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A BDDC PRECONDITIONER FOR A SYMMETRIC INTERIOR PENALTY GALERKIN METHOD*

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Abstract. We develop a nonoverlapping domain decomposition preconditioner for the symmetric interior penalty Galerkin method for heterogeneous elliptic problems. The preconditioner is based on balancing domain decomposition by constraints (BDDC). We show that the condition number of the preconditioned system satisfies similar estimates as those for conforming finite element methods. Corroborating numerical results are also presented.

Key words. nonoverlapping domain decomposition, BDDC preconditioner, symmetric interior penalty method

AMS subject classifications. 65N55, 65N30

1. Introduction.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and $\Omega_1, \ldots, \Omega_J$ be polygonal subdomains of Ω that form a nonoverlapping decomposition of Ω. Given $f \in L_2(\Omega)$, consider the following model problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \rho \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega), \tag{1.1}$$

where ρ equals a positive constant ρ_j on the subdomain Ω_j for $1 \le j \le J$. Let $\mathcal{T}_h$ be a simplicial triangulation of Ω aligned with $\Omega_1, \ldots, \Omega_J$ and

$$X_h = \{v \in L_2(\Omega) : v|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h\} \tag{1.2}$$

be the discontinuous P_1 finite element space associated with $\mathcal{T}_h$. The model problem (1.1) can be discretized by the following symmetric interior penalty Galerkin (SIPG) method [5, 15, 19, 20, 35]: Find $u_h \in X_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in X_h, \tag{1.3}$$

where

$$a_h(v, w) = \sum_{T \in \mathcal{T}_h} \int_T \rho \nabla v \cdot \nabla w \, dx + \eta \sum_{e \in \mathcal{E}_h} \frac{\rho_e}{|e|} \int_e [\![v]\!] \cdot [\![w]\!] \, ds$$
$$- \sum_{e \in \mathcal{E}_h} \int_e \big(\{\!\{\rho \nabla v\}\!\} \cdot [\![w]\!] + \{\!\{\rho \nabla w\}\!\} \cdot [\![v]\!] \big) \, ds.$$

Here η is a positive penalty parameter, $\mathcal{E}_h$ is the set of the edges of $\mathcal{T}_h$ and $|e|$ is the length of the edge e. The weight ρ_e is the harmonic average of ρ with respect to the triangles sharing the edge e: For the interior edge e shared by the triangles $T_\pm$, we have

$$\rho_e = \frac{2\rho_- \rho_+}{\rho_- + \rho_+}, \tag{1.4}$$

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where $\rho_\pm = \rho\big|_{T_\pm}$. For an edge along $\partial\Omega$ we have $\rho_e = \rho$.

The jump $[\![v]\!]$ on the edge shared by the triangles $T_\pm$ is the vector defined by

$$[\![v]\!] = v_+\mathbf{n}_+ + v_-\mathbf{n}_-, \tag{1.5}$$

where $v_\pm = v|_{T_\pm}$ and $\mathbf{n}_\pm$ are the unit outer normals along $\partial T_\pm$. On an edge of $\mathcal{T}_h$ along $\partial\Omega$, we define $[\![v]\!] = v\mathbf{n}$, where $\mathbf{n}$ is the unit normal pointing towards the outside of Ω.

Finally, the mean $\{\!\{\rho\nabla v\}\!\}$ is defined as follows. For an interior edge shared by the triangles $T_\pm$, we have

$$\{\!\{\rho\nabla v\}\!\} = \beta_+(\rho_-\nabla v_-) + \beta_-(\rho_+\nabla v_+) = \frac{\rho_-\rho_+}{\rho_- + \rho_+}(\nabla v_- + \nabla v_+), \tag{1.6}$$

where

$$\beta_+ = \frac{\rho_+}{\rho_- + \rho_+}, \quad \beta_- = \frac{\rho_-}{\rho_- + \rho_+}.$$

On an edge of $\mathcal{T}_h$ along $\partial\Omega$, we define $\{\!\{\rho\nabla v\}\!\}$ to be $\rho\nabla v$.

Let $v \in X_h$ be arbitrary and e be an interior edge shared by the triangles $T_\pm$ in $\mathcal{T}_h$. We can easily show

$$|e| \, \| \{\!\{\rho\nabla v\}\!\} \|^2_{L_2(e)} \lesssim \rho_e^2\Big(|v|^2_{H^1(T_-)} + |v|^2_{H^1(T_+)}\Big), \tag{1.7}$$

by using the fact that ∇v is a constant vector on e; cf. [5, 15, 20, 30].

REMARK 1.1. To avoid the proliferation of constants, throughout the paper we will use $A \lesssim B$ and $A \gtrsim B$ to represent the statements that $A \le (\text{constant})B$ and $A \ge (\text{constant})B$, where the positive constant is independent of the mesh size, the subdomain size, the number of subdomains, and ρ. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$.

Let ϵ be an arbitrary positive constant. From (1.7) and the relation

$$\rho_e < 2\rho_\pm, \tag{1.8}$$

we have

$$\Big|\int_e \{\!\{\rho\nabla v\}\!\} \cdot [\![v]\!] \, ds\Big| \lesssim \epsilon\Big(\rho_-|v|^2_{H^1(T_-)} + \rho_+|v|^2_{H^1(T_+)}\Big) + \epsilon^{-1}\frac{\rho_e}{|e|}\| [\![v]\!] \|^2_{L_2(e)}. \tag{1.9}$$

Similarly we have, for an edge $e \subset \partial\Omega$ of a triangle T in $\mathcal{T}_h$,

$$\Big|\int_e \{\!\{\rho\nabla v\}\!\} \cdot [\![v]\!] \, ds\Big| \lesssim \epsilon\rho|v|^2_{H^1(T)} + \epsilon^{-1}\frac{\rho}{|e|}\| [\![v]\!] \|^2_{L_2(e)}. \tag{1.10}$$

Combining (1.9) and (1.10), we obtain

$$a_h(v,v) \ge (1 - C\epsilon)\sum_{T\in\mathcal{T}_h} \rho|v|^2_{H^1(T)} + (\eta - C\epsilon^{-1})\sum_{e\in\mathcal{E}_h} \frac{\rho_e}{|e|}\| [\![v]\!] \|^2_{L_2(e)}.$$

Hence, the following coercivity property holds: There exists a constant η_0 such that, for any $\eta > \eta_0$,

$$a_h(v,v) \gtrsim \sum_{T\in\mathcal{T}_h} \rho|v|^2_{H^1(T)} + \sum_{e\in\mathcal{E}_h} \frac{\rho_e}{|e|}\| [\![v]\!] \|^2_{L_2(e)}. \tag{1.11}$$

The SIPG method is one of the best known discontinuous Galerkin (DG) methods; cf. [6] and the references therein. In this paper we will develop a nonoverlapping domain decomposition preconditioner for the SIPG method that is based on the balancing domain decomposition by constraints (BDDC) approach. The performance of our preconditioner (cf. Theorem 4.14 and Theorem 4.15) is similar to the performance of BDDC preconditoners for conforming finite element methods [14, 18, 27, 28].

REMARK 1.2. The discrete problem (1.3) can also be defined for higher order finite elements and the extension of our results to such methods is straightforward. For the case of discontinuous P_1 finite element, one can also replace the jumps in the penalty term by their projections to the space of constant functions on the edges. Such methods have been investigated in [7, 8], where a decomposition of the discontinuous finite element space led to a block diagonal structure for the stiffness matrix that can be exploited for the construction of fast solvers.

There is a growing literature on domain decomposition preconditioners for discontinuous finite element methods [1, 2, 3, 4, 9, 11, 16, 17, 21, 22, 24, 25, 26, 31]. Among these papers, the work in [11, 17, 22] are closest to the work in this paper. Below we will briefly describe the differences.

For conforming finite element methods, the bilinear forms for the discrete problems can be written as the sum of bilinear forms defined on the subdomains that only involve the degrees of freedom (dofs) on the respective subdomains. Therefore in the BDDC or the FETI-DP (finite element tearing and interconnecting primal-dual) approach, these subdomain bilinear forms are decoupled along the interface of the subdomains. But this is not the case for DG methods due to the terms in the DG bilinear forms that penalize the jumps of the discontinuous finite element functions across the element boundaries.

In [11], where we consider the weakly over-penalized symmetric interior penalty (WOPSIP) method [12], we overcome this difficulty by introducing a decomposition of the discontinuous finite element space X_h so that the BDDC preconditioner is needed only for a subspace of X_h whose members are continuous across the interface of the subdomains. We follow the same approach in this paper; cf. Section 3. However the treatment of the SIPG method in this paper is more challenging than the treatment of the WOPSIP method in [11] due to the stronger coupling of the SIPG method. Moreover in this paper we also consider the more general cases of heterogeneous coefficients and nonconforming meshes.

A FETI-DP domain decomposition preconditioner is developed and analyzed in [22] for the same heterogeneous problem treated in this paper, with conforming meshes. There the authors overcome the difficulty of the DG coupling across the interface by enlarging the number of dofs of a subdomain to include those from the neighboring subdomains that share an edge with it. As a result, the number of unknowns for the subdomain Schur complement problem is doubled.

BDDC preconditioners are developed in [17] for several DG methods on conforming meshes. For the SIPG method, the authors also enlarge the number of dofs of the subdomains and consequently the number of unknowns for the subdomain Schur complement problem is doubled. However we must confess that the presentation in [17] is very concise and it is difficult for us to truly understand the subtleties of either the algorithms or the analysis in that paper.

The rest of the paper is organized as follows. In Section 2 we introduce the subspace decomposition. We then design a BDDC preconditioner for the reduced problem in Section 3. The condition number estimates are carried out in Section 4. Furthermore, we discuss the extension to the case with nonconforming triangulations in Section 5. Finally, we report numerical results in Section 6 that illustrate the performance of the proposed preconditioner and corroborate the theoretical estimates.

For the convenience of the readers, we also include a table that provides references to the notations throughout the paper.

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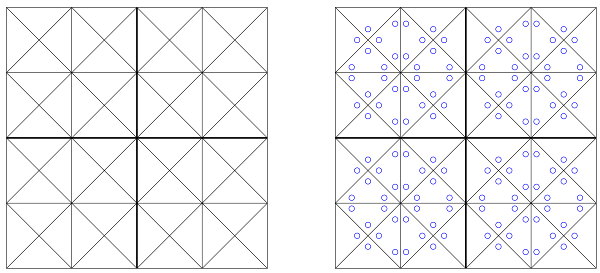
2. A subspace decomposition. In this section we introduce a subspace decomposition of the discontinuous finite element space, which yields an intermediate preconditioner for the discrete problem resulting from the SIPG method.

Let $\Gamma = \left(\bigcup_{j=1}^{J} \partial\Omega_j\right) \setminus \partial\Omega$ be the interface of the subdomains; cf. Fig. 2.1(a). We assume that the subdomains are shape-regular polygons; cf. [13, Section 7.5]. We denote the diameter of Ω_j by H_j and define H to be $\max_{1\le j\le J} H_j$. Let $\mathcal{E}_{h,\Gamma}$ be the subset of $\mathcal{E}_h$ containing the edges on Γ. In order to control the effects of the high contrast among the ρ_j's, we assume that none of the triangles in $\mathcal{T}_h$ contains more than one edge in $\mathcal{E}_{h,\Gamma}$; cf. Fig. 2.1(a).



(a) (b)

FIG. 2.1. *(a) A triangulation $\mathcal{T}_h$ and a nonoverlapping domain partition of Ω with the interface Γ in thick lines. (b) $\mathcal{V}_I$, the set of the interior vertices.*

REMARK 2.1. For the interior penalty method investigated in [17, 22], the average $\{\!\{\rho\nabla v\}\!\}$ is given by $(\rho_-\nabla v_- + \rho_+\nabla v_+)/2$ and the weight ρ_e is given by $(\rho_- + \rho_+)/2$. The condition that each triangle can have at most one edge on the interface Γ is not needed for this formulation of the discrete problem because the effects of the high contrast among the ρ_j's can be controlled by the stronger weight $\rho_e = (\rho_- + \rho_+)/2$. Incidentally the two weights coincide when ρ is a constant on Ω and hence in this case our method does not require this condition on the mesh.

