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IMPROVED PERTURBATION BOUNDS FOR THE CONTINUOUS-TIME H_∞-CONTROL PROBLEM*

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Abstract. New local perturbation bounds for the continuous-time H_∞-control problem are obtained, which are nonlinear functions of the data perturbations and are tighter than the existing condition number-based local bounds. These nonlinear local bounds are then incorporated into nonlocal perturbation bounds which are less conservative than the existing nonlocal perturbation estimates for the H_∞-control problem.

Key words. H_∞-control, perturbation analysis, Riccati equations

AMS subject classifications. 93B36, 65F35, 93B35

1. Introduction.

In this paper we present a complete perturbation analysis of the H_∞-control problem for continuous-time linear multivariable systems. Nonlinear local perturbation bounds are first obtained for the matrix equations determining the problem solution. These local bounds are tighter than the condition number-based perturbation bounds.

Using the nonlocal perturbation analysis techniques developed in [8, 9], nonlocal perturbation bounds are then derived. The new nonlocal bounds are less conservative than the existing nonlocal perturbation estimates for the H_∞-control problem and are rigorously valid in contrast to the local bounds.

The following notations are used: $\mathbb{R}^{m\times n}$ denotes the space of real $m \times n$ matrices, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, I_n the unit $n \times n$ matrix, A^T the transpose of A, $\|A\|_2 = \sigma_{\max}(A)$ the spectral norm of A, where $\sigma_{\max}(A)$ denotes the largest singular value of A, $\|A\|_{\rm F} = \sqrt{\mathrm{tr}(A^TA)}$ is the Frobenius norm of A, $\|\,.\,\|$ is any of the above norms, $\mathrm{vec}(A) \in \mathbb{R}^{mn}$ denotes the column-wise vector representation of $A \in \mathbb{R}^{m\times n}$, $\Pi \in \mathbb{R}^{n^2\times n^2}$ the vec-permutation matrix so that $\mathrm{vec}(A^T) = \Pi\mathrm{vec}(A)$ for $A \in \mathbb{R}^{n\times n}$, and $A \otimes B$ denotes the Kronecker product of the matrices A and B. The notation ":=" stands for "equal by definition".

2. Statement of the problem.

Consider the linear multivariable continuous-time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ev(t),\\ y(t) &= Cx(t) + w(t),\\ z(t) &= \begin{bmatrix} Dx(t)\\ u(t)\end{bmatrix}, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$, and $z(t) \in \mathbb{R}^p$ are the system state, input, output, and performance vectors, respectively, $v(t) \in \mathbb{R}^l$ and $w(t) \in \mathbb{R}^r$ are disturbances, and A, B, C, D, E are constant matrices of compatible dimensions.

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The H_∞-control problem is stated as follows: given the system (2.1) and a constant $\lambda > 0$, find a stabilizing controller

$$u(t) = -K\hat{x}(t),$$
$$\dot{\hat{x}}(t) = \hat{A}\hat{x} + L(y(t) - C\hat{x}(t)),$$

which satisfies

$$\|H\|_\infty := \sup_{\mathrm{Re}\, s \geq 0} \|H(s)\|_2 < \lambda,$$

where $H(s)$ is the closed-loop transfer matrix from v, w to z.

If such a controller exists, then it holds that [10]

$$K = B^T X_0,$$
$$\hat{A} = A - Y_0(C^T C - D^T D/\lambda^2),$$
$$L = Z_0 Y_0 C^T,$$

where $X_0 \geq 0$ and $Y_0 \geq 0$ are the stabilizing solutions to the Riccati equations

$$A^T X + XA - X(BB^T - EE^T/\lambda^2)X + D^T D = 0, \quad (2.2)$$
$$AY + YA^T - Y(C^T C - D^T D/\lambda^2)Y + EE^T = 0,$$

and the matrix Z_0 is defined by

$$Z_0 = (I - Y_0 X_0/\lambda^2)^{-1} \quad (2.3)$$

under the assumption $\|Y_0 X_0\|_2 < \lambda^2$.

In the sequel we shall write equations (2.2) as

$$A^T X + XA - XSX + Q = 0, \quad (2.4)$$
$$AY + YA^T - YRY + T = 0, \quad (2.5)$$

where $Q = D^T D$, $T = EE^T$, $S = BB^T - T/\lambda^2$, $R = C^T C - Q/\lambda^2$.

