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TOPOLOGICAL SOLVABILITY AND DAE-INDEX CONDITIONS FOR MASS FLOW CONTROLLED PUMPS IN LIQUID FLOW NETWORKS*

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Abstract. This work is devoted to the analysis of a model for the thermal management in liquid flow networks consisting of pipes and pumps. The underlying model equation for the liquid flow is not only governed by the equation of motion and the continuity equation, describing the mass transfer through the pipes, but also includes thermodynamic effects in order to cover cooling and heating processes. The resulting model gives rise to a differential-algebraic equation (DAE), for which a proof of unique solvability and an index analysis is presented. For the index analysis, the concepts of the *Strangeness Index* is pursued. Exploring the network structure of the liquid flow network via graph-theoretical approaches allow us to develop network topological criteria for the existence of solutions and the DAE-index. The topological criteria are explained by various examples.

Key words. differential-algebraic equations, topological index criteria, hydraulic network

AMS subject classifications. 65L80, 94C15

1. Introduction.

Increasingly demanding emissions legislation specifies the performance requirements for the next generation of products from vehicle manufacturers. Conversely, the stringent emissions legislation is coupled with the trend in increased power, drivability, and safety expectations from the consumer market. Promising approaches to meet these requirements are downsizing the internal combustion engines (ICE), the application of turbochargers, variable valve timing, advanced combustion systems, or comprehensive exhaust aftertreatment, but also different variants of combinations of the ICE with an electrical engine in terms of hybridization or even a purely electric propulsion. The challenges in the development of future powertrains do not only lie in the design of individual components but in the assessment of the powertrain as a whole. On a system engineering level it is required to optimize individual components globally and to balance the interaction of different subsystems. A typical system engineering model comprises several subsystems. For instance in case of a hybrid propulsion these can be the vehicle chassis, the drive line, the air path of the ICE including combustion and exhaust aftertreatment, the cooling system of the ICE, the electrical propulsion system including the engine and a battery pack, and finally the corresponding control systems. Similar to the ICE, the battery pack requires a cooling system as well. Both cooling systems are typically represented by an corresponding hydraulic network in the overall model. The simulation and optimization of hydraulic networks have been studied in various works, including [4, 7, 12, 13, 23] and the references therein. The considered models are motivated by drinking water supply systems, where the main target is to circulate an amount of water at any time, assuring a certain pressure at extraction points. The aim of this work is to consider and analyze hydraulic networks used for thermal management systems. Examples in automotive applications are the above mentioned cooling systems.

In contrast to water transportation networks, the primary interest is not the pressure distribution across the whole system, but the temperature distribution. Consequently, the models have to be equipped with energy balance laws in order to model the thermodynamic effects. The purpose of this work is to extend the results, which are already available for water

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transportation networks [12, 13], to cooling and heating systems used for thermal management and to networks including mass flow-controlled pumps. Thermal flow networks consisting solely of pipes have been analyzed in [2]. The extension to networks of pipes and pumps is not straightforward since additionally to Kirchhoff's first law, also Kirchhoff's second law has to be considered. In particular, Kirchhoff's second law restricts the allowed pump constellations for a valid liquid flow model.

The model under consideration is a quasi-stationary pipe network, cf. [13], equipped with energy balance laws. This model is suited to describe circuits which are filled with incompressible fluids (e.g., water). Here incompressible means that density changes with respect to temperature changes or pressure changes are neglected.

While general networks consist of various types of elements (pipes, pumps, valves) [23], the model here is restricted to pipes and pumps only. Despite this simplification, the demanding issues are caused by the arbitrary network structure of the underlying model. Since valves can change the topology of the underlying network due to their discrete nature, they have to be treated separately.

State-of-the-art modeling and simulation packages such as Dymola[1], Matlab/Simulink[2], Flowmaster[3], Amesim[4], SimulationX[5], or Cruise M[6] offer many excellent concepts for the automatic generation of dynamic system models, including hydraulic networks. Modeling is done in a modularized way, based on a network of subsystems which again consists of simple standardized subcomponents. The network structure (topology) carries the core information of the network properties and therefore is predestinated to be exploited for the analysis and numerical simulation of those. In many applications, the equations describing the network are differential-algebraic equations (DAEs). Hence, the analysis of existence and uniqueness of solutions as well as rank considerations are a delicate issue.