We will define the set $\mathcal{V}_h$ of the vertices of the triangles in $\mathcal{T}_h$ by

$$\mathcal{V}_h = \{(p,T) : p \text{ is a vertex of the triangle } T \text{ in } \mathcal{T}_h\}\,.$$

The value of v at a vertex is understood to be $v_T(p)$, where $v_T = v|_T$. The set $\mathcal{V}_I$ of interior vertices (cf. Fig. 2.1(b)) is defined by

$$\mathcal{V}_I = \left\{(p,T) \in \mathcal{V}_h :\ \text{both edges that share } p \text{ are disjoint from } \bigcup_{j=1}^{J} \partial\Omega_j\right\}. \tag{2.1}$$

The set $\mathcal{V}_h \setminus \mathcal{V}_I$ can be partitioned into three disjoint subsets as follows:

$$\mathcal{V}_h \setminus \mathcal{V}_I = \mathcal{V}_{\mathcal{C}} \cup \mathcal{V}_\Gamma \cup \mathcal{V}_{\partial\Omega},$$

where

$$\begin{aligned} \mathcal{V}_{\mathcal{C}} &= \{(p,T) \in \mathcal{V}_h : p \text{ is a corner of one of the subdomains and one of the edges} \\ &\qquad \text{of } T \text{ that contains } p \text{ is on } \Gamma\}\,, \\ \mathcal{V}_\Gamma &= \{(p,T) \in \mathcal{V}_h \setminus \mathcal{V}_{\mathcal{C}} : \text{one of the edges of } T \text{ that contains } p \text{ is on } \Gamma\}\,, \\ \mathcal{V}_{\partial\Omega} &= \{(p,T) \in \mathcal{V}_h \setminus \mathcal{V}_{\mathcal{C}} : \text{at least one of the edges of } T \text{ that contains } p \text{ is on } \partial\Omega\}\,; \end{aligned} \tag{2.2}$$

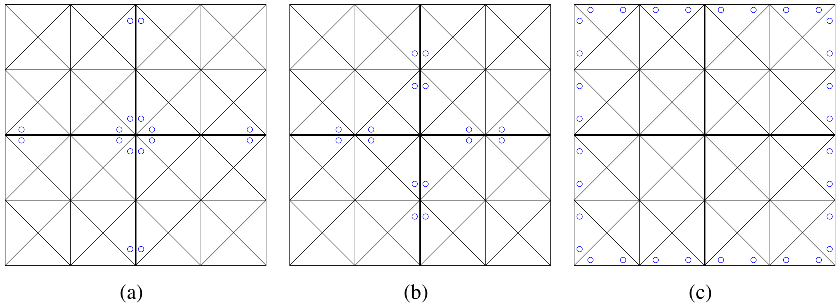


FIG. 2.2. *(a) $\mathcal{V}_{\mathcal{C}}$, the set of the corner vertices. (b) $\mathcal{V}_\Gamma$, the set of the interface vertices on Γ. (c) $\mathcal{V}_{\partial\Omega}$, the set of the boundary vertices on $\partial\Omega$.*

cf. Fig. 2.2. For simplicity, we will refer to a vertex in $\mathcal{V}_{\mathcal{C}}$ (resp. $\mathcal{V}_\Gamma$ and $\mathcal{V}_{\partial\Omega}$) as a corner (resp. interface and boundary) vertex.

First we decompose X_h into two subspaces as follows:

$$X_h = X_{h,C} \oplus X_{h,D}, \tag{2.3}$$

where

$$X_{h,C} = \left\{ v \in X_h : [\![v]\!] = 0 \text{ on the edges in } \mathcal{E}_h \text{ that are subsets of } \bigcup_{j=1}^{J} \partial\Omega_j \right\} \tag{2.4}$$

and

$$X_{h,D} = \{ v \in X_h : \{\!\{v\}\!\} = 0 \text{ on the edges in } \mathcal{E}_{h,\Gamma} \text{ and } v \text{ vanishes at the vertices in } \mathcal{V}_I \}. \tag{2.5}$$

Here the weighted mean $\{\!\{v\}\!\}$ for an edge e in $\mathcal{E}_{h,\Gamma}$ is defined by

$$\{\!\{v\}\!\} = \beta_- v|_{T_-} + \beta_+ v|_{T_+}, \tag{2.6}$$

where the edge e is shared by the triangles $T_\pm$.

REMARK 2.2. Let $v = v_C + v_D$ be the decomposition of $v \in X_h$. Then v_C vanishes at the vertices in $\mathcal{V}_{\partial\Omega}$, v_C agrees with v at the vertices in $\mathcal{V}_I$ and v_C equals $\{\!\{v\}\!\}$ at the vertices in $\mathcal{V}_{\mathcal{C}} \cup \mathcal{V}_\Gamma$. On the other hand, v_D vanishes at the vertices in $\mathcal{V}_I$, $v_D = v$ at the vertices in $\mathcal{V}_{\partial\Omega}$, and at a pair of neighboring vertices (p, T_+) and (p, T_-) in $\mathcal{V}_{\mathcal{C}} \cup \mathcal{V}_\Gamma$,

$$v_D(p, T_+) = \beta_-\big(v(p, T_+) - v(p, T_-)\big) \quad \text{and} \quad v_D(p, T_-) = \beta_+\big(v(p, T_-) - v(p, T_+)\big).$$

Accordingly, the finite element function v_C in $X_{h,C}$ has one degree of freedom (dof) associated with each pair of neighboring vertices in $\mathcal{V}_{\mathcal{C}} \cup \mathcal{V}_\Gamma$, which is represented by '•–•' in Fig. 2.3(a), and one dof at each vertex in $\mathcal{V}_I$, which is represented by '∘' in Fig. 2.3(a). The dofs for a function v_D in $X_{h,D}$ are associated with the neighboring vertices in $\mathcal{V}_{\mathcal{C}} \cup \mathcal{V}_\Gamma$, which are represented by '∘–∘' in Fig. 2.3(b), and the vertices in $\mathcal{V}_{\partial\Omega}$, which are represented by '∘' in Fig. 2.3(b).

Let $A_h : X_h \longrightarrow X_h{}'$ be the symmetric positive-definite (SPD) operator defined by

$$\langle A_h v, w \rangle = a_h(v, w) \qquad \forall\, v, w \in X_h, \tag{2.7}$$

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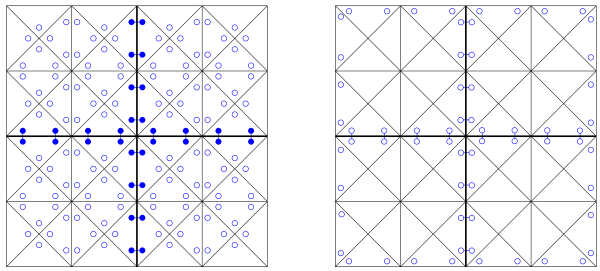

(a) (b)

FIG. 2.3. *(a) Degrees of freedom of* $X_{h,C}$*. (b) Degrees of freedom of* $X_{h,D}$*.*

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form between a vector space and its dual. Similarly, we define the SPD operators $A_{h,D} : X_{h,D} \longrightarrow X'_{h,D}$ and $A_{h,C} : X_{h,C} \longrightarrow X'_{h,C}$ by

$$\langle A_{h,D} v, w \rangle = a_h(v, w) \qquad \forall\, v, w \in X_{h,D}, \tag{2.8}$$

$$\langle A_{h,C} v, w \rangle = a_h(v, w) \qquad \forall\, v, w \in X_{h,C}. \tag{2.9}$$

REMARK 2.3. Let N_c be the number of corners of the subdomains. The system involving $A_{h,D}$ can be reduced to a system of dimension $\approx N_c \times N_c$ by solving a block diagonal system where each block is 2×2 and symmetric positive definite. Thus the solve $A_{h,D}^{-1}$ can be efficiently implemented.

REMARK 2.4. Since the functions in $X_{h,C}$ are continuous across the edges in $\mathcal{E}_{h,\Gamma}$ and vanish on $\partial\Omega$, it holds that

$$a_h(v, w) = \sum_{j=1}^{J} a_{h,j}(v_j, w_j) \qquad \forall\, v, w \in X_{h,C},$$

where $v_j = v\big|_{\Omega_j}$, $w_j = w\big|_{\Omega_j}$ and

$$\begin{aligned} a_{h,j}(v_j, w_j) = &\sum_{\substack{T\in\mathcal{T}_h \\ T\subset\Omega_j}} \int_T \rho_j \nabla v_j \cdot \nabla w_j \, dx + \eta \sum_{\substack{e\in\mathcal{E}_h \\ e\subset\Omega_j}} \frac{\rho_j}{|e|} \int_e [\![v_j]\!] \cdot [\![w_j]\!] \, ds \\ &- \sum_{\substack{e\in\mathcal{E}_h \\ e\subset\Omega_j}} \int_e \Big(\{\!\{\rho_j \nabla v_j\}\!\} \cdot [\![w_j]\!] + \{\!\{\rho_j \nabla w_j\}\!\} \cdot [\![v_j]\!] \Big) \, ds. \end{aligned} \tag{2.10}$$

Note that the bilinear form $a_{h,j}(\cdot,\cdot)$ is local to the subdomain Ω_j.

REMARK 2.5. Let $v \in X_h$ be arbitrary. For the localized bilinear form in (2.10), it follows from (1.9) and (1.10) that

$$a_{h,j}(v_j, v_j) \approx \rho_j \Big(\sum_{\substack{T\in\mathcal{T}_h \\ T\subset\Omega_j}} |v_j|^2_{H^1(T)} + \sum_{\substack{e\in\mathcal{E}_h \\ e\subset\Omega_j}} |e|^{-1} \, \| \, [\![v_j]\!] \, \|^2_{L_2(e)} \Big), \tag{2.11}$$

where $v_j = v|_{\Omega_j}$.

REMARK 2.6. Let e be an edge shared by two triangles in $\mathcal{T}_h$. It can be easily shown that

$$| \, [\![v]\!] \, (p^*)|^2 \lesssim |e|^{-1} \, \| \, [\![v]\!] \, \|^2_{L_2(e)} \qquad \forall\, v \in X_h, \tag{2.12}$$

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where p^* is any convex combination of the two endpoints of the edge e. Moreover, (2.12) also holds for an edge on $\partial\Omega$.

LEMMA 2.7. *Let $v = v_C + v_D$ be the decomposition of $v \in X_h$ according to* (2.3). *We have*

$$\langle A_h v, v\rangle \approx \langle A_{h,D} v_D, v_D\rangle + \langle A_{h,C} v_C, v_C\rangle \qquad \forall\, v \in X_h. \tag{2.13}$$

Proof. From the Cauchy-Schwarz inequality and (2.8)–(2.9), we have

$$\langle A_h v, v\rangle \le 2\left(\langle A_{h,D} v_D, v_D\rangle + \langle A_{h,C} v_C, v_C\rangle\right).$$

In the other direction, based on the relation

$$\langle A_{h,D} v_D, v_D\rangle \lesssim \langle A_h v, v\rangle + \langle A_{h,C} v_C, v_C\rangle,$$

it suffices to show that $\langle A_{h,C} v_C, v_C\rangle \lesssim \langle A_h v, v\rangle$. In view of Remark 2.2, we have

$$\begin{aligned}
&\langle A_{h,C} v_C, v_C\rangle \\
&= \sum_{T\in\mathcal{T}_h} \rho |v_C|^2_{H^1(T)} + \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\cup_{j=1}^J \Omega_j}} \left(\frac{\eta\rho_e}{|e|}\int_e |\,[\![v_C]\!]\,|^2\,ds - 2\int_e \{\!\{\rho\nabla v_C\}\!\}\cdot[\![v_C]\!]\,ds\right) \qquad (2.14)\\
&\lesssim \sum_{T\in\mathcal{T}_h} \rho |v_C|^2_{H^1(T)} + \eta \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\cup_{j=1}^J \Omega_j}} \frac{\rho_e}{|e|}\int_e |\,[\![v_C]\!]\,|^2\,ds,
\end{aligned}$$

and then it suffices to estimate the terms on the right-hand side of (2.14). From Remark 2.2 we know that $v_C - v$ vanishes at the vertices in $\mathcal{V}_I$. It then follows from a standard inverse estimate and scaling that

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} |v_C|^2_{H^1(T)} &\lesssim \sum_{T\in\mathcal{T}_h} |v|^2_{H^1(T)} + \sum_{T\in\mathcal{T}_h} |v_C - v|^2_{H^1(T)} \\
&\lesssim \sum_{T\in\mathcal{T}_h} |v|^2_{H^1(T)} + \sum_{(p,T)\in\mathcal{V}_h} [v_C(p) - v(p,T)]^2 \\
&= \sum_{T\in\mathcal{T}_h} |v|^2_{H^1(T)} + \sum_{(p,T)\in\mathcal{V}_\mathcal{C}\cup\mathcal{V}_\Gamma} [v_C(p) - v(p,T)]^2 + \sum_{(p,T)\in\mathcal{V}_{\partial\Omega}} |v_T(p)|^2 \\
&= \sum_{T\in\mathcal{T}_h} |v|^2_{H^1(T)} + \sum_{(p,T)\in\mathcal{V}_\mathcal{C}\cup\mathcal{V}_\Gamma} \beta^2_{T'} |\,[\![v]\!]\,(p)|^2 + \sum_{(p,T)\in\mathcal{V}_{\partial\Omega}} |v_T(p)|^2,
\end{aligned}$$

where $\beta_{T'} = \frac{\rho|_{T'}}{\rho|_T + \rho|_{T'}} < 1$ and T' is the triangle that shares a common edge with T along Γ. Note that the last equality follows from (2.6) and Remark 2.2.