Suppose that the matrices $A, \ldots, E$ in (2.1) are subject to perturbations $\Delta A, \ldots, \Delta E$. Then we have the perturbed equations

$$(A + \Delta A)^T X + X(A + \Delta A) - X(S + \Delta S)X + Q + \Delta Q = 0, \quad (2.6)$$
$$(A + \Delta A)Y + Y(A + \Delta A)^T - Y(R + \Delta R)Y + T + \Delta T = 0, \quad (2.7)$$
$$Z = (I - YX/\lambda^2)^{-1}, \quad (2.8)$$

where

$$\Delta Q = \Delta D^T D + D^T \Delta D + \Delta D^T \Delta D,$$
$$\Delta T = \Delta E E^T + E \Delta E^T + \Delta E \Delta E^T,$$
$$\Delta S = \Delta B B^T + B \Delta B^T + \Delta B \Delta B^T - \Delta T/\lambda^2,$$
$$\Delta R = \Delta C^T C + C^T \Delta C + \Delta C^T \Delta C - \Delta Q/\lambda^2.$$

Denote by $\Delta_M = \|\Delta M\|$ the absolute perturbation of a matrix M. It is natural to use the Frobenius norm $\|\,.\,\|_F$ identifying the matrix perturbations with their vector-wise representations.

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Since the Fréchet derivatives of the left-hand sides of (2.4), (2.5) in X and Y at $X = X_0$ and $Y = Y_0$ are invertible (see the next section), then, according to the implicit function theorem [3], the perturbed equations (2.6), (2.7) have unique solutions $X = X_0 + \Delta X$ and $Y = Y_0 + \Delta Y$ in a neighborhood of X_0 and Y_0, respectively. Assume that $\|YX\| < \lambda^2$, and denote by $Z = Z_0 + \Delta Z$ the corresponding solution of the perturbed equation (2.8).

The sensitivity analysis of the H_∞-control problem aims at determining perturbation bounds for the solutions X, Y, and Z of equations (2.4), (2.5), and (2.3) as functions of the perturbations in the data A, S, Q, R, T.

Using the approach developed in [4, 6], local perturbation bounds for the H_∞-control problem have been obtained in [1] based on the condition numbers of equations (2.4), (2.5), and (2.3). However, using condition numbers for those local estimates may eventually produce too pessimistic results. At the same time it is possible to derive local, first order homogeneous estimates which are tighter in general [9]. In this paper, we use the local perturbation analysis technique developed in [9] to establish such bound that are tighter than those in [1].

Local perturbation bounds have a serious drawback: they are valid in a usually small neighborhood of the data $A, \ldots, T$, i.e., for $\Delta = [\Delta_A, \ldots, \Delta_T]^T$ asymptotically small. In practice, however, the perturbations in the data are always finite. Hence, the use of local estimates remains (at least theoretically) unjustified unless an additional analysis of the neglected terms is done, which in most cases is a difficult task. In fact, obtaining bounds for the neglected nonlinear terms means getting a nonlocal perturbation bound.

Nonlocal perturbation bounds for the continuous-time H_∞-control problem have been first obtained in [1] using the Banach fixed point principle. In this paper, applying the method of nonlinear perturbation analysis [8, 9], we derive new nonlocal perturbation bounds for the problem considered which are less conservative than those in [1].

3. Local perturbation analysis. Consider first the local sensitivity analysis of the Riccati equation (2.4). Denote by

$$F(X, \Sigma) = F(X, A, S, Q)$$

the left-hand side of (2.4), where

$$\Sigma = (A, S, Q) \in \mathbb{R}^{n.n} \times \mathbb{R}^{n.n} \times \mathbb{R}^{n.n}.$$

Then $F(X_0, \Sigma) = 0$.