Topology-based index analysis for networks connects the research fields of *Analysis for DAEs* [22] and *Graph Theory* [8] in order to provide the appropriate basis to analyze DAEs stemming from automatic generated system models. So far it has been established for various types of networks, including electric circuits [24], gas supply networks [10], and water supply networks [12, 13, 23]. Although all those networks share some similarities, an individual investigation is required due to their different physical nature. Recently, a unified modeling approach for different types of flow networks has been introduced in [14], aiming at a unified topology-based index analysis for the different physical domains on an abstract level.

The structure of this work is the following: in Section 2, the main two concepts required for the analysis are introduced. First, an introduction to graph theory is given, then the application to equations imposed on networks is described, and the core tools for the following analysis are proven. The network model and the arising DAEs are formulated in Section 3. Furthermore, some basic properties are derived, which lead to the full DAE analysis in Section 4. Beside existence and uniqueness results, DAE-index considerations are performed to ensure an accurate and efficient numerical simulation. Throughout the analysis, the sufficient algebraic conditions are linked to necessary conditions imposed on the network structure. Those topological conditions are explained in terms of examples. A summary of the results with comments on their practical relevance in commercial simulation software concludes the paper in Section 5.

[1] http://www.dynasim.com
[2] http://www.mathworks.com
[3] http://www.mentor.com
[4] http://www.plm.automation.siemens.com
[5] http://www.iti.de
[6] http://www.avl.com

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2. Graphs and their application in network dynamics. In this section, we introduce the notation and graph-theoretical concepts that we need in our analysis and prove some additional results in Lemma 2.1 and Lemma 2.2.

For a detailed introduction to graph theory, we refer the reader to, e.g., [3, 8]. A *graph* $\mathcal{G}$ is a pair $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ of subsets $\mathcal{V}, \mathcal{E} \subset \mathbb{N}$ such that $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, i.e., each element $e_j \in \mathcal{E}$ corresponds to a pair $(v_{i_1}, v_{i_2}) \in \mathcal{V} \times \mathcal{V}$ [8, p. 2]. If the pairs $(v_i, v_k) \in \mathcal{E}$ are *ordered*, then $\mathcal{G}$ is called an *oriented* graph [8, p. 25]. If $\mathcal{G}$ is oriented, then v_i and v_k are called the originating and terminating vertex of the edge $e_j = (v_i, v_k)$, respectively, [8, p. 25]. If $\mathcal{G}$ contains no self-loops or parallel edges, then $\mathcal{G}$ is called *simple*, cf. [8, p. 25].

Two vertices $v_i, v_k \in \mathcal{V}$ are called *adjacent* if there exists an edge $e_j \in \mathcal{E}$ such that $e_j = (v_i, v_k)$ [8, p. 13]. The edge e_j is called *incident* to v_i and v_k, respectively [8, p. 13]. Two edges $e_j, e_l \in \mathcal{E}$ are called *adjacent* if they are incident to a common vertex v_i [8, p. 13]. For $v_i \in \mathcal{V}$, the incident edges are summarized in the set

$$\mathcal{E}_{inc}(v_i) := \{e_j \in \mathcal{E} \mid \exists\, v_k \in \mathcal{V} \colon e_j = (v_i, v_k)\}.$$

If $\mathcal{E}_{inc}(v_i) = \emptyset$, then v_i is *isolated*, and if $|\mathcal{E}_{inc}(v_i)| = 1$, then v_i is an *end vertex* [8, p. 2].

The connection structure of $\mathcal{G}$ is described by the *incidence matrix* A, which, if $\mathcal{G}$ is oriented, is defined as

$$A_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the left vertex of } e_j, \\ -1, & \text{if } v_i \text{ is the right vertex of } e_j, \\ 0, & \text{else.} \end{cases}$$

A subset $\mathcal{G}_s = \{\mathcal{V}_s, \mathcal{E}_s\}$ with $\mathcal{V}_s \subset \mathcal{V}$ is a *subgraph* of $\mathcal{G}$ if $\mathcal{E}_s \subset \mathcal{V}_s \times \mathcal{V}_s$ [8, p. 3]. If $\mathcal{V}_s = \mathcal{V}$, then the subgraph $\mathcal{G}_s$ *spans* $\mathcal{G}$ [8, p. 3]. The incidence matrix of $\mathcal{G}_s$ is given by $A_s = [A_{ij}]_{(v_i,e_j)\in\mathcal{V}_s\times\mathcal{E}_s}$ [8, p. 3].