In view of (2.12) and the obvious estimate

$$\rho|_T \beta^2_{T'} < \frac{\rho|_T\rho|_{T'}}{\rho|_T + \rho|_{T'}} = \frac{1}{2}\rho_e, \tag{2.15}$$

we find

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \rho |v_C|^2_{H^1(T)} &\lesssim \sum_{T\in\mathcal{T}_h} \rho |v|^2_{H^1(T)} + \sum_{e\in\mathcal{E}_{h,\Gamma}} \frac{\rho_e}{|e|} \|\,[\![v]\!]\,\|^2_{L_2(e)} \\
&\quad + \sum_{j=1}^J \sum_{e\in\partial\Omega_j\setminus\Gamma} \frac{\rho_j}{|e|} \|\,[\![v]\!]\,\|^2_{L_2(e)}.
\end{aligned} \tag{2.16}$$

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Similarly, we can establish the estimate

$$\sum_{\substack{e\in\mathcal{E}_h \\ e\subset\cup_{j=1}^J\Omega_j}} \frac{\rho_e}{|e|}\int_e |\,[\![v_C]\!]\,|^2\,ds \lesssim \sum_{j=1}^J \sum_{e\in\partial\Omega_j\setminus\Gamma} \frac{\rho_j}{|e|}\|\,[\![v]\!]\,\|^2_{L_2(e)}. \tag{2.17}$$

Combining (1.11), (2.7), and (2.14)–(2.17), we see that $\langle A_{h,C}v_C, v_C\rangle \lesssim \langle A_h v, v\rangle$. ∎

REMARK 2.8. The estimate $\langle A_{h,C}v_C, v_C\rangle \lesssim \langle A_h v, v\rangle$ in the proof of Lemma 2.7 depends crucially on the condition that any triangle in $\mathcal{T}_h$ can have at most one edge on the interface Γ. Without this condition one would have to introduce a weighted mean $\{\!\{v\}\!\}$ that involves the values of ρ on three or more of the subdomains and to establish the analog of (2.15) for T and T' that do not share a common edge on Γ. Such "long distance" estimates are difficult to achieve without additional assumptions on the values of ρ that are involved in the definition of $\{\!\{v\}\!\}$.

Based on Remark 2.3 and Lemma 2.7, our goal is to construct an efficient preconditioner for $A_{h,C}$. For this purpose we next decompose $X_{h,C}$ into two subspaces $X_{h,C}(\Omega\setminus\Gamma)$ and $X_{h,C}(\Gamma)$:

$$X_{h,C}(\Omega\setminus\Gamma) = \{v \in X_{h,C} : v = 0 \text{ at all the vertices in } \mathcal{V}_h\setminus\mathcal{V}_I\} \tag{2.18}$$

and

$$X_{h,C}(\Gamma) = \{v \in X_{h,C} : a_h(v,w) = 0 \quad \forall w \in X_{h,C}(\Omega\setminus\Gamma)\}. \tag{2.19}$$

The functions in $X_{h,C}(\Gamma)$ are discrete harmonic functions, which are uniquely determined by their values at the vertices in $\mathcal{V}_h\setminus\mathcal{V}_I$.

Let the symmetric positive definite operators $A_{h,\Omega\setminus\Gamma} : X_{h,C}(\Omega\setminus\Gamma) \longrightarrow X_{h,C}(\Omega\setminus\Gamma)'$ and $S_h : X_{h,C}(\Gamma) \longrightarrow X_{h,C}(\Gamma)'$ be defined by

$$\langle A_{h,\Omega\setminus\Gamma}v, w\rangle = a_h(v,w) \quad \forall v,w \in X_{h,C}(\Omega\setminus\Gamma), \tag{2.20}$$

$$\langle S_h v, w\rangle = a_h(v,w) \quad \forall v,w \in X_{h,C}(\Gamma). \tag{2.21}$$

From (2.19)–(2.21) we have

$$\langle A_{h,C}v_C, v_C\rangle = \langle A_{h,\Omega\setminus\Gamma}v_{C,\Omega\setminus\Gamma}, v_{C,\Omega\setminus\Gamma}\rangle + \langle S_h v_{C,\Gamma}, v_{C,\Gamma}\rangle \quad \forall\, v_C \in X_{h,C}, \tag{2.22}$$

where $v_C = v_{C,\Omega\setminus\Gamma} + v_{C,\Gamma}$ is the unique decomposition of v_C with respect to $X_{h,C}(\Omega\setminus\Gamma)$ and $X_{h,C}(\Gamma)$.

We can now define a preconditioner $B_1 : X_h' \longrightarrow X_h$ for A_h by

$$B_1 = I_D A_{h,D}^{-1} I_D^t + I_{\Omega\setminus\Gamma} A_{h,\Omega\setminus\Gamma}^{-1} I_{\Omega\setminus\Gamma}^t + I_\Gamma S_h^{-1} I_\Gamma^t, \tag{2.23}$$

where

$$I_D : X_{h,D} \longrightarrow X_h,\ I_{\Omega\setminus\Gamma} : X_{h,C}(\Omega\setminus\Gamma) \longrightarrow X_h \text{ and } I_\Gamma : X_{h,C}(\Gamma) \longrightarrow X_h \tag{2.24}$$

are natural injections, and $I_D^t, I_{\Omega\setminus\Gamma}^t$, and I_Γ^t are the transposes of these injections defined as

$$\begin{aligned}
\langle I_D^t\phi, v\rangle &= \langle \phi, I_D v\rangle && \forall \phi \in X_h', v \in X_{h,D},\\
\langle I_{\Omega\setminus\Gamma}^t\phi, v\rangle &= \langle \phi, I_{\Omega\setminus\Gamma} v\rangle && \forall \phi \in X_h', v \in X_{h,C}(\Omega\setminus\Gamma),\\
\langle I_\Gamma^t\phi, v\rangle &= \langle \phi, I_\Gamma v\rangle && \forall \phi \in X_h', v \in X_{h,C}(\Gamma).
\end{aligned}$$

Given any $v \in X_h$, it follows from (2.13) and (2.22) that

$$\langle A_h v, v\rangle \approx \langle A_{h,D} v_D, v_D\rangle + \langle A_{h,\Omega\backslash\Gamma} v_{C,\Omega\backslash\Gamma}, v_{C,\Omega\backslash\Gamma}\rangle + \langle S_h v_{C,\Gamma}, v_{C,\Gamma}\rangle, \tag{2.25}$$

where $v = I_D v_D + I_{\Omega\backslash\Gamma} v_{C,\Omega\backslash\Gamma} + I_\Gamma v_{C,\Gamma}$ is the unique decomposition with respect to the spaces $X_{h,D}$, $X_{h,C}(\Omega\backslash\Gamma)$ and $X_{h,C}(\Gamma)$. Therefore by the theory of additive Schwarz preconditioners we have

$$\kappa(B_1 A_h) = \frac{\lambda_{\max}(B_1 A_h)}{\lambda_{\min}(B_1 A_h)} \approx 1. \tag{2.26}$$

REMARK 2.9. From Remark 2.4, we see that $A^{-1}_{h,\Omega\backslash\Gamma}$ can be implemented by solving subdomain problems in parallel. On the other hand, the global solve S_h^{-1} in B_1 needs to be replaced by a good parallel preconditioner.

3. A BDDC preconditioner. In this section we construct a preconditioner for the Schur complement operator S_h based on the BDDC methodology. Let

(3.1) $X_{h,j}$ be the space of discontinuous P_1 finite element functions on Ω_j with resect to $\mathcal{T}_{h,j}$,

where $\mathcal{T}_{h,j}$ is the restriction of $\mathcal{T}_h$ to Ω_j, i.e.,

$$\mathcal{T}_{h,j} = \{T \in \mathcal{T}_h : T \subset \Omega_j\}, \tag{3.2}$$

and

$X_h(\Omega_j)$ be the subspace of $X_{h,j}$ whose members vanish on $\partial\Omega_j$.

We denote by $\mathcal{H}_j$ the space of local discrete harmonic functions defined by

$$\mathcal{H}_j = \{v \in X_{h,j} : v = 0 \text{ on } \partial\Omega_j \setminus \Gamma \text{ and } a_{h,j}(v,w) = 0 \quad \forall\, w \in X_h(\Omega_j)\}\,, \tag{3.3}$$

where the subdomain bilinear form $a_{h,j}(\cdot,\cdot)$ is given by (2.10).

REMARK 3.1. Let $v \in \mathcal{H}_j$ and $w \in X_{h,j}$ such that $v = w$ at all the vertices in $\mathcal{T}_{h,j}$ that do not belong to $\mathcal{V}_I$. Then v satisfies the following minimum energy property:

$$a_{h,j}(v,v) \le a_{h,j}(w,w).$$

The space $\mathcal{H}_{\mathcal{C}}$ is defined by gluing the spaces $\mathcal{H}_j$ together along the interface by imposing continuity at the corner vertices:

$$\mathcal{H}_{\mathcal{C}} = \Big\{v \in L_2(\Omega) : v\big|_{\Omega_j} \in \mathcal{H}_j \text{ for } 1 \le j \le J \text{ and } v \text{ is continuous across } \Gamma \text{ at the vertices in } \mathcal{V}_{\mathcal{C}}\Big\}\,, \tag{3.4}$$

and we equip $\mathcal{H}_{\mathcal{C}}$ with the bilinear form:

$$a_h^C(v,w) = \sum_{1\le j\le J} a_{h,j}(v_j,w_j) \qquad v,w \in \mathcal{H}_{\mathcal{C}}, \tag{3.5}$$

where $v_j = v\big|_{\Omega_j}$ and $w_j = w\big|_{\Omega_j}$.

To construct a BDDC preconditioner for S_h, we introduce a decomposition of $\mathcal{H}_{\mathcal{C}}$:

$$\mathcal{H}_{\mathcal{C}} = \overset{\circ}{\mathcal{H}} \oplus \mathcal{H}_0, \tag{3.6}$$

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where

$$\overset{\circ}{\mathcal{H}} = \{v \in \mathcal{H}_{\mathcal{C}} : v \text{ vanishes at the vertices in } \mathcal{V}_{\mathcal{C}}\}\,, \tag{3.7}$$

$$\mathcal{H}_0 = \left\{v \in \mathcal{H}_{\mathcal{C}} : a_h^C(v,w) = 0 \quad \forall\, w \in \overset{\circ}{\mathcal{H}}\right\}. \tag{3.8}$$

Note that $X_{h,C}(\Gamma)$ is a subspace of $\mathcal{H}_{\mathcal{C}}$ and there exists a projection $P_\Gamma : \mathcal{H}_{\mathcal{C}} \longrightarrow X_{h,C}(\Gamma)$ defined by the weighted averaging:

$$P_\Gamma v = \{\!\{v\}\!\} \quad \text{on an edge in } \mathcal{E}_{h,\Gamma}. \tag{3.9}$$

The SPD operator $S_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}_0'$ is defined by

$$\langle S_0 v, w\rangle = a_h^C(v,w) \quad \forall\, v, w \in \mathcal{H}_0. \tag{3.10}$$

Let $\overset{\circ}{\mathcal{H}}_j$ be the subspace of $\mathcal{H}_j$ whose members vanish at the corner vertices in Ω_j. We define the SPD operator $S_j : \overset{\circ}{\mathcal{H}}_j \longrightarrow \overset{\circ}{\mathcal{H}}_j'$ by

$$\langle S_j v, w\rangle = a_{h,j}(v,w) \quad \forall\, v, w \in \overset{\circ}{\mathcal{H}}_j. \tag{3.11}$$

REMARK 3.2. The positive definiteness of S_j is due to the fact that functions in $\overset{\circ}{\mathcal{H}}_j$ vanish at the vertices in $\mathcal{V}_{\mathcal{C}}$, and the positive definiteness of S_0 results from the fact that the functions in $\mathcal{H}_{\mathcal{C}}$ are continuous across Γ at the vertices in $\mathcal{V}_{\mathcal{C}}$.