Setting $X = X_0 + \Delta X$, the perturbed equation (2.6) may be written as

$$\begin{aligned} & F(X_0 + \Delta X, \Sigma + \Delta\Sigma) \\ & \quad = F(X_0, \Sigma) + F_X(\Delta X) + F_A(\Delta A) + F_S(\Delta S) + F_Q(\Delta Q) \\ & \qquad + \; G(\Delta X, \Delta\Sigma) = 0, \end{aligned} \tag{3.1}$$

where $F_X(.)$, $F_A(.)$, $F_S(.)$, and $F_Q(.)$ are the Fréchet derivatives of $F(X, \Sigma)$ in the corresponding matrix arguments evaluated at $X = X_0$, and $G(\Delta X, \Delta\Sigma)$ contains the second and higher order terms in ΔX, $\Delta\Sigma$. A straightforward calculation leads to

$$\begin{aligned} F_X(M) &= A_c^T M + M A_c, \\ F_A(M) &= X_0 M + M^T X_0, \\ F_S(M) &= -X_0 M X_0, \\ F_Q(M) &= M, \end{aligned}$$

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where

$$A_c = A - (BB^T - EE^T/\lambda^2)X_0\,.$$

Denote by $M_X \in \mathbb{R}^{n^2\times n^2}$, $M_A \in \mathbb{R}^{n^2\times n^2}$, $M_S \in \mathbb{R}^{n^2\times n^2}$ the matrix representations of the operators $F_X(.)$, $F_A(.)$, $F_S(.)$,

$$\begin{aligned} M_X &= A_c^T \otimes I_n + I_n \otimes A_c^T\,, \\ M_A &= I_n \otimes X_0 + (X_0 \otimes I_n)\Pi, \\ M_S &= -X_0 \otimes X_0, \end{aligned} \tag{3.2}$$

where $\Pi \in \mathbb{R}^{n^2\times n^2}$ is the permutation matrix such that $\mathrm{vec}(M^T) = \Pi\mathrm{vec}(M)$ for each $M \in \mathbb{R}^{n\times n}$, and $\mathrm{vec}(M) \in \mathbb{R}^{n^2}$ is the column-wise vector representation of M.

It follows from (3.1) that

$$F_X(\Delta X) = -F_A(\Delta A) - F_S(\Delta S) - \Delta Q - G(\Delta X, \Delta\Sigma). \tag{3.3}$$

Since A_c is stable, the operator $F_X(.)$ is invertible, and (3.3) yields

$$\Delta X = -F_X^{-1} \circ F_A(\Delta A) - F_X^{-1} \circ F_S(\Delta S) - F_X^{-1}(\Delta Q) - F_X^{-1}(G(\Delta X, \Delta\Sigma)). \tag{3.4}$$

The operator equation (3.4) may be written in vector form as

$$\begin{aligned} \mathrm{vec}(\Delta X) \;=\; & N_1\mathrm{vec}(\Delta A) + N_2\mathrm{vec}(\Delta S) + N_3\mathrm{vec}(\Delta Q) \\ & - M_X^{-1}\mathrm{vec}(G(\Delta X, \Delta\Sigma)), \end{aligned} \tag{3.5}$$

where $N_1 = -M_X^{-1}M_A$, $N_2 = -M_X^{-1}M_S$, $N_3 = -M_X^{-1}$.

It is easy to show that the well-known condition number-based perturbation bound [1] is a corollary of (3.5). Indeed, it follows from (3.5) that

$$\begin{aligned} \|\mathrm{vec}(\Delta X)\|_2 \le \|N_1\|_2\|\mathrm{vec}(\Delta A)\|_2 + \|N_2\|_2\|\mathrm{vec}(\Delta S)\|_2 + \|N_3\|_2\|\mathrm{vec}(\Delta Q)\|_2 \\ + \mathrm{O}(\|\tilde\Delta\|^2). \end{aligned}$$

Having in mind that $\|\mathrm{vec}(\Delta M)\|_2 = \|\Delta M\|_F = \Delta_M$ and denoting

$$K_A^X = \|N_1\|_2, \quad K_S^X = \|N_2\|_2, \quad K_Q^X = \|N_3\|_2,$$

we obtain

$$\Delta_X \le K_A^X\Delta_A + K_S^X\Delta_S + K_Q^X\Delta_Q + \mathrm{O}(\|\tilde\Delta\|^2)\,, \tag{3.6}$$

where K_A^X, K_S^X, K_Q^X are the individual condition numbers of (2.4) and

$$\tilde\Delta = [\Delta_A, \Delta_S, \Delta_Q]^T.$$

Denoting $\Delta_{\max} = \max\{\Delta_A, \Delta_S, \Delta_Q\}$ and taking into account the inequalities

$$\begin{aligned} K_A^X &\le 2K_Q^X\|X_0\|\,, \\ K_S^X &\le K_Q^X\|X_0\|^2, \end{aligned}$$