In our analysis, we consider a simple, oriented graph $\mathcal{G}$ whose vertices $\mathcal{V}$ and edges $\mathcal{E}$ are composed from subsets $\mathcal{V}_1, \dots, \mathcal{V}_{\hat{v}}$ and $\mathcal{E}_1, \dots, \mathcal{E}_{\hat{e}}$ such that $\mathcal{V} = \bigcup_{I=1}^{\hat{v}} \mathcal{V}_{\mathrm{I}}$, $\mathcal{E} = \bigcup_{J=1}^{\hat{e}} \mathcal{E}_{\mathrm{J}}$. Accordingly, the incidence matrix A is composed of submatrices A_{IJ} describing the connection structure of the subsets $\mathcal{G}_{\mathrm{IJ}} := \{\mathcal{V}_{\mathrm{I}}, \mathcal{E}_{\mathrm{J}}\}$. In general, a set $\mathcal{G}_{\mathrm{IJ}}$ is not a proper subgraph of $\mathcal{G}$ as the edges $\mathcal{E}_{\mathrm{J}}$ may be incident to vertices outside $\mathcal{V}_{\mathrm{I}}$. Then, the connection matrix A_{IJ} does not have the usual pattern of two non-zero entries per column. To characterize the fundamental subspaces of A_{IJ}, we partition the edges into

$$\mathcal{E}_{\mathrm{J}} = \mathcal{E}_{\mathrm{IJ}}^{\mathrm{inner}} \cup \mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}} \cup \mathcal{E}_{\mathrm{IJ}}^{\mathrm{isolated}},$$

where $\mathcal{E}_{\mathrm{J}}^{\mathrm{inner}}$ contains the edges incident to vertices in $\mathcal{V}_{\mathrm{I}}$, i.e.,

$$\mathcal{E}_{\mathrm{J}}^{\mathrm{inner}} := \{e_j \in \mathcal{E}_{\mathrm{J}} \mid e_j = (v_{j_1}, v_{j_2}) \text{ with } v_{j_1}, v_{j_2} \in \mathcal{V}_{\mathrm{I}}\},$$

$\mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}}$ contains the *loose edges* incident to a vertex in $\mathcal{V}_{\mathrm{I}}$ and a vertex outside $\mathcal{V}_{\mathrm{I}}$, i.e.,

$$\mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}} := \{e_j \in \mathcal{E}_{\mathrm{J}} \mid e_j = (v_{j_1}, v_{j_2}) \text{ with } v_{j_1} \in \mathcal{V}_{\mathrm{I}}, v_{j_2} \in \mathcal{V}\backslash\mathcal{V}_{\mathrm{I}}\},$$

and $\mathcal{E}_{\mathrm{IJ}}^{\mathrm{isolated}}$ contains the *isolated edges* incident to vertices outside $\mathcal{V}_{\mathrm{I}}$, i.e.,

$$\mathcal{E}_{\mathrm{IJ}}^{\mathrm{isolated}} := \{e_j \in \mathcal{E}_{\mathrm{J}} \mid e_j = (v_{j_1}, v_{j_2}) \text{ with } v_{j_1}, v_{j_2} \in \mathcal{V}\backslash\mathcal{V}_{\mathrm{I}}\}.$$

For simplicity, we assume that there is at most one loose edge per vertex. Using an equivalence relation, the following results can be extend to the case of multiple loose edges per vertex. Furthermore, we set

$$\mathcal{V}_{\mathrm{IJ}}^{\mathrm{outer}} := \mathcal{V}_{\mathrm{I}} \cup \mathcal{V}_{\mathrm{IJ}}^{\mathrm{c}},$$

where $\mathcal{V}_{\mathrm{IJ}}^{\mathrm{c}}$ contains the vertices outside $\mathcal{V}_{\mathrm{I}}$ that are incident to edges in $\mathcal{E}_{\mathrm{J}}$, i.e.,

$$\mathcal{V}_{\mathrm{IJ}}^{\mathrm{c}} := \{v_i \in \mathcal{V}\backslash\mathcal{V}_{\mathrm{I}} \,|\, \mathcal{E}_{\mathrm{adj}}(v_i) \cap \mathcal{E}_{\mathrm{J}} \neq \varnothing\}.$$