We can now define the BDDC preconditioner B_{BDDC} for S_h:

$$B_{BDDC} = (P_\Gamma I_0)\, S_0^{-1}\, (P_\Gamma I_0)^t + \sum_{j=1}^{J} (P_\Gamma \mathbb{E}_j)\, S_j^{-1}\, (P_\Gamma \mathbb{E}_j)^t\,, \tag{3.12}$$

where $I_0 : \mathcal{H}_0 \to \mathcal{H}_{\mathcal{C}}$ is the natural injection and $\mathbb{E}_j : \overset{\circ}{\mathcal{H}}_j \to \mathcal{H}_{\mathcal{C}}$ is the trivial extension defined by

$$\mathbb{E}_j \mathring{v}_j = \begin{cases} \mathring{v}_j & \text{on } \Omega_j \\ 0 & \text{on } \Omega \setminus \Omega_j \end{cases} \qquad \forall\, \mathring{v}_j \in \overset{\circ}{\mathcal{H}}_j. \tag{3.13}$$

Finally, we construct the preconditioner $B_2 : X_h{}' \longrightarrow X_h$ for A_h by replacing S_h^{-1} with the preconditioner B_{BDDC}:

$$B_2 = I_D A_{h,D}^{-1} I_D^t + I_{\Omega\setminus\Gamma} A_{h,\Omega\setminus\Gamma}^{-1} I_{\Omega\setminus\Gamma}^t + I_\Gamma B_{BDDC} I_\Gamma^t. \tag{3.14}$$

4. Condition number estimates. In this section we analyze the condition number of $B_2 A_h$, whose key ingredient is the condition number estimate

$$\kappa(B_{BDDC} S_h) \lesssim \left(1 + \ln \frac{H}{h}\right)^2.$$

First note that

$$X_{h,C}(\Gamma) = P_\Gamma I_0 \mathcal{H}_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \overset{\circ}{\mathcal{H}}_j. \tag{4.1}$$

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In detail, given any $v \in X_{h,C}(\Gamma) \subset \mathcal{H}_{\mathcal{C}}$, there exists a unique decomposition of v from (3.6):

$$v = v_0 + \mathring{v} = I_0 v_0 + \sum_{j=1}^{J} \mathbb{E}_j \mathring{v}_j \qquad v_0 \in \mathcal{H}_0,\ \mathring{v} \in \mathring{\mathcal{H}}, \tag{4.2}$$

where $\mathring{v}_j = \mathring{v}\big|_{\Omega_j}$, and then we have

$$v = P_\Gamma v = (P_\Gamma I_0) v_0 + \sum_{j=1}^{J} (P_\Gamma \mathbb{E}_j) \mathring{v}_j$$

since v is continuous across the edges in $\mathcal{E}_{h,\Gamma}$.

Therefore, by the theory of additive Schwarz preconditioners, e.g., [23, 29, 33, 34, 37], it follows from (4.1) that the BDDC preconditioner B_{BDDC} is SPD and the minimum and maximum eigenvalues of $B_{BDDC} S_h$ are characterized by

$$\lambda_{\min}(B_{BDDC} S_h) = \min_{\substack{v \in X_{h,C}(\Gamma) \\ v \neq 0}} \frac{\langle S_h v, v\rangle}{\displaystyle\min_{\substack{v = P_\Gamma I_0 v_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \mathring{v}_j \\ v_0 \in \mathcal{H}_0, \mathring{v}_j \in \mathring{\mathcal{H}}_j}} \Big(\langle S_0 v_0, v_0\rangle + \sum_{j=1}^{J} \langle S_j \mathring{v}_j, \mathring{v}_j\rangle \Big)}, \tag{4.3}$$

$$\lambda_{\max}(B_{BDDC} S_h) = \max_{\substack{v \in X_{h,C}(\Gamma) \\ v \neq 0}} \frac{\langle S_h v, v\rangle}{\displaystyle\min_{\substack{v = P_\Gamma I_0 v_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \mathring{v}_j \\ v_0 \in \mathcal{H}_0, \mathring{v}_j \in \mathring{\mathcal{H}}_j}} \Big(\langle S_0 v_0, v_0\rangle + \sum_{j=1}^{J} \langle S_j \mathring{v}_j, \mathring{v}_j\rangle \Big)}. \tag{4.4}$$

4.1. A lower bound for $\lambda_{\min}(B_{BDDC} S_h)$. In this section a lower bound for the minimum eigenvalue is obtained from the decomposition (4.2).

LEMMA 4.1. *We have*

$$\lambda_{\min}(B_{BDDC} S_h) \geq 1. \tag{4.5}$$

Proof. Let $v \in X_{h,C}(\Gamma)$ be arbitrary. For the decomposition of v given in (4.2), it follows from Remark 2.4 and (3.8)–(3.11) that

$$\langle S_h v, v\rangle = a_h^C(v,v) = a_h^C(v_0,v_0) + a_h^C(\mathring{v},\mathring{v}) = \langle S_0 v_0, v_0\rangle + \sum_{j=1}^{J} \langle S_j \mathring{v}_j, \mathring{v}_j\rangle.$$

Therefore we have

$$\langle S_h v, v\rangle \geq \min_{\substack{v = P_\Gamma I_0 v_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \mathring{v}_j \\ v_0 \in \mathcal{H}_0, \mathring{v}_j \in \mathring{\mathcal{H}}_j}} \Big(\langle S_0 v_0, v_0\rangle + \sum_{j=1}^{J} \langle S_j \mathring{v}_j, \mathring{v}_j\rangle \Big),$$

which, in view of (4.3), implies (4.5). □

4.2. A trace norm. In this section we construct a trace norm for the space $\mathcal{H}_j$ that is equivalent to the energy norm. This equivalence relation is the key to establishing an upper bound for the maximum eigenvalue of $B_{BDDC} S_h$ in Section 4.3.

Let T be a triangle in $\mathcal{T}_{h,j}$. We consider a set of ten nodes on T that determines a cubic polynomial in $P_3(T)$: three vertex nodes, six edge nodes and one center node; cf. Fig. 4.1. Let

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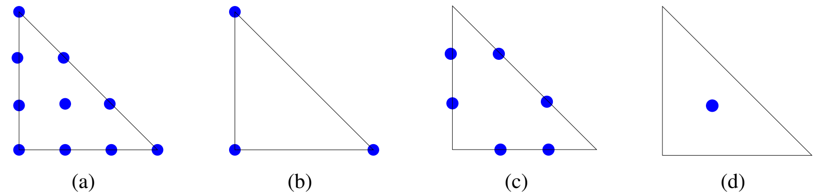

FIG. 4.1. *(a) Nodes of T in $\mathcal{T}_h$. (b) Vertex nodes of T. (c) Edge nodes of T. (d) A center node of T.*

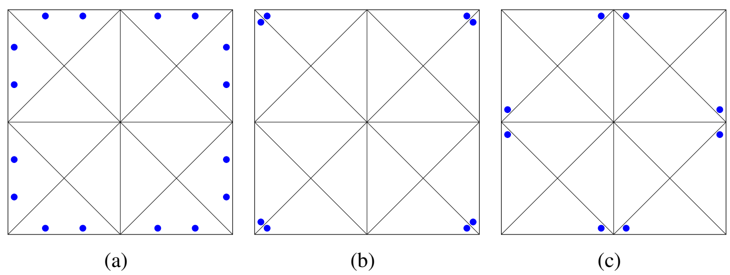

FIG. 4.2. *(a) Edge nodes in (i) of Definition 4.2. (b) Vertex nodes in (ii) of Definition 4.2. (c) Vertex nodes in (iii) of Definition 4.2.*

$\mathcal{N}_{\text{vertex}}$ (resp. $\mathcal{N}_{\text{edge}}$ and $\mathcal{N}_{\text{center}}$) be the set of the vertex nodes (resp. edge nodes and center nodes) in $\mathcal{T}_{h,j}$.

DEFINITION 4.2. *Given any $v \in X_{h,j}$, we define a continuous piecewise cubic polynomial v_* along $\partial\Omega_j$ according to the following rules (cf. Fig. 4.2):*

(i) v_* equals v at the edge nodes in $\mathcal{N}_{\text{edge}}$ that are on $\partial\Omega_j$.

(ii) At a vertex node p in $\mathcal{N}_{\text{vertex}}$ that is a corner of Ω_j, v_* equals one of the values of v at (p,T) in $\mathcal{V}_{\mathcal{C}}$.

(iii) At a vertex node p in $\mathcal{N}_{\text{vertex}}$ that is not a corner of Ω_j, $v_*(p)$ is defined as

$$v_*(p) = \frac{1}{2}\big(v(p,T_1) + v(p,T_2)\big)$$

where (p,T_1) and (p,T_2) are the two vertices associated with p that do not belong to $\mathcal{V}_I$.

REMARK 4.3. Step (i) in Definition 4.2 guarantees that all the information for v is stored in v_* and we can reconstruct v from v_*. This is the reason for using cubic polynomials in the construction of v_*.

REMARK 4.4. Note that v_* is determined by the nodal values of v at the nodes along $\partial\Omega_j$ that do not belong to $\mathcal{V}_I$. In particlular, if $v \in X_{h,j}$ vanishes at all the nodes associated with a closed edge of Ω_j that do not belong to $\mathcal{V}_I$, then v_* also vanishes on that edge.

REMARK 4.5. In view of rule (ii) in Definition 4.2, the function v_* is not unique. The flexibility in assigning the value of v_* at the corner vertices will become useful later.

The following lemma establishes the equivalence between the energy norm and a trace norm for $\mathcal{H}_j$.

LEMMA 4.6. *We have*

$$a_{h,j}(v,v) \approx \rho_j |v_*|^2_{H^{1/2}(\partial\Omega_j)} \qquad \forall\, v \in \mathcal{H}_j,$$

where v_* is the continuous piecewise cubic polynomial on $\partial\Omega_j$ constructed from v according to Definition 4.2.

We will prove Lemma 4.6 through an enriching process that connects $X_{h,j}$ to a cubic Lagrange finite element space:

$$\tilde{X}_{h,j} = \{\tilde{v} \in C(\bar{\Omega}_j) : \tilde{v}\big|_T \in P_3(T) \quad \forall T \in \mathcal{T}_{h,j}\}.$$

DEFINITION 4.7. *We define the enriching operator $\mathsf{E}_j : X_{h,j} \longrightarrow \tilde{X}_{h,j}$ by the following rules:*

(i) $\mathsf{E}_j v$ equals v_* on $\partial\Omega_j$.
(ii) $\mathsf{E}_j v$ equals v at the center nodes in $\mathcal{N}_{\text{center}}$.
(iii) At an edge node p in $\mathcal{N}_{\text{edge}}$ that is not on $\partial\Omega_j$, $(\mathsf{E}_j v)(p)$ is the average of the values of v from the two sides of the edge containing p.
(iv) At a vertex node p in $\mathcal{N}_{\text{vertex}}$ that is interior to Ω_j, $(\mathsf{E}_j v)(p)$ is the average of the values of v at p.

LEMMA 4.8. *We have*

$$\rho_j |\mathsf{E}_j v|^2_{H^1(\Omega_j)} \lesssim a_{h,j}(v, v) \qquad \forall\, v \in X_{h,j}.$$

Proof. Let $v \in X_{h,j}$ be arbitrary. It follows from a standard inverse estimate and scaling that

$$\begin{aligned} |\mathsf{E}_j v|^2_{H^1(\Omega_j)} &= \sum_{T\in\mathcal{T}_{h,j}} |\mathsf{E}_j v|^2_{H^1(T)} \\ &\lesssim \sum_{T\in\mathcal{T}_{h,j}} |\mathsf{E}_j v - v|^2_{H^1(T)} + \sum_{T\in\mathcal{T}_{h,j}} |v|^2_{H^1(T)} \\ &\lesssim \sum_{T\in\mathcal{T}_{h,j}} \sum_{p\in\mathcal{N}(T)} [(\mathsf{E}_j v - v_T)(p)]^2 + \sum_{T\in\mathcal{T}_{h,j}} |v_T|^2_{H^1(T)}, \end{aligned} \tag{4.6}$$

where $\mathcal{N}(T) = \mathcal{N}_{\text{center}}(T) \cup \mathcal{N}_{\text{edge}}(T) \cup \mathcal{N}_{\text{vertex}}(T)$ is the set of the nodes that determines a cubic polynomial.