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we get

$$\Delta_X \leq K_Q^X (1 + \|X_0\|)^2 \Delta_{\max} \,, \tag{3.7}$$

where $K_Q^X (1 + \|X_0\|)^2$ is the overall condition number of (2.4). Relation (3.5) also gives

$$\Delta_X \leq \|\tilde{N}\|_2 \|\tilde{\Delta}\|_2 + \mathrm{O}(\|\tilde{\Delta}\|^2) \,, \tag{3.8}$$

where $\tilde{N} = [N_1, N_2, N_3]$. Depending of the value of $\tilde{\Delta}$, the bound in (3.8) can be larger or smaller than that in (3.6).

There is also a third bound, which is always smaller or equal to the bound in (3.6). We have

$$\Delta_X \leq \sqrt{\tilde{\Delta}^T U(\tilde{N}) \tilde{\Delta}} + \mathrm{O}(\|\tilde{\Delta}\|^2),$$

where $U(\tilde{N})$ is the 3×3 matrix with elements $u_{ij}(\tilde{N}) = \|N_i^T N_j\|_2$. Since

$$\left\|N_i^T N_j\right\|_2 \leq \|N_i\|_2 \|N_j\|_2,$$

we get

$$\sqrt{\tilde{\Delta}^T U(\tilde{N}) \tilde{\Delta}} \leq \|N_1\|_2 \Delta_A + \|N_2\|_2 \Delta_S + \|N_3\|_2 \Delta_Q .$$

Hence, we have the overall estimate

$$\Delta_X \leq f(\tilde{\Delta}) + \mathrm{O}(\|\tilde{\Delta}\|^2), \qquad \tilde{\Delta} \to 0, \tag{3.9}$$

where

$$f(\tilde{\Delta}) = \min\left\{ \|\tilde{N}\|_2 \|\tilde{\Delta}\|_2, \sqrt{\tilde{\Delta}^T U(\tilde{N}) \tilde{\Delta}} \right\}$$

is a first order homogeneous and piecewise real analytic function in $\tilde{\Delta}$.

The local sensitivity of the Riccati equation (2.5) may be determined using the duality of (2.4) and (2.5). For the estimate of Δ_Y, we have

$$\Delta_Y \leq g(\hat{\Delta}) + \mathrm{O}(\|\hat{\Delta}\|^2), \qquad \hat{\Delta} \to 0, \tag{3.10}$$

where

$$g(\hat{\Delta}) = \min\left\{ \|\hat{N}\|_2 \|\hat{\Delta}\|_2, \sqrt{\hat{\Delta}^T U(\hat{N}) \hat{\Delta}} \right\},$$

$\hat{\Delta} = [\Delta_A, \Delta_R, \Delta_T]^T$, and $\hat{N}$ is determined by replacing in (3.2) A_c and X_0 by $\hat{A}^T$ and Y_0, respectively.

Consider finally the local sensitivity analysis of equation (2.3). In view of (2.8), we have

$$\begin{aligned} \Delta Z &= [I_n - (Y_0 + \Delta Y)(X_0 + \Delta X)/\lambda^2]^{-1} - Z_0 \\ &= Z_0 W Z_0 + \mathrm{O}(\|W\|^2), \end{aligned} \tag{3.11}$$

where $W = (Y_0 \Delta X + \Delta Y X_0 + \Delta Y \Delta X)/\lambda^2$. It follows form (3.11) that

$$\Delta_Z \leq \|Z_0^T \otimes Z_0\|_2 \|W\|_F + \mathrm{O}(\|W\|^2),$$

and denoting $\zeta_0 = \|Z_0^T \otimes Z_0\|_2$, we get

$$\begin{aligned} \Delta_Z &\leq \zeta_0 (\|Y_0\|_2 \Delta_X + \|X_0\|_2 \Delta_Y)/\lambda^2 + \mathrm{O}(\|(\Delta X, \Delta Y)\|^2) \\ &\leq \zeta_0 (\|Y_0\|_2 f(\tilde{\Delta}) + \|X_0\|_2 g(\hat{\Delta}))/\lambda^2 + \mathrm{O}(\|\Delta\|^2). \end{aligned} \tag{3.12}$$

The relations (3.9), (3.10), and (3.12) give local first order perturbation bounds for the continuous-time H_∞-control problem.