With this notation, we set

$$\mathcal{G}_{\mathrm{IJ}}^{\mathrm{outer}} := \{\mathcal{V}_{\mathrm{IJ}}^{\mathrm{outer}}, \mathcal{E}_{\mathrm{J}}\}, \qquad \mathcal{G}_{\mathrm{IJ}}^{\mathrm{inner}} := \{\mathcal{V}_{\mathrm{I}}, \mathcal{E}_{\mathrm{IJ}}^{\mathrm{inner}}\},$$

where $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{outer}}$ is the minimal subgraph containing $\mathcal{G}_{\mathrm{IJ}}$ and $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ is the maximal subgraph contained in $\mathcal{G}_{\mathrm{IJ}}$. Using $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{outer}}$, $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ we can straightforwardly extend the standard definitions of graphs, cf., e.g., [6, 8], to the set $\mathcal{G}_{\mathrm{IJ}}$.

A subset $\mathcal{P} := \{\mathcal{V}_{\mathcal{P}}, \mathcal{E}_{\mathcal{P}}\} \subset \mathcal{G}_{\mathrm{IJ},k}$ is called a *path* in $\mathcal{G}_{\mathrm{IJ}}$ if it is a path in $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{outer}}$, i.e., if the vertices in $\mathcal{V}_{\mathcal{P}}$ are pairwise distinct and there exists a numbering such that v_i, e_j are adjacent to v_{i+1}, e_{j+1} for $(i,j) \in \{1,\ldots,|\mathcal{V}_{\mathcal{P}}|-1\} \times \{1,\ldots,|\mathcal{E}_{\mathcal{P}}|-1\}$, respectively. If $\mathcal{G}$ is oriented, with respect to this numbering, we assign a *sign* to every edge $e_j \in \mathcal{P}$ by

$$\mathrm{sgn}_{\mathcal{P}}(e_j) = \begin{cases} 1, & e_j = (v_i, v_{i+1}), \\ -1, & e_j = (v_{i+1}, v_i), \end{cases}$$

and define the *path matrix* $P = \sum_{e_j \in \mathcal{E}_{\mathcal{P}}} \mathrm{sgn}_{\mathcal{P}}(e_j) e_j$, where $e_1, \ldots, e_{|\mathcal{E}|} \in \mathbb{R}^{|\mathcal{E}|}$ denotes the standard canonical basis.

If $\mathrm{sgn}_{\mathcal{P}}(e_j) = \mathrm{sgn}_{\mathcal{P}}(e_l)$ for $e_j, e_l \in \mathcal{E}_{\mathcal{P}}$, then $\mathcal{P}$ is called *directed*. If $v_1, v_{|\mathcal{E}_{\mathcal{P}}|} \in \mathcal{V}_{\mathrm{IJ}}^{\mathrm{c}}$, then $\mathcal{P}$ is called a *crossing path*. If $v_1, v_{|\mathcal{E}_{\mathcal{P}}|} \in \mathcal{V}_{\mathrm{I}}$ with $v_1 = v_{|\mathcal{E}_{\mathcal{P}}|}$, then $\mathcal{C} := \mathcal{P}$ is called a *cycle* in $\mathcal{G}_{\mathrm{IJ}}$.

The set $\mathcal{G}_{\mathrm{IJ}}$ is *connected* if $\mathcal{E}_{\mathrm{IJ}}^{\mathrm{isolated}} = \varnothing$ and $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ is connected, i.e., if every pair of vertices $v_i, v_k \in \mathcal{V}_{\mathrm{I}}$ can be connected by a path. If $\mathcal{G}_{\mathrm{IJ}}$ is not connected, then it is composed of connected components $\mathcal{G}_{\mathrm{IJ},k} = \{\mathcal{V}_{\mathrm{IJ},k}, \mathcal{E}_{\mathrm{IJ},k}\}$, $k = 1,\ldots,K$, containing the connected components $\mathcal{G}_{\mathrm{IJ},k}^{\mathrm{inner}}$ of $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{inner}}$, respectively, plus loose edges $\mathcal{E}_{\mathrm{IJ},k}^{\mathrm{loose}}$ incident to vertices in $\mathcal{G}_{\mathrm{IJ},k}^{\mathrm{inner}}$.