Let $T \in \mathcal{T}_{h,j}$ be arbitrary. According to rule (ii) of Definition 4.7, we have

$$(\mathsf{E}_j v - v_T)(p) = 0 \quad \text{at the center } p \text{ of } T,$$

and hence

$$\sum_{p\in\mathcal{N}(T)} [(\mathsf{E}_j v - v_T)(p)]^2 = \sum_{p\in\mathcal{N}_{\text{edge}}(T)\cup\mathcal{N}_{\text{vertex}}(T)} [(\mathsf{E}_j v - v_T)(p)]^2. \tag{4.7}$$

Consider first an edge node $p \in \mathcal{N}_{\text{edge}}(T)$. Due to rule (i) of Definition 4.2 and Definition 4.7, it suffices to focus on a node $p \in \mathcal{N}_{\text{edge}}(T)$ that is not on $\partial\Omega_j$. Assume that p is on the common edge e shared by triangles T and T' in $\mathcal{T}_{h,j}$. It follows from rule (iii) of Definition 4.7, scaling and (2.12) that

$$[(\mathsf{E}_j v - v_T)(p)]^2 = \Big[\frac{1}{2}\big(v_{T'}(p) - v_T(p)\big)\Big]^2 \lesssim |e|^{-1} \| [\![v]\!] \|^2_{L_2(e)}. \tag{4.8}$$

Consider next a vertex node p in $\mathcal{N}_{\text{vertex}}(T)$, which does not belong to $\partial\Omega_j$. According to rule (iv) of Definition 4.7, we have

$$(\mathsf{E}_j v)(p) - v_T(p) = \frac{1}{|\mathcal{T}_p|} \sum_{T'\in\mathcal{T}_p\setminus\{T\}} \Big(v(p, T') - v(p, T)\Big), \tag{4.9}$$

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where $\mathcal{T}_p$ is the set of the triangles in $\mathcal{T}_{h,j}$ sharing the vertex p. We consider a chain of triangles $T_0, \dots, T_\ell$ such that $T_0 = T, T_\ell = T'$, and the consecutive triangle T_{i-1} and T_i in this chain share a common edge e_i; cf. [9]. Then, using the Cauchy-Schwarz inequality and (2.12), we obtain

$$
\begin{aligned}
\text{(4.10)} \qquad \big[v(p,T) - v(p,T')\big]^2 &\lesssim \sum_{i=1}^{\ell} [v(p,T_{i-1}) - v(p,T_i)]^2 \\
&\lesssim \sum_{i=1}^{\ell} |e_i|^{-1} \| [\![v]\!] \|_{L_2(e_i)}^2 \lesssim \sum_{e \in \mathcal{E}_p} |e|^{-1} \| [\![v]\!] \|_{L_2(e)}^2,
\end{aligned}
$$

where $\mathcal{E}_p$ is the set of all the edges in $\mathcal{T}_{h,j}$ sharing p. Combining (4.9) and (4.10), we find

$$
\text{(4.11)} \qquad [(\mathsf{E}_j v - v_T)(p)]^2 \lesssim \sum_{e \in \mathcal{E}_p} |e|^{-1} \| [\![v]\!] \|_{L_2(e)}^2.
$$

Similarly, for a vertex node p in V_T that belongs to $\partial\Omega_j$, it holds that

$$
\text{(4.12)} \qquad [(\mathsf{E}_j v - v_T)(p)]^2 \lesssim \sum_{\substack{e \in \mathcal{E}_p \\ e \not\subset \partial\Omega_j}} |e|^{-1} \| [\![v]\!] \|_{L_2(e)}^2.
$$

The lemma follows from (2.11), (4.6)–(4.8), and (4.11)–(4.12). ☐

DEFINITION 4.9. *We define a map $\mathsf{F}_j : \tilde{X}_{h,j} \longrightarrow X_{h,j}$ triangle by triangle as follows. Let $T \in \mathcal{T}_{h,j}$ and $\mathcal{N}(T)$ be the set of the nodes that determines the cubic Lagrange finite element on T.*

(i) If T does not have an edge on $\partial\Omega_j$, then $\mathsf{F}_j\tilde{v}$ equals $\tilde{v}$ at the three vertex nodes in $\mathcal{N}(T)$.

(ii) If T has two edges on $\partial\Omega_j \setminus \Gamma$, then $\mathsf{F}_j\tilde{v}$ equals $\tilde{v}$ at three of the four edge nodes in $\mathcal{N}(T)$ that belong to $\partial\Omega_j$.

(iii) If T has only one edge e on $\partial\Omega_j$, then $\mathsf{F}_j\tilde{v}$ equals $\tilde{v}$ at the two edge nodes on e and at the center of T.

REMARK 4.10. Let $v \in X_{h,j}$ be arbitrary and $w = \mathsf{E}_j(v)$. According to Remark 4.3 and rules (ii) and (iii) of Definition 4.9, $\mathsf{F}_j(w)$ equals v at all the vertices on $\partial\Omega_j$ that do not belong to $\mathcal{V}_I$.

Since the bilinear form $a_{h,j}(\cdot,\cdot)$ only involves edges interior to Ω_j, it follows from Definition 4.9 and a direct local calculation that

$$
\text{(4.13)} \qquad a_{h,j}(\mathsf{F}_j(\tilde{v}), \mathsf{F}_j(\tilde{v})) \lesssim \rho_j |\tilde{v}|_{H^1(\Omega_j)}^2 \quad \forall\, \tilde{v} \in \tilde{X}_{h,j}.
$$

Proof of Lemma 4.6. Let $v \in \mathcal{H}_j$ $(\subset X_{h,j})$ be arbitrary and v_* be the continuous piecewise cubic polynomial on $\partial\Omega_j$ constructed from v according to Definition 4.2. Based on (i) of Definition 4.7, v_* is identical to the trace of $\mathsf{E}_j v \in \tilde{X}_{h,j}$. Then, combining the trace theorem and Lemma 4.8, we arrive at

$$
\rho_j |v_*|_{H^{1/2}(\partial\Omega_j)}^2 \lesssim \rho_j |\mathsf{E}_j v|_{H^1(\Omega_j)}^2 \lesssim a_{h,j}(v,v).
$$

In the other direction v_* can be extended to a finite element function in $\tilde{X}_{h,j}$ such that

$$
\text{(4.14)} \qquad |v_*|_{H^1(\Omega_j)}^2 \lesssim |v_*|_{H^{1/2}(\partial\Omega_j)}^2;
$$

cf. [32, 36], [13, Section 7.5]. Note that $w = \mathsf{F}_j(v_*) \in X_{h,j}$ equals v at all vertices in $\mathcal{T}_{h,j}$ that do not belong to $\mathcal{V}_I$; cf. Remark 4.10. Then, by Remark 3.1 and (4.13), we have that

$$a_{h,j}(v,v) \le a_{h,j}(w,w) \lesssim \rho_j |v_*|^2_{H^1(\Omega_j)},$$

which together with (4.14) implies

$$a_{h,j}(v,v) \lesssim \rho_j |v_*|^2_{H^{1/2}(\partial\Omega_j)}. \qquad \square$$

4.3. An upper bound for $\lambda_{\max}(B_{BDDC}S_h)$. In this section we analyze the maximum eigenvalue of $B_{BDDC}S_h$ by using the trace norm constructed in Section 4.2.

The following lemmas establish the key estimates for an upper bound for $\lambda_{\max}(B_{BDDC}S_h)$.

LEMMA 4.11. *We have*

$$a_h^C(P_\Gamma I_0 v_0, P_\Gamma I_0 v_0) \lesssim \Big(1+\ln\frac{H}{h}\Big)^2 a_h^C(v_0,v_0) \qquad \forall\, v_0 \in \mathcal{H}_0. \tag{4.15}$$

Proof. Let $v_0 \in \mathcal{H}_0$ be arbitrary, $z = v_0 - P_\Gamma I_0 v_0 \in \mathring{\mathcal{H}}$ and $z_j = z\big|_{\Omega_j} \in \mathring{\mathcal{H}}_j$. In view of (3.5) it suffices to focus on $a_{h,j}(z_j, z_j)$.

Let $E_1, \dots, E_{N_j}$ be the edges of the subdomain Ω_j. (Note that N_j is limited by the shape regularity of the subdomains.) Since $z_j \in \mathcal{H}_j$ vanishes at all the corner vertices in Ω_j, we can write

$$z_j = \sum_{\ell=1}^{N_j} z_{j,E_\ell},$$

where $z_{j,E_\ell} \in \mathcal{H}_j$ agrees with z_j at all the vertices on the edge E_ℓ that do not belong to $\mathcal{V}_I$ and z_{j,E_ℓ} vanishes at all the vertices on the other edges of Ω_j that do not belong to $\mathcal{V}_I$.

Let $v_{0,j} = v_0\big|_{\Omega_j} \in \mathcal{H}_j$ for $1 \le j \le J$. It suffices to show that

$$a_{h,j}(z_{j,E}, z_{j,E}) \lesssim \Big(1+\ln\frac{H}{h}\Big)^2 \big(a_{h,j}(v_{0,j}, v_{0,j}) + a_{h,k}(v_{0,k}, v_{0,k})\big), \tag{4.16}$$

where E is the common edge shared by the subdomains Ω_j and Ω_k. Summing up (4.16) over all the edges of Ω_j and over all the subdomains, we have

$$a_h^C(z,z) \lesssim \Big(1+\ln\frac{H}{h}\Big)^2 \sum_{j=1}^{J} a_{h,j}(v_{0,j}, v_{0,j}),$$

which then implies (4.15).

For the estimate (4.16), we first apply Lemma 4.6 to obtain

$$a_{h,j}(z_{j,E}, z_{j,E}) \lesssim \rho_j |\widetilde{z_{j,E}}|^2_{H^{1/2}(\partial\Omega_j)}, \tag{4.17}$$

where $\widetilde{z_{j,E}}$ is the continuous piecewise cubic polynomial on $\partial\Omega_j$ constructed from $z_{j,E}$ according to Definition 4.2. According to Remark 4.4, the function $\widetilde{z_{j,E}}$ vanishes on all the edges of Ω_j except the edge E and it also vanishes at the endpoints of the edge E because $z_{j,E} \in \mathring{\mathcal{H}}_j$. Consequently we can apply a standard truncation estimate for piecewise polynomials (cf. [10, Section 3], [34, Section 4.6], [13, Section 7.5]) to conclude that

$$|\widetilde{z_{j,E}}|^2_{H^{1/2}(\partial\Omega_j)} \lesssim \Big(1+\ln\frac{H}{h}\Big) \|\widetilde{z_{j,E}}\|^2_{L_\infty(E)} + |\widetilde{z_{j,E}}|^2_{H^{1/2}(E)}. \tag{4.18}$$

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The definition of P_Γ (cf. (2.6) and (3.9)) implies

$$z_{j,E} = \beta_k(v_{0,j} - v_{0,k}),$$

where $\beta_k = \frac{\rho_k}{\rho_j+\rho_k} < 1$.