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4. Nonlocal perturbation analysis. The local perturbation bounds are obtained by neglecting terms of order $\mathrm{O}(\|\Delta\|^2)$, i.e., they are valid only asymptotically for $\Delta \to 0$. That is why their application for possibly small but nevertheless finite perturbations Δ requires additional justification. This disadvantage may be overcome using the methods of nonlinear perturbation analysis [7, 12]. As a result, we obtain nonlocal (and in general nonlinear) perturbation bounds, which guarantee that the perturbed problem still has a solution, and are valid rigorously unlike the local bounds [5, 9]. However, in some cases the nonlocal bounds may not exist or may be too pessimistic.

Consider first the nonlocal perturbation analysis of the Riccati equation (2.4). The perturbed equation (3.4) can be rewritten in the form

$$\Delta X = \Psi(\Delta X), \tag{4.1}$$

where $\Psi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is determined by the right-hand side of (3.4). For $\rho > 0$, denote by $\mathcal{B}(\rho) \subset \mathbb{R}^{n\times n}$ the set of all matrices $M \in \mathbb{R}^{n\times n}$ satisfying $\|M\|_F \le \rho$. For $U, V \in \mathcal{B}(\rho)$, we have

$$\|\Psi(U)\|_F \le a_0(\tilde{\Delta}) + a_1(\tilde{\Delta})\rho + a_2(\tilde{\Delta})\rho^2$$

and

$$\|\Psi(U) - \Psi(V)\|_F \le (a_1(\tilde{\Delta}) + 2a_2(\tilde{\Delta})\rho)\|U - V\|_F,$$

where

$$\begin{aligned}
a_0(\tilde{\Delta}) &:= f(\tilde{\Delta}),\\
a_1(\tilde{\Delta}) &:= 2\|M_X^{-1}\|_2\Delta_A + (\|M_X^{-1}(X_0 \otimes I_n)\|_2 + \|M_X^{-1}(I_n \otimes X_0)\|_2)\Delta_S,\\
a_2(\tilde{\Delta}) &:= \|M_X^{-1}\|_2(\|S\|_2 + \Delta_S).
\end{aligned}$$

Hence, the function

$$h(\rho, \tilde{\Delta}) = a_0(\tilde{\Delta}) + a_1(\tilde{\Delta})\rho + a_2(\tilde{\Delta})\rho^2$$

is a Lyapunov majorant [2] for equation (4.1), and the majorant equation for determining a nonlocal bound $\rho = \rho(\tilde{\Delta})$ for Δ_X is

$$a_2(\tilde{\Delta})\rho^2 - (1 - a_1(\tilde{\Delta}))\rho + a_0(\tilde{\Delta}) = 0. \tag{4.2}$$

Suppose that $\tilde{\Delta} \in \tilde{\Omega}$, where

$$\tilde{\Omega} = \left\{ \tilde{\Delta} \succeq 0 : a_1(\tilde{\Delta}) + 2\sqrt{a_0(\tilde{\Delta})a_2(\tilde{\Delta})} \le 1 \right\}.$$

Then, equation (4.2) has nonnegative roots [5] $\rho_1 \le \rho_2$ with

$$\rho_1 = \phi(\tilde{\Delta}) := \frac{2a_0(\tilde{\Delta})}{1 - a_1(\tilde{\Delta}) + \sqrt{(1 - a_1(\tilde{\Delta}))^2 - 4a_0(\tilde{\Delta})a_2(\tilde{\Delta})}}. \tag{4.3}$$

The operator Ψ maps the closed convex set

$$\mathcal{B}(\tilde{\Delta}) = \left\{ M \in \mathbb{R}^{n\times n} : \|M\|_F \le \phi(\tilde{\Delta}) \right\} \subset \mathbb{R}^{n\times n}$$

into itself, and according to the Schauder fixed point principle, there exists a solution $\Delta X \in \mathcal{B}(\tilde{\Delta})$ of equation (4.1) for which

$$\Delta_X \le \phi(\tilde{\Delta}), \qquad \tilde{\Delta} \in \tilde{\Omega}\,. \tag{4.4}$$

The elements of ΔX are continuous functions of the elements of $\Delta\Sigma$.

