \geq 2$. Thus there exists a point $z \in \mathbb{R}$ such that

$$s_{2n}(z) = \min_{x \in \mathbb{R}} s_{2n}(x)$$

and

$$s'_{2n}(z) = 0.$$

Therefore, it follows from Lemma 2.5 that

$$s_{2n-1}(z) = \frac{s'_{2n}(z)}{2n\left[b_1 + (2n-1)\,a_2\right]} = 0.$$

Hence one can apply the recurrent formula (2.4) and Lemma 2.5 to obtain

$$\begin{aligned} s_{2n}(z) &= -(2n-1)\left[b_1 + (2n-2)\,a_2\right] s_{2n-2}(z)\,A(z) \\ &\quad + s_{2n-1}(z)\left[B(z) + (2n-1)\,A'(z)\right] \\ &= -(2n-1)\left[b_1 + (2n-2)\,a_2\right] s_{2n-2}(z)\,A(z) > 0. \end{aligned}$$

Note that the last inequality follows immediately from the inductive hypotesis $s_{2n-2}(x) > 0$ ($x \in \mathbb{R}$), the positivity of $A(x)$ and from the fact that $b_1 + (2n-2)\,a_2$ is a negative factor of δ_{2n}; cf. the cases (a)-(f) above. Since the point z is a global minimum of $s_{2n}(x)$, the proof is completed for all six classes of polynomials listed in (i)-(vi) of Section 1. The other cases follow directly from the fact that a linear change of variable $x \to d_1 x + d_0$ ($d_1 > 0$) preserves the positivity and convexity of the function $\tilde{s}_{2n}(x) = s_{2n}(d_1 x + d_0)$, $x \in \mathbb{R}$. □

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3. Proofs of the main results. Let the function

$$\lambda_{w_1}(x) = \lambda_{w_1}(x; x_1, \cdots, x_n) = \sum_{k=1}^{n} \frac{l_k^2(x)}{w_1(x_k)} \tag{3.1}$$

be defined as in Section 1, i.e., let the Lagrange fundamental polynomials

$$l_k(x) = \frac{p_n(x)}{(x - x_k)\, p_n'(x_k)},\ p_n(x) = (x - x_1)\cdots(x - x_n), \tag{3.2}$$

of degree $n-1$ be defined by the interpolating conditions

$$l_k(x_i) = \delta_{ki},\ i = 1, \cdots, n. \tag{3.3}$$

Moreover, let

$$\Lambda_{w_1}(x_1, \cdots, x_n) = \sup_{a<x<b} w_1(x)\, \lambda_{w_1}(x; x_1, \cdots, x_n). \tag{3.4}$$

Below we will also need the formulae

$$l_k'(x_k) = \frac{p_n''(x_k)}{2p_n'(x_k)},\ k = 1, \cdots, n, \tag{3.5}$$

which can be obtained by an application of the l'Hospital's rule to the derivative of the formula (3.2) for $l_k(x)$.

Proof of Theorem 2.1. Necessity.
If $\Lambda_{w_1}(x_1, \cdots, x_n) = 1$ for some $x_1 < \cdots < x_n$ in (a, b), then in view of (3.1), (3.4) and (3.3) we have

$$0 \le w_1(x)\, \lambda_{w_1}(x) \le 1,\ a < x < b,$$

and

$$w_1(x_i)\, \lambda_{w_1}(x_i) = 1,\ i = 1, \cdots, n.$$

Hence it follows that

$$\left[w_1(x)\, \lambda_{w_1}(x)\right]'\Big|_{x=x_i} = 0,\ i = 1, \cdots, n.$$

On the other hand, by (3.3) and (3.5) we have

$$\begin{aligned}
\left[w_1(x)\, \lambda_{w_1}(x)\right]'\Big|_{x=x_i} &= \sum_{k=1}^{n} \frac{\left[w_1(x)\, l_k^2(x)\right]'\Big|_{x=x_i}}{w_1(x_k)} \\
&= \frac{\left[w_1(x)\, l_i^2(x)\right]'\Big|_{x=x_i}}{w_1(x_i)} \\
&= \frac{w_1'(x)\, l_i^2(x)\Big|_{x=x_i} + 2w_1(x)\, l_i(x)\, l_i'(x)\Big|_{x=x_i}}{w_1(x_i)} \\
&= \frac{w_1'(x_i)}{w_1(x_i)} + \frac{p_n''(x_i)}{p_n'(x_i)} = 0,
\end{aligned}$$