Let $\widetilde{v_{0,j}}$ (resp. $\widetilde{v_{0,k}}$) be the continuous piecewise cubic polynomial on $\partial\Omega_j$ (resp. $\partial\Omega_k$) constructed according to Definition 4.2. Note that according to rule (ii) of Definition 4.2, the value of $\widetilde{v_{0,j}}$ (resp. $\widetilde{v_{0,k}}$) at the endpoints of E can be assigned to equal the values of $v_{0,j}$ (resp. $v_{0,k}$) at the corner vertices shared by Ω_k and Ω_j due to the flexibility mentioned in Remark 4.5. Therefore we have

$$\widetilde{z_{j,E}} = \beta_k(\widetilde{v_{0,j}} - \widetilde{v_{0,k}}) \quad \text{on } E. \tag{4.19}$$

Let us now estimate the first term on the right-hand side of (4.18). Since v_0 is continuous at the vertices in $\mathcal{V}_\mathcal{C}$, there exists a number α such that $v_{0,j}(p) = \alpha = v_{0,k}(p)$ at an endpoint p of the edge E. Hence, both $\widetilde{v_{0,j}} - \alpha$ and $\widetilde{v_{0,k}} - \alpha$ vanish at the endpoint p of E. It then follows from (1.8), (4.19), Lemma 4.6, and a discrete Sobolev inequality (cf. [10, Lemma 3.4], [34, Section 4.6]) that

$$\begin{aligned}
\rho_j\|\widetilde{z_{j,E}}\|^2_{L_\infty(E)} &\lesssim \rho_j\beta_k^2\Big(\|\widetilde{v_{0,j}} - \alpha\|^2_{L_\infty(E)} + \|\alpha - \widetilde{v_{0,k}}\|^2_{L_\infty(E)}\Big)\\
&= \rho_j\|\widetilde{v_{0,j}} - \alpha\|^2_{L_\infty(E)} + \frac{\rho_j\rho_k}{\rho_j+\rho_k}\beta_k\|\alpha - \widetilde{v_{0,k}}\|^2_{L_\infty(E)}\\
&\le \rho_j\|\widetilde{v_{0,j}} - \alpha\|^2_{L_\infty(E)} + \rho_k\|\alpha - \widetilde{v_{0,k}}\|^2_{L_\infty(E)}\\
&\lesssim \Big(1+\ln\frac{H}{h}\Big)\big(\rho_j|\widetilde{v_{0,j}} - \alpha|^2_{H^{1/2}(\partial\Omega_j)} + \rho_k|\alpha - \widetilde{v_{0,k}}|^2_{H^{1/2}(\partial\Omega_k)}\big)\\
&\lesssim \Big(1+\ln\frac{H}{h}\Big)\big(a_{h,j}(v_{0,j},v_{0,j}) + a_{h,k}(v_{0,k},v_{0,k})\big).
\end{aligned} \tag{4.20}$$

Using (1.8), (4.19), and Lemma 4.6, we can estimate the second term on the right-hand side of (4.18) by

$$\begin{aligned}
\rho_j|\widetilde{z_{j,E}}|^2_{H^{1/2}(E)} &\lesssim \rho_j|\widetilde{v_{0,j}}|^2_{H^{1/2}(E)} + \rho_k|\widetilde{v_{0,k}}|^2_{H^{1/2}(E)}\\
&\lesssim a_{h,j}(v_{0,j},v_{0,j}) + a_{h,k}(v_{0,k},v_{0,k}).
\end{aligned} \tag{4.21}$$

The estimate (4.16) follows from (4.17)–(4.18) and (4.20)–(4.21). ☐

LEMMA 4.12. *We have*

$$a_h^C(P_\Gamma\mathbb{E}_j\mathring{v}_j, P_\Gamma\mathbb{E}_j\mathring{v}_j) \lesssim \Big(1+\ln\frac{H}{h}\Big)^2 a_{h.j}(\mathring{v}_j,\mathring{v}_j) \qquad \forall\,\mathring{v}_j \in \mathring{\mathcal{H}}_j. \tag{4.22}$$

Proof. Let $\mathring{v}_j \in \mathring{\mathcal{H}}_j$ be arbitrary and $z = P_\Gamma\mathbb{E}_j\mathring{v}_j$. From the definitions of P_Γ and $\mathbb{E}_j$, it is noted that z is supported in the union of Ω_j and all the subdomains which share an edge with Ω_j.

We first observe that

$$z_j = z\big|_{\Omega_j} = \beta_j\mathring{v}_j, \tag{4.23}$$

where $\beta_j = \frac{\rho_j}{\rho_j+\rho_k} < 1$ and hence

$$a_{h,j}(z_j,z_j) = \beta_j^2 a_{h,j}(\mathring{v}_j,\mathring{v}_j) \le a_{h,j}(\mathring{v}_j,\mathring{v}_j).$$

Let Ω_k be a subdomain which shares an edge E with Ω_j. We will show that

$$a_{h,k}(z_k, z_k) \lesssim \Big(1+\ln\frac{H}{h}\Big)^2 a_{h,j}(\mathring{v}_j, \mathring{v}_j), \tag{4.24}$$

where $z_k = z\big|_{\Omega_k} \in \mathcal{H}_k$. Note that $z_k = z_j$ at the vertices on the closure of E that do not belong to $\mathcal{V}_I$ and z_k vanishes at all the vertices on all the other edges of Ω_k that do not belong to $\mathcal{V}_I$.

Let $\widetilde{z}_k$ be the continuous piecewise cubic polynomial on $\partial\Omega_k$ constructed according to Definition 4.2. The derivation of (4.24) is analogous to the derivation of (4.16) in the proof of Lemma 4.11. In fact it is simpler because $\widetilde{z}_k$ vanishes at the endpoints of E according to rule (ii) of Definition 4.2. By combining a standard truncation estimate, a discrete Sobolev inequality Lemma 4.6 and (4.23), we find

$$\begin{aligned}
a_{h,k}(z_k, z_k) &\lesssim \rho_k |\widetilde{z}_k|^2_{H^{1/2}(\partial\Omega_k)} \\
&\lesssim \rho_k \Big(\Big(1+\ln\frac{H}{h}\Big)\|\widetilde{z}_k\|^2_{L_\infty(E)} + |\widetilde{z}_k|^2_{H^{1/2}(E)}\Big) \\
&= \rho_k \Big(\Big(1+\ln\frac{H}{h}\Big)\|\widetilde{z}_j\|^2_{L_\infty(E)} + |\widetilde{z}_j|^2_{H^{1/2}(E)}\Big) \\
&\lesssim \rho_j \Big(\Big(1+\ln\frac{H}{h}\Big)\|\widetilde{\mathring{v}}_j\|^2_{L_\infty(E)} + |\widetilde{\mathring{v}}_j|^2_{H^{1/2}(E)}\Big) \\
&\lesssim \rho_j \Big(1+\ln\frac{H}{h}\Big)^2 |\widetilde{\mathring{v}}_j|^2_{H^{1/2}(\partial\Omega_j)} \\
&\lesssim \Big(1+\ln\frac{H}{h}\Big)^2 a_{h,j}(\mathring{v}_j, \mathring{v}_j). \qquad \square
\end{aligned}$$

The following lemma results immediately from Lemma 4.11 and Lemma 4.12.

LEMMA 4.13. *There exists a positive constant C independent of h, H, J, and ρ such that*

$$\lambda_{\max}(B_{BDDC} S_h) \le C\Big(1+\ln\frac{H}{h}\Big)^2. \tag{4.25}$$

Proof. Let $v \in X_{h,C}(\Gamma)$ be arbitrary. Consider a decomposition of v in the form of

$$v = P_\Gamma I_0 v_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \mathring{v}_j \qquad v_0 \in \mathcal{H}_0,\ \mathring{v}_j \in \mathring{\mathcal{H}}_j\ (1 \le j \le J).$$

Based on the characterization of the maximum eigenvalue in (4.4), we need to find an upper bound for $\langle S_h v, v\rangle$ in terms of $\langle S_0 v_0, v_0\rangle$ and $\langle S_j \mathring{v}_j, \mathring{v}_j\rangle$.

Since $P_\Gamma \mathbb{E}_j$ is supported in the union of Ω_j and the subdomains which share an edge with Ω_j, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
\langle S_h v, v\rangle &= a_h^C\Big(P_\Gamma I_0 v_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \mathring{v}_j, P_\Gamma I_0 v_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \mathring{v}_j\Big) \\
&\lesssim a_h^C(P_\Gamma I_0 v_0, P_\Gamma I_0 v_0) + \sum_{j=1}^{J} a_h^C(P_\Gamma \mathbb{E}_j \mathring{v}_j, P_\Gamma \mathbb{E}_j \mathring{v}_j),
\end{aligned}$$

which together with (4.15) and (4.22) implies

$$\langle S_h v, v\rangle \lesssim \Big(1+\ln\frac{H}{h}\Big)^2 \Big(\langle S_0 v_0, v_0\rangle + \sum_{j=1}^{J} \langle S_j \mathring{v}_j, \mathring{v}_j\rangle\Big) \tag{4.26}$$

for an arbitrary decomposition of $v \in X_{h,C}(\Gamma)$.

The estimate (4.25) follows from (4.4) and (4.26). ∎

4.4. Condition number estimates for $B_{BDDC}S_h$ and B_2A_h. The following bound for the condition number of $B_{BDDC}S_h$ is a direct consequence of Lemma 4.1 and Lemma 4.13.

THEOREM 4.14. *There exists a positive constant C independent of h, H, J, and ρ such that*

$$\kappa(B_{BDDC}S_h) = \frac{\lambda_{\max}(B_{BDDC}S_h)}{\lambda_{\min}(B_{BDDC}S_h)} \leq C\Big(1 + \ln\frac{H}{h}\Big)^2.$$

We also have a similar estimate for B_2A_h.

THEOREM 4.15. *There exists a positive constant C independent of h, H, J, and ρ such that*

$$\kappa(B_2A_h) = \frac{\lambda_{\max}(B_2A_h)}{\lambda_{\min}(B_2A_h)} \leq C\Big(1 + \ln\frac{H}{h}\Big)^2.$$

Proof. From the estimates (4.5) and (4.25) for the extreme eigenvalues of $B_{BDDC}S_h$, it follows that

$$\begin{aligned}\langle B_{BDDC}^{-1} v_{C,\Gamma}, v_{C,\Gamma}\rangle &\leq \langle S_h v_{C,\Gamma}, v_{C,\Gamma}\rangle \\ &\lesssim \Big(1 + \ln\frac{H}{h}\Big)^2 \langle B_{BDDC}^{-1} v_{C,\Gamma}, v_{C,\Gamma}\rangle \quad \forall\, v_{C,\Gamma} \in X_{h,C}(\Gamma).\end{aligned} \tag{4.27}$$

Combining (2.25) and (4.27), we have

$$\begin{aligned}&\langle A_{h,D} v_D, v_D\rangle + \langle A_{h,\Omega\setminus\Gamma} v_{C,\Omega\setminus\Gamma}, v_{C,\Omega\setminus\Gamma}\rangle + \langle B_{BDDC}^{-1} v_{C,\Gamma}, v_{C,\Gamma}\rangle \\ &\lesssim \langle A_h v, v\rangle \\ &\lesssim \Big(1 + \ln\frac{H}{h}\Big)^2 \big(\langle A_{h,D} v_D, v_D\rangle + \langle A_{h,\Omega\setminus\Gamma} v_{C,\Omega\setminus\Gamma}, v_{C,\Omega\setminus\Gamma}\rangle + \langle B_{BDDC}^{-1} v_{C,\Gamma}, v_{C,\Gamma}\rangle\big)\end{aligned} \tag{4.28}$$

for any $v \in X_h$, where $v = I_D v_D + I_{\Omega\setminus\Gamma} v_{C,\Omega\setminus\Gamma} + I_\Gamma v_{C,\Gamma}$ is the unique decomposition of v with respect to $X_{h,D}$, $X_{h,C}(\Omega\setminus\Gamma)$, and $X_{h,C}(\Gamma)$. It then follows from (3.14), (4.28), and the theory of additive Schwarz preconditioners that

$$1 \lesssim \lambda_{\min}(B_2A_h) \quad \text{and} \quad \lambda_{\max}(B_2A_h) \lesssim \Big(1 + \ln\frac{H}{h}\Big)^2. \tag{4.29}$$ ∎

5. The case of nonconforming meshes. In this section we extend our preconditioning techniques to the case of nonconforming meshes. For simplicity we will focus on the modification of the algorithm for the model problem (1.1) with $\rho = 1$. But the case where ρ is piecewise constant can also be treated in a similar fashion.

Let $\mathcal{T}_h$ be a nonconforming simplicial mesh for Ω, where hanging nodes occur only along the interface; cf. Fig. 5.1 (a). We assume that if an edge of a triangle in $\mathcal{T}_h$ has a hanging node, then the edge is the union of the edges of other triangles in $\mathcal{T}_h$; which of course are from the other side of Γ. Let $e \subset \Gamma$ be an edge of a triangle in $\mathcal{T}_h$. Then e belongs to the set $\mathcal{E}_{h,\Gamma}$ if (i) e is the common edge of two triangles in $\mathcal{T}_h$, or (ii) e contains at least one hanging node. In other words, the set $\mathcal{E}_{h,\Gamma}$ consists of only the long edges, such as the red line segment in Fig. 5.1 (b).