If $\tilde{\Delta} \in \tilde{\Omega}_1$, where

$$\tilde{\Omega}_1 = \left\{ \tilde{\Delta} \succeq 0 : a_1(\tilde{\Delta}) + 2\sqrt{a_0(\tilde{\Delta})a_2(\tilde{\Delta})} < 1 \right\} \subset \tilde{\Omega},$$

then $\rho_1 < \rho_2$, and the operator Ψ is a contraction on $\mathcal{B}(\tilde{\Delta})$. Hence, according to the Banach fixed point principle, the solution ΔX for which the estimate (4.4) holds true is unique. This means that the perturbed equation has an isolated solution $X = X_0 + \Delta X$. In this case, the elements of ΔX are analytic functions of the elements of $\Delta\Sigma$.

In a similar way, replacing A_c with $\hat{A}^T$, S with R, Q with T, and X_0 with Y_0, we obtain a nonlocal perturbation bound for ΔY. Suppose that $\hat{\Delta} \in \hat{\Omega}$, where

$$\hat{\Omega} = \left\{ \hat{\Delta} : b_1(\hat{\Delta}) + 2\sqrt{b_0(\hat{\Delta})b_2(\hat{\Delta})} \le 1 \right\} \subset \mathbb{R}^3_+$$

and

$$\begin{aligned}
b_0(\hat{\Delta}) &= g(\hat{\Delta}),\\
b_1(\hat{\Delta}) &= 2\|M_Y^{-1}\|_2\Delta_{\hat{A}} + \left(\|M_Y^{-1}((Y_0 \otimes I_n))\|_2 + \|M_Y^{-1}((I_n \otimes Y_0))\|_2\right)\Delta_R,\\
b_2(\hat{\Delta}) &= \|M_Y^{-1}\|_2\left(\|R\|_2 + \Delta_R\right).
\end{aligned}$$

Then,

$$\Delta_Y \le \psi(\hat{\Delta}), \qquad \hat{\Delta} \in \hat{\Omega}, \tag{4.5}$$

where

$$\psi(\hat{\Delta}) = \frac{2b_0(\hat{\Delta})}{1 - b_1(\hat{\Delta}) + \sqrt{(1 - b_1(\hat{\Delta}))^2 - 4b_0(\hat{\Delta})b_2(\hat{\Delta})}}\,.$$

Finally, the nonlinear perturbation bound for ΔZ is obtained by using (3.5) and (4.3), (4.4). If $1 \notin \mathrm{spect}(WZ_0)$, then we have

$$\Delta Z = Z_0 W Z_0 (I_n - WZ_0)^{-1}.$$

Hence,

$$\Delta_Z \le \zeta_0 \|W\|_F \|(I_n - WZ_0)^{-1}\|_2.$$

If $\|W\|_2 < 1/\|Z_0\|_2$, then we have

$$\Delta_Z \le \frac{\zeta_0\|W\|_F}{1 - \|Z_0\|_2\|W\|_2}.$$

It is realistic to estimate $\|W\|$ when $\Delta X, \Delta Y$ vary independently. In this case, one has to assume that

$$\|Y_0\|_2\phi(\tilde{\Delta}) + \|X_0\|_2\psi(\hat{\Delta}) + \phi(\tilde{\Delta})\psi(\hat{\Delta}) < \lambda^2/\|Z_0\|_2$$

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and

$$\Delta_Z \leq \frac{\zeta_0 \lambda^2 \xi_0}{\lambda^2 - \|Z_0\|_2 \xi_0}, \tag{4.6}$$

where

$$\xi_0 = \|Y_0\|_2 \phi(\tilde{\Delta}) + \|X_0\|_2 \psi(\hat{\Delta}) + \phi(\tilde{\Delta})\psi(\hat{\Delta}).$$

Relations (4.4), (4.5), and (4.6) give nonlocal perturbation bounds for the continuous-time H_∞-control problem.

Note finally that one has to ensure the inequality

$$\|YX\|_2 < \lambda^2. \tag{4.7}$$

Since the unperturbed inequality $\|Y_0 X_0\|_2 < \lambda^2$ holds true, a sufficient condition for (4.7) to be valid is

$$\|Y_0\|_2 \phi(\tilde{\Delta}) + \|X_0\|_2 \psi(\hat{\Delta}) + \phi(\tilde{\Delta})\psi(\hat{\Delta}) < \lambda^2 - \|Y_0 X_0\|_2 \,.$$

Note that $\tilde{\Delta}, \hat{\Delta}$ depend on λ^2 through Δ_S, Δ_R.