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where $p_n(x)$ is as in (3.2). This in conjunction with the identity

$$\frac{w_1'(x)}{w_1(x)} = \frac{B(x)}{A(x)},$$

obtained from the Pearson differential equation (1.2), yields

$$A(x_i)\, p_n''(x_i) + B(x_i)\, p_n'(x_i) = 0,\ i = 1, \cdots, n.$$

Since the polynomial $A(x)\, p_n''(x) + B(x)\, p_n'(x)$ has degree n, the last identities show that $p_n(x) = (x - x_1)\cdots(x - x_n)$ is a solution of the following differential equation

$$A(x)\, p_n''(x) + B(x)\, p_n'(x) = \lambda_n p_n(x),\ a < x < b,\ \lambda_n = n\left[(n-1)\, a_2 + b_1\right],$$

which is equivalent to the generic Sturm-Liouville differential equation (1.1). Thus the polynomial $p_n(x)$ is the classical monic polynomial, orthogonal with respect to the classical weight $w(x)$.

Sufficiency.
Let $x_1 < \cdots < x_n$ be the roots of the classical orthogonal polynomial $q_n(x)$ in the interval (a, b). Then it follows from the Sturm-Liouville differential equation (1.1) that

$$w_1'(x_i)\, q_n'(x_i) + w_1(x_i)\, q_n''(x_i) = 0,\ i = 1, \cdots, n.$$

Hence we conclude that

$$\begin{aligned} \lambda_{w_1}'(x_i) &= \sum_{k=1}^{n} \frac{2 l_k(x_i)\, l_k'(x_i)}{w_1(x_k)} = \frac{q_n''(x_i)}{q_n'(x_i)} \frac{1}{w_1(x_i)} \\ &= -\frac{w_1'(x_i)}{w_1^2(x_i)} = \left.\left(\frac{1}{w_1(x)}\right)'\right|_{x = x_i}. \end{aligned}$$

This in conjunction with the obvious identities

$$\lambda_{w_1}(x_i) = \frac{1}{w_1(x_i)},\ i = 1, \cdots, n, \tag{3.6}$$

implies that $\lambda_{w_1}(x)$ is the Hermite interpolating polynomial for the function $1/w_1(x)$ at knots $x_1 < \cdots < x_n$. Hence the remainder formula [8] for Hermite interpolation yields

$$\frac{1}{w_1(x)} - \lambda_{w_1}(x) = \left.\left(\frac{1}{w_1(x)}\right)^{(2n)}\right|_{x=\xi} \frac{q_n^2(x)}{(2n)!}$$

for some $\xi \in (a, b)$. Further, in view of Lemmas 2.4 and 2.6, we have

$$\left(\frac{1}{w_1(x)}\right)^{(2n)} > 0,\ a < x < b.$$

Thus we conclude that

$$0 \le w_1(x)\, \lambda_{w_1}(x) \le 1.$$

This inequality together with the identity (3.6) shows that

$$\inf_{a < z_1 < \cdots < z_n < b} \Lambda_{w_1}(z_1, \cdots, z_n) = \Lambda_{w_1}(x_1, \cdots, x_n) = 1,$$

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which completes the proof. □

Proof of Theorem 2.3. The fact that the $\widehat{l}_k(x)$, $k = 1, \cdots, n$, form the w_1-stable system is equivalent to the inequalities

$$0 \le w_1(x) \frac{\widehat{l}_k(x)}{w_1(x_k)} \le w_1(x) \sum_{k=1}^{n} \frac{\widehat{l}_k(x)}{w_1(x_k)} \le 1, \; a < x < b. \tag{3.7}$$