The construction of the intermediate preconditioner B_1 involves only $X_{h,C}$ and $X_{h,D}$. The definition of $X_{h,C}$ remains the same under the new definition of $\mathcal{E}_{h,\Gamma}$. This means that for a function $v \in X_{h,C}$ associated with the nonconforming mesh in Fig. 5.1 (b), its value at vertex 1

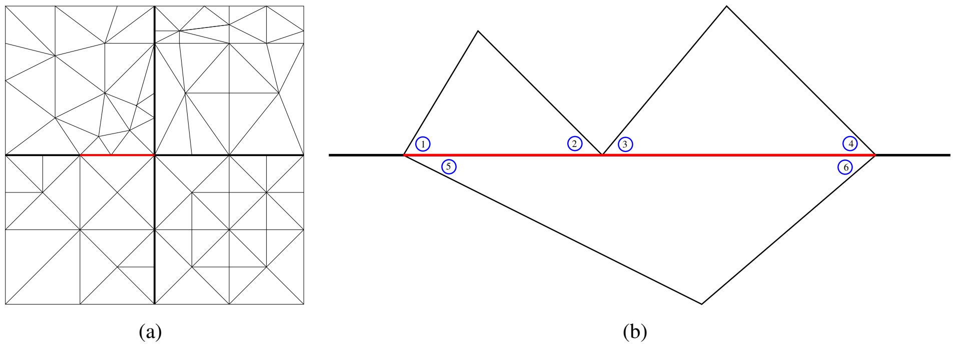

FIG. 5.1. *(a) A nonconforming triangulation with hanging nodes along* Γ *where the red line segment depicts one edge in* $\mathcal{E}_{h,\Gamma}$. *(b) Nonconforming meshes along a long edge in* $\mathcal{E}_{h,\Gamma}$.

matches its value at vertex 5, its value at vertex 4 matches its value at vertex 6, and its values at vertices 2 and 3 are determined by the requirement that v is continuous at the red line segment.

The definition of $X_{h,D}$ is modified as follows. A function $v \in X_h$ belongs to $X_{h,D}$ if (i) v vanishes at the vertices in $\mathcal{V}_I$, and (ii) $\{\!\{v\}\!\} = 0$ at the vertices of the edges of $\mathcal{E}_{h,\Gamma}$ that do not belong to $\mathcal{V}_I$. This means that for a function $v \in X_{h,D}$ associated with the nonconforming mesh in Fig. 5.1(b), its value at vertex 1 is $(-1)\times$ its value at vertex 5, its value at vertex 4 is $(-1)\times$ its value at vertex 6, and its values at the vertices 2 and 3 are unconstrained. Consequently, $A_{h,D}$ is a block diagonal matrix where each block corresponds to an edge in $\mathcal{E}_{h,\Gamma}$ and its dimension is 2 plus the number of hanging nodes that appear on the edge.

The proof of the condition number estimate (2.26) remains the same.

The preconditioner B_2 involves the BDDC preconditioner, whose construction remains the same under the new definition of $\mathcal{E}_{h,\Gamma}$. Here we only illustrate the meaning of the projection operator $P_\Gamma : \mathcal{H}_\mathcal{C} \longrightarrow X_{h,C}(\Gamma)$ for the non-conforming mesh in Fig. 5.1 (b): the values of $P_\Gamma v$ at the two vertices of the red line segment are given by the mean of its values at the vertices 1 and 5 and the mean of its values at the vertices 4 and 6. The values of P_Γ at the vertices 2 and 3 are then determined by the requirement that $P_\Gamma v$ is continuous at the red line segment.

The key ingredient for the condition number estimates in Section 4 is Lemma 4.6, whose proof relies on Lemma 4.8. Below we sketch the idea behind the extension of Lemma 4.8 to the case of nonconforming meshes.

Let $v \in X_{h,k}$. We define $v^\dagger \in X_{h,k}$ so that v and $v^\dagger$ have the same values at all the vertices except the ones on $\partial\Omega_k \cap \Gamma$ that are interior to the edges in $\mathcal{E}_{h,\Gamma}$, and the values of $v^\dagger$ at these vertices are chosen so that the restriction of $v^\dagger$ to $\partial\Omega_k \cap \Gamma$ is piecewise linear with respect to the edges in $\mathcal{E}_{h,\partial\Omega_k\cap\Gamma} = \{e \in \mathcal{E}_{h,\Gamma} : e \subset \partial\Omega_k \cap \Gamma\}$. For example (cf. Fig. 5.1(b)), the value of $v^\dagger$ at the vertices 1 and 4 equal to the values of v. But the values of $v^\dagger$ at the vertices 2 and 3 are determined by the condition that $v^\dagger$ restricted to the red line segment is a linear polynomial. Let $\mathcal{E}_{h,k}$ be the set of the edges of triangles in $\mathcal{T}_{h,k}$. Then we have

$$\sum_{T\in\mathcal{T}_{h,k}} |v - v^\dagger|^2_{H^1(T)} + \sum_{\substack{e\in\mathcal{E}_{h,k}\\ e\subset\Omega_k}} |e|^{-1}\|v - v^\dagger\|^2_{L_2(e)} \le C a_{h,k}(v,v), \tag{5.1}$$

where the positive constant C depends on the shape regularity of $\mathcal{T}_{h,k}$ and the maximum number of hanging nodes that can appear on the edges in $\mathcal{E}_{h,\partial\Omega_k\cap\Gamma}$.

The proof of Lemma 4.8 carries over to the function $v^\dagger$ and hence Lemma 4.8 also holds for v because of (5.1).

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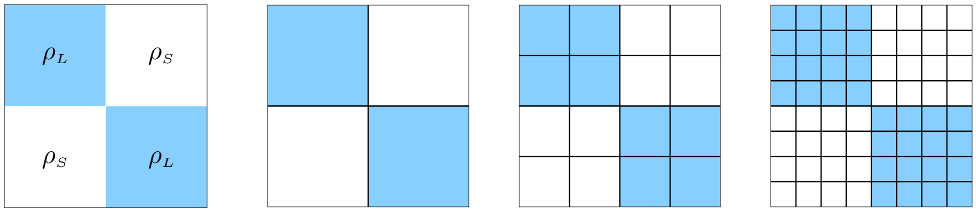

(a) (b)

FIG. 6.1. *(a) Piecewise constant coefficients ρ in a checkerboard pattern. (b) Decompositions of Ω into J subdomains for $J = 2^2, 4^2, 8^2$.*

Consequently, Theorem 4.14 and Theorem 4.15 remain valid but now the constants in the estimates also depend on the maximum number of hanging nodes on the edges in $\mathcal{E}_{h,\Gamma}$.

6. Numerical results. In this section we present some numerical results to illustrate the performance of the preconditioners B_{BDDC} and B_2

We consider the model problem (1.1) on the unit square $\Omega = (0,1) \times (0,1)$. The coefficient ρ is distributed in a checkboard pattern with two different constants ρ_S and ρ_L for $\rho_S \leq \rho_L$; cf. Fig. 6.1(a). The domain Ω is divided into J nonoverlapping squares so that ρ is a constant on each subdomain and the length of the horizontal/vertical edges of the squares is denoted by H; cf. Fig. 6.1(b). We use a uniform triangulation $\mathcal{T}_h$ of Ω, where h denotes the mesh size; cf. Fig. 2.1(a).

The discrete problem resulting from the SIPG method is solved by the preconditioned conjugate gradient algorithm. For comparison, the conjugate gradient iteration is also carried out without preconditioning. The iteration is stopped when the relative residual is less than 10^{-6}. In Table 6.1 and Table 6.4, the sign $-$ in the CG Iter columns indicates that the conjugate gradient iteration fails to stop before the maximum number of iterations ($=$ the total number of unknowns of the discrete problem) is reached.

Numerical results for the preconditioner B_2 are presented in Table 6.1. For comparison, the results for case of $\rho = 1$ are also presented in Table 6.2. Both set of results are in agreement with the theoretical estimates in (4.29).

TABLE 6.1
Performance of the preconditioner B_2 in case of discontinuous coefficients ρ where $\rho_S = 1$, $\rho_L = 10^5$, and $\eta = 5$.

J	$1/h$	H/h	PCG Iter	κ	$\lambda_{\min}(B_2A_h)$	$\lambda_{\max}(B_2A_h)$	CG Iter
2^2	6	3	18	5.6914	2.9889e-1	1.7011	$-$
	12	6	17	6.3193	2.7325e-1	1.7267	$-$
	24	12	17	6.5063	2.6644e-1	1.7336	$-$
4^2	12	3	20	8.0552	2.8408e-1	2.2884	$-$
	24	6	19	9.6797	2.7016e-1	2.6151	$-$
8^2	24	3	23	8.2283	2.8408e-1	2.3375	$-$

Table 6.3 shows that the performance of the preconditioner B_2 is independent of the choice of η.

The robustness of the preconditioner B_2 with respect to the jump in ρ is demonstrated in Table 6.4, where $\rho_S = 1$ and ρ_L ranges between 1 and 10^5; cf. Fig. 6.1(a).

TABLE 6.2
Performance of the preconditioner B_2 in case of $\rho = 1$ where $\eta = 5$.

J	$1/h$	H/h	PCG Iter	κ	$\lambda_{\min}(B_2A_h)$	$\lambda_{\max}(B_2A_h)$	CG Iter
	6	3	16	5.6874	2.9915e-1	1.7014	44
2^2	12	6	16	6.3172	2.7334e-1	1.7267	79
	24	12	16	6.5849	2.6645e-1	1.7545	149
4^2	12	3	16	6.4992	2.9873e-1	1.9415	79
	24	6	18	9.0475	2.7332e-1	2.4729	149
8^2	24	3	18	7.0336	2.9854e-1	2.0998	149

TABLE 6.3
Dependence of the preconditioner B_2 on η where $\rho = 1$, $J = 4^2$, and $h = 1/16$.

η	B_2A_h		A_h	
	κ	PCG Iter	κ	CG Iter
5	7.5117	17	7.7883e+2	102
10	5.9025	16	1.5611e+3	137
50	8.1257	20	7.7914e+3	253
100	8.8201	20	1.5578e+4	299

TABLE 6.4
Robustness of the preconditioner B_2 with respect to the jump in the coefficient ρ where $\rho_S = 1$, $\eta = 5$, $J = 4^2$, and $h = 1/12$.

ρ_L	B_2A_h		A_h	
	κ	PCG Iter	κ	CG Iter
1	6.4992	16	4.2383e+2	83
10^1	7.7321	17	1.3226e+3	200
10^2	8.0202	18	1.1169e+4	482
10^3	8.0517	18	1.1042e+5	–
10^4	8.0549	19	1.1030e+6	–
10^5	8.0552	20	1.1029e+7	–

TABLE 6.5
Results for the preconditioners B_{BDDC} and B_2 for $\rho_S = 1$, and $\rho_L = 10$ where $\eta = 5$ and $J = 3^2$.

$1/h$	H/h	$B_{BDDC}S_h$			B_2A_h		
		κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$
9	3	1.3082	1.0000	1.3082	5.9399	2.8824e-1	1.7121
18	6	1.4640	1.0000	1.4640	6.4698	2.7129e-1	1.7551
30	10	1.5872	1.0000	1.5872	6.8798	2.6693e-1	1.8365
36	12	1.6328	1.0000	1.6328	7.0218	2.6611e-1	1.8686
48	18	1.7066	1.0000	1.7066	7.2479	2.6525e-1	1.9225

From a comparison between the results in Table 6.5, the decisive effect of the preconditioner B_{BDDC} on the performance of the preconditioner B_2 can be observed, which agrees with the analysis in Theorem 4.15.

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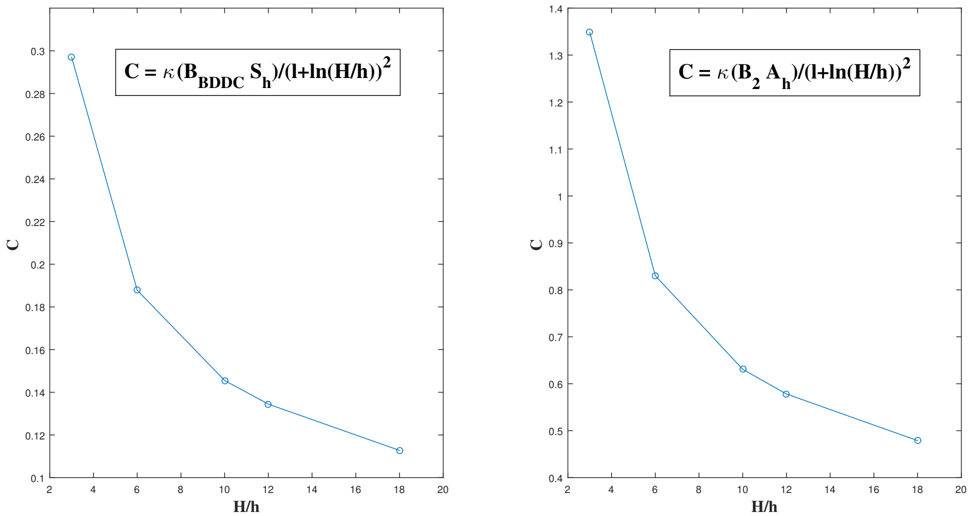

FIG. 6.2. *Left figure: the behavior of* $\kappa\left(B_{BDDC}S_h\right)/\left(1+\ln(H/h)\right)^2$ *for the BDDC preconditioner; Right figure: the behavior of* $\kappa\left(B_2A_h\right)/\left(1+\ln(H/h)\right)^2$ *for the preconditioner* B_2.