5. Numerical example. Consider a third order Riccati equation of type (2.4) with matrices

$$A = VA^*V, \quad S = VS^*V \quad Q = VQ^*V,$$

where

$$V = I_3 - 2vv^T/3, \quad v = [1,1,1]^T,$$

and

$$A^* = \text{diag}(1, -0.1, -1), \quad S^* = \text{diag}(0.2, 1, 10), \quad Q^* = \text{diag}(0.1, 0.1, 0.1).$$

The solution is given by

$$X = VX^*V, \quad X^* = \text{diag}(x_1, x_2, x_3),$$

where

$$x_i = \frac{a_i + \sqrt{a_i^2 + s_i q_i}}{s_i}$$

and a_i, s_i, and q_i are the corresponding diagonal elements of A^*, S^*, and Q^*.

The perturbations considered in the data satisfy

$$\Delta A = V\Delta A^*V, \quad \Delta S = V\Delta S^*V, \quad \Delta Q = V\Delta Q^*V,$$

where

$$\Delta F^* = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -9 \\ 0 & -9 & 5 \end{bmatrix} \times 10^{-i},$$

$$\Delta S^* = \begin{bmatrix} 10 & -5 & 7 \\ -5 & 1 & 3 \\ 7 & 3 & 10 \end{bmatrix} \times 10^{-i-1},$$

$$\Delta Q^* = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 10 \end{bmatrix} \times 10^{-i}, \qquad \text{for } i = 12, 11, \ldots, 4.$$

The perturbed solution $X + \Delta X$ of the Riccati equation is computed by the Schur method [11, 13] with relative arithmetic precision $\varepsilon = 2^{-52} \approx 2.22 \times 10^{-16}$.

The perturbations $\Delta_X = \|\Delta X\|_F$ in the solution are estimated by the well-known linear bound (3.7), the new nonlinear homogeneous bound (3.9), and the nonlocal bound (4.4). The results obtained for different values of i are shown in Table 5.1. The actual variations in the solution are close to the quantities predicted by the improved sensitivity analysis. The case when the conditions for existence of a nonlocal estimate are violated is denoted by an asterisk.

TABLE 5.1

i	Δ_X	Est. (3.7)	Est. (3.9)	Est. (4.4)
12	$2.1\ 10^{-11}$	$2.6\ 10^{-9}$	$2.5\ 10^{-10}$	$2.5\ 10^{-10}$
11	$2.1\ 10^{-10}$	$2.6\ 10^{-8}$	$2.5\ 10^{-9}$	$2.5\ 10^{-9}$
10	$2.1\ 10^{-9}$	$2.6\ 10^{-7}$	$2.5\ 10^{-8}$	$2.5\ 10^{-8}$
9	$2.1\ 10^{-8}$	$2.6\ 10^{-6}$	$2.5\ 10^{-7}$	$2.5\ 10^{-7}$
8	$2.1\ 10^{-7}$	$2.6\ 10^{-5}$	$2.5\ 10^{-6}$	$2.5\ 10^{-6}$
7	$2.1\ 10^{-6}$	$2.6\ 10^{-4}$	$2.5\ 10^{-5}$	$2.5\ 10^{-5}$
6	$2.1\ 10^{-5}$	$2.6\ 10^{-3}$	$2.5\ 10^{-4}$	$2.5\ 10^{-4}$
5	$2.1\ 10^{-4}$	$2.6\ 10^{-2}$	$2.5\ 10^{-3}$	$2.6\ 10^{-3}$
4	$2.1\ 10^{-3}$	$2.6\ 10^{-1}$	$2.5\ 10^{-2}$	$*$

6. Conclusions. A complete perturbation analysis of the H_∞-control problem for continuous-time linear systems has been presented. First, new local and nonlocal perturbation bounds have been obtained for the matrix equations determining the solution of the problem. The new local bounds are nonlinear functions of the data perturbations and are tighter than the existing condition number-based local bounds. Then, using the nonlinear perturbation analysis technique developed by the authors, nonlocal perturbation bounds have been derived. These bounds guarantee—unlike the local perturbation bounds—that the perturbed problem still has a solution. The new nonlocal bounds are less conservative than the existing nonlocal perturbation bounds for the H_∞-control problem.

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