Since the classical weights $w_1(x) = A(x)\, w(x)$ satisfy the boundary conditions

$$\lim_{x \downarrow a} w_1(x) = \lim_{x \uparrow b} w_1(x) = 0,$$

it follows from the identities $\widehat{l}_k(x_i) = \delta_{ki}$ that points $x_1 < \cdots < x_{k-1} < x_{k+1} < \cdots < x_n$ have to be zeros of polynomials $\widehat{l}_k(x)$ of even multiplicity. Hence we have $\deg\left(\widehat{l}_k(x)\right) \ge 2(n-1)$ and so

$$\min \sum_{k=1}^{n} \deg\left(\widehat{l}_k(x)\right) \ge 2n(n-1).$$

For the proof of sufficiency, we note that the last inequality shows that the system

$$\widehat{l}_k(x) = \left[\frac{q_n(x)}{(x - x_k)\, q_n'(x_k)}\right]^2, \; k = 1, \cdots, n,$$

has the minimal sum

$$\sum_{k=1}^{n} \deg\left(\widehat{l}_k(x)\right) = 2n(n-1)$$

of degrees. Moreover, Theorem 2.1 yields inequality (3.7). Hence this system is w_1-stable and most economical. On the other hand, if system $\widehat{l}_k(x)$ is w_1-stable and most economical, then the necessity part of the Theorem 2.3 follows directly from the necessity party of Theorem 2.1. □

4. Numerical aspects of optimal interpolation. A numerical evaluation of the optimal, stable and most economical interpolating operator requires an algorithm to compute zeros $a < x_1 < \cdots < x_n < b$ of the monic classical orthogonal polynomials $q_n(x)$ in the interval (a, b). For this purpose one can use the following recurrence relation [24, 33]

$$q_0(x) = 1, \; q_1(x) = x - c_0,$$
$$q_{k+1}(x) = (x - c_k)\, q_k(x) - d_k q_{k-1}(x), \; k = 1, 2, \cdots, n-1,$$

with coefficients c_k and d_k equal to

$$\begin{aligned} c_k &= -\frac{2k a_1 r_{k-1} - b_0(2a_2 - b_1)}{r_{2k-2} r_{2k}}, \; k = 0, 1, \cdots, n-1, \\ d_k &= k r_{k-2} \frac{s_{k-1}(r_{k-1} a_1 - a_2 b_0) - a_0 r_{2k-2}^2}{r_{2k-3} r_{2k-2}^2 r_{2k-1}}, \; k = 1, 2, \cdots, n-1, \end{aligned} \tag{4.1}$$

where

$$r_\nu = \nu a_2 + b_1 \text{ and } s_\nu = \nu a_1 + b_0.$$

It should be noticed that the formulae for c_0 and d_1 given in (4.1) are also valid for Jacobi weight functions with $\alpha+\beta=0$ and $\alpha+\beta=-1$, whenever we assume that $0/0=1$.

Since the cost of computation of the value $q_n(x)$ at x is equal to $O(n)$, it follows that the zeros of $q_n(x)$ can be computed by the well-known bisection algorithm for the Sturm's sequences [13], which has the cost $O\left(n^2\right)$. Of course, this cost depends also on the precision of computation of the zeros. Moreover, it can be reduced to $O(n)$ by applying the fast algorithm from the paper [12]. It is interesting that these zeros give the solution $(x_1,\cdots,x_n)$ of the electrostatic equilibrium problem, which is well-known for the Hermite, Jacobi and Laguerre polynomials [21, 42], and has been proved recently for the remaining three classes (iv), (v) and (vi) of classical orthogonal polynomials in [37].

Finally, after computing the zeros of $q_n(x)$, one can easily adopt the barycentric algorithm for the Lagrange interpolation [5, 16] in order to evaluate the Hermite or Lagrange interpolating polynomials with knots equal to the zeros $x_1,\cdots,x_n$ of $q_n(x)$. Alternatively, one can use the fast algorithm due to Tygert [43].

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