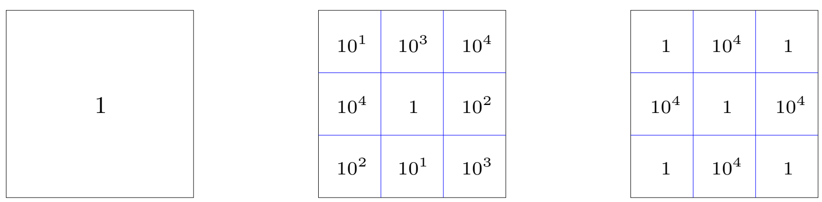

FIG. 6.3. *Left figure: Type A; Center figure: Type B; Right figure: Type C.*

TABLE 6.6
Results for the preconditioners B_{BDDC} *and* B_2 *for three different types of coefficient distribution where* $\eta = 5$ *and* $J = 3^2$.

$1/h$	Type	$\kappa(B_{BDDC}S_h)$	$\kappa(B_2A_h)$	$\kappa(A_h)$
9	A	1.6324	6.0782	2.4113e+2
	B	1.3932	6.0386	3.4076e+5
	C	1.0007	6.0401	3.3000e+6
36	A	2.7886	10.6070	3.9386e+3
	B	2.1798	8.4656	4.5804e+6
	C	1.0016	6.5222	4.4116e+6

The numbers $\kappa\left(B_{BDDC}S_h\right)/\left(1+\ln(H/h)\right)^2$ and $\kappa\left(B_2A_h\right)/\left(1+\ln(H/h)\right)^2$ are plotted in Fig. 6.2 against H/h. As H/h increases, these two numbers settle down around 0.1 and 0.5, which indicates that the condition number estimates in Theorem 4.14 and Theorem 4.15 are sharp. Table 6.6 shows how the performance of the preconditioners B_{BDDC} and B_2 is affected by a variation of the jump of ρ in three different patterns depicted in Fig. 6.3. The minimum of the multiplicative jump across the edges of subdomains is 1, 10, and 10^4 for Type A, Type B, and Type C, respectively.

The numerical results in Table 6.7 present the performance of the preconditioners B_{BDDC}

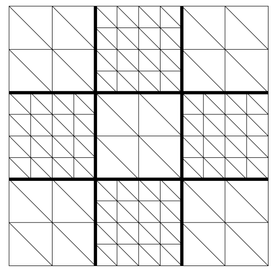

FIG. 6.4. *A nonconforming triangulation of* Ω *for case of* $\rho = 1$*, where* $J = 3^2$.

TABLE 6.7
Results for the preconditioners B_{BDDC} *and* B_2 *in case of a nonconforming triangulation where* $\rho = 1$ *and* $\eta = 10$.

J	$1/h_c$	H/h_c	$B_{BDDC}S_h$			B_2A_h			A_h
			κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ
3^2	6	2	1.2423	1.0000	1.2423	8.1014	2.2008e-1	1.7830	3.3226e+2
	12	4	1.5755	1.0000	1.5755	8.0913	2.2066e-1	1.7854	1.3113e+3
	24	8	2.0866	1.0000	2.0866	9.5509	2.2067e-1	2.1076	5.2346e+3
6^2	12	2	1.5470	1.0000	1.5470	8.1482	2.1967e-1	1.7899	1.3259e+3
	24	4	2.0908	1.0000	2.0908	9.5502	2.2064e-1	2.1071	5.2601e+3

and B_2 for the case of nonconforming meshes (cf. Fig. 6.4), which agrees with the discussion in Section 5.

Appendix A. References for the notations. For the convenience of the readers, we provide references in Table A.1 for the notations that appear in multiple sections.

TABLE A.1
References for the notations.

Notation	Reference	Notation	Reference	Notation	Reference
A_h	(2.7)	$A_{h,C}$	(2.9)	$A_{h,D}$	(2.8)
$a_{h,j}(\cdot,\cdot)$	(2.10)	$A_{h,\Omega\setminus\Gamma}$	(2.20)	a_h^C	(3.5)
B_{BDDC}	(3.12)	B_1	(2.23)	B_2	(3.14)
$\mathbb{E}_j$	(3.13)	$\mathcal{H}_\mathcal{C}$	(3.4)	$\mathcal{H}_j$	(3.3)
$\mathring{\mathcal{H}}$	(3.7)	$\mathcal{H}_0$	(3.8)	I_D	(2.24)
I_Γ	(2.24)	$I_{\Omega\setminus\Gamma}$	(2.24)	P_Γ	(3.9)
ρ_e	(1.4)	$\{\!\{\rho\nabla v\}\!\}$	(1.6)	S_h	(2.21)
S_j	(3.11)	S_0	(3.10)	$\mathcal{T}_{h,j}$	(3.2)
$[\![v]\!]$	(1.5)	$\mathcal{V}_I$	(2.1)	$\mathcal{V}_\mathcal{C}$	(2.2)
X_h	(1.2)	$X_{h,C}$	(2.4)	$X_{h,C}(\Gamma)$	(2.19)
$X_{h,C}(\Omega\setminus\Gamma)$	(2.18)	$X_{h,D}$	(2.5)	$X_{h,j}$	(3.1)

REFERENCES

[1] P. F. ANTONIETTI AND B. AYUSO, *Schwarz domain decomposition preconditioners for discontinuous Galerkin approximations of elliptic problems: non-overlapping case*, M2AN Math. Model. Numer. Anal., 41 (2007), pp. 21–54.

[2] ———, *Two-level Schwarz preconditioners for super penalty discontinuous Galerkin methods*, Commun. Comput. Phys., 5 (2009), pp. 398–412.

[3] P. F. ANTONIETTI, B. AYUSO DE DIOS, S. C. BRENNER, AND L.-Y. SUNG, *Schwarz methods for a preconditioned WOPSIP method for elliptic problems*, Comput. Methods Appl. Math., 12 (2012), pp. 241–272.

[4] P. F. ANTONIETTI AND P. HOUSTON, *A class of domain decomposition preconditioners for hp-discontinuous Galerkin finite element methods*, J. Sci. Comput., 46 (2011), pp. 124–149.

[5] D. N. ARNOLD, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., 19 (1982), pp. 742–760.

[6] D. N. ARNOLD, F. BREZZI, B. COCKBURN, AND L. D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2001/02), pp. 1749–1779.

[7] B. AYUSO DE DIOS, M. HOLST, Y. ZHU, AND L. ZIKATANOV, *Multilevel preconditioners for discontinuous, Galerkin approximations of elliptic problems, with jump coefficients*, Math. Comp., 83 (2014), pp. 1083–1120.

[8] B. AYUSO DE DIOS AND L. ZIKATANOV, *Uniformly convergent iterative methods for discontinuous Galerkin discretizations*, J. Sci. Comput., 40 (2009), pp. 4–36.

[9] A. T. BARKER, S. C. BRENNER, E.-H. PARK, AND L.-Y. SUNG, *Two-level additive Schwarz preconditioners for a weakly over-penalized interior penalty method*, J. Sci. Comput., 47 (2011), pp. 27–49.

[10] J. H. BRAMBLE, J. E. PASCIAK, AND A. H. SCHATZ, *The construction of preconditioners for elliptic problems by substructuring. I*, Math. Comp., 47 (1986), pp. 103–134.

[11] S. BRENNER, E.-H. PARK, AND L.-Y. SUNG, *A balancing domain decomposition by constraints preconditioner for a weakly over-penalized symmetric interior penalty method*, Numer. Linear Algebra Appl., 20 (2013), pp. 472–491.

[12] S. C. BRENNER, L. OWENS, AND L.-Y. SUNG, *A weakly over-penalized symmetric interior penalty method*, Electron. Trans. Numer. Anal., 30 (2008), pp. 107–127.
http://etna.ricam.oeaw.ac.at/vol.30.2008/pp107-127.dir/pp107-127.pdf

[13] S. C. BRENNER AND L. R. SCOTT, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Springer, New York, 2008.

[14] S. C. BRENNER AND L.-Y. SUNG, *BDDC and FETI-DP without matrices or vectors*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1429–1435.

[15] E. BURMAN AND P. ZUNINO, *A domain decomposition method based on weighted interior penalties for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., 44 (2006), pp. 1612–1638.

[16] C. CANUTO, L. F. PAVARINO, AND A. B. PIERI, *BDDC preconditioners for continuous and discontinuous Galerkin methods using spectral/hp elements with variable local polynomial degree*, IMA J. Numer. Anal., 34 (2014), pp. 879–903.

[17] L. T. DIOSADY AND D. L. DARMOFAL, *A unified analysis of balancing domain decomposition by constraints for discontinuous Galerkin discretizations*, SIAM J. Numer. Anal., 50 (2012), pp. 1695–1712.

[18] C. R. DOHRMANN, *A preconditioner for substructuring based on constrained energy minimization*, SIAM J. Sci. Comput., 25 (2003), pp. 246–258.

[19] J. DOUGLAS, JR. AND T. DUPONT, *Interior penalty procedures for elliptic and parabolic Galerkin methods*, in Computing Methods in Applied Sciences (Second Internat. Sympos., Versailles, 1975), R. Glowinski and J. L. Lions, eds., Lecture Notes in Phys., Vol. 58, Springer, Berlin, 1976, pp. 207–216.

[20] M. DRYJA, *On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients*, Comput. Methods Appl. Math., 3 (2003), pp. 76–85.

[21] M. DRYJA, J. GALVIS, AND M. SARKIS, *BDDC methods for discontinuous Galerkin discretization of elliptic problems*, J. Complexity, 23 (2007), pp. 715–739.

[22] ———, *The analysis of a FETI-DP preconditioner for a full DG discretization of elliptic problems in two dimensions*, Numer. Math., 131 (2015), pp. 737–770.

[23] M. DRYJA AND O. B. WIDLUND, *An additive variant of the Schwarz alternating method in the case of many subregions*, Tech. Report 339, Department of Computer Science, Courant Institute, 1987.

[24] X. FENG AND O. A. KARAKASHIAN, *Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems*, SIAM J. Numer. Anal., 39 (2001), pp. 1343–1365.

[25] ———, *Two-level non-overlapping Schwarz preconditioners for a discontinuous Galerkin approximation of the biharmonic equation*, J. Sci. Comput., 22/23 (2005), pp. 289–314.

[26] H. H. KIM, E. T. CHUNG, AND C. S. LEE, *A BDDC algorithm for a class of staggered discontinuous Galerkin methods*, Comput. Math. Appl., 67 (2014), pp. 1373–1389.

[27] J. LI AND O. B. WIDLUND, *FETI-DP, BDDC, and block Cholesky methods*, Internat. J. Numer. Methods Engrg., 66 (2006), pp. 250–271.
[28] J. MANDEL AND C. R. DOHRMANN, *Convergence of a balancing domain decomposition by constraints and energy minimization*, Numer. Linear Algebra Appl., 10 (2003), pp. 639–659.
[29] T. P. A. MATHEW, *Domain Decomposition Methods for the Numerical Solution of Partial Differential Equations*, Springer, Berlin, 2008.
[30] B. RIVIÈRE, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*, SIAM, Philadelphia, 2008.
[31] J. SCHÖBERL AND C. LEHRENFELD, *Domain decomposition preconditioning for high order hybrid discontinuous Galerkin methods on tetrahedral meshes*, in Advanced Finite Element Methods and Applications, T. Apel and O. Steinbach, eds., Lect. Notes Appl. Comput. Mech., 66, Springer, Heidelberg, 2013, pp. 27–56.
[32] L. R. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.
[33] B. F. SMITH, P. E. BJØRSTAD, AND W. D. GROPP, *Domain Decomposition*, Cambridge University Press, Cambridge, 1996.
[34] A. TOSELLI AND O. WIDLUND, *Domain Decomposition Methods—Algorithms and Theory*, Springer, Berlin, 2005.
[35] M. F. WHEELER, *An elliptic collocation-finite element method with interior penalties*, SIAM J. Numer. Anal., 15 (1978), pp. 152–161.
[36] O. B. WIDLUND, *An extension theorem for finite element spaces with three applications*, in Numerical Techniques in Continuum Mechanics, W. Hackbusch and K. Witsch, eds., Vieweg, Braunschweig, 1987, pp. 110–122.
[37] J. XU, *Iterative methods by space decomposition and subspace correction*, SIAM Rev., 34 (1992), pp. 581–613.