A subgraph $\mathcal{T}_1 \subset \mathcal{G}$ of a connected graph $\mathcal{G}$ that contains no cycles and spans $\mathcal{G}$ is called a *spanning tree* [8, p. 13]. Every connected graph has at least one spanning tree [8, p. 14]. If $\mathcal{G}_{\mathrm{IJ}}$ is connected, then a subset $\mathcal{T}_1 \subset \mathcal{G}_{\mathrm{IJ}}$ is called a *spanning tree* if $\mathcal{T}_1 = \mathcal{T}_1^{\mathrm{inner}} \cup \{e_{j_0}\}$, where $\mathcal{T}_1^{\mathrm{inner}}$ is a spanning tree of $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ and $e_{j_0} \in \mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}}$ is a *reference loose edge*. The complement $\mathcal{T}_2$ is called the *chord set*. For the incidence matrix, the associated edges are selected by the permutation $\Pi = [\Pi_1, \Pi_2]$ with $\Pi_i = [e_j]_{e_j \in \mathcal{T}_i}$, $i = 1, 2$, where $e_1, \ldots, e_{|\mathcal{E}_{\mathrm{J}}|} \in \mathbb{R}^{|\mathcal{E}_{\mathcal{J}}|}$ denotes the standard canonical basis. Every interior chord $e_k \in \mathcal{E}_{\mathcal{T}_2} \cap \mathcal{E}_{\mathrm{IJ}}^{\mathrm{inner}}$ defines a unique *fundamental cycle* $\mathcal{C}_k = \{\mathcal{V}_k, \mathcal{E}_k\}$ with $\mathcal{E}_k \backslash \{e_k\} \subset \mathcal{T}_1 \cap \mathcal{E}_{\mathrm{IJ}}^{\mathrm{inner}}$. The *fundamental cycle matrix* is defined as $C = [C_1, \ldots, C_c]$, where C_k is the cycle matrix of $\mathcal{C}_k$. Similarly, every loose chord $e_k \in \mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}} \backslash \{e_{j_0}\}$ defines a unique *fundamental crossing path* $\mathcal{P}_k$ starting and ending (or vice versa) at e_k and e_{j_0}, respectively. The *fundamental crossing path matrix* is defined as $P_{\mathrm{IJ}} = [P_1, \ldots, P_{|\mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}}|-1}]$, where P_k denotes the path matrix of $\mathcal{P}_k$.

If $\mathcal{G}_{\mathrm{IJ}}$ is connected and $\mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}} = \varnothing$, then after choosing a *ground node* $v_{i_0} \in \mathcal{V}$ we set $\mathcal{V}_2 := \{v_{i_0}\}$ and denote the associated *reduced vertex set* by $\mathcal{V}_1 := \mathcal{V}\backslash\mathcal{V}_2$. If $\mathcal{E}_{\mathrm{IJ}}^{\mathrm{loose}} \neq \varnothing$, then $\mathcal{V}_2 = \varnothing$, and the reduced vertex set is given by $\mathcal{V}_1 = \mathcal{V}_{\mathrm{I}}$. For the incidence matrix, these vertices are selected by the permutation $\Gamma = [\Gamma_1, \Gamma_2]$ with $\Gamma_i = [e_k]_{v_k \in \mathcal{V}_i}$, $i = 1, 2$, where $e_1, \ldots, e_{|\mathcal{V}|} \in \mathbb{R}^{|\mathcal{V}_{\mathrm{I}}|}$ denotes the standard canonical basis.

For a subset $\mathcal{V}_s \subset \mathcal{V}_{\mathrm{I}}$, the *vertex identification* of $\mathcal{V}_s$ merges the elements of $\mathcal{V}_s$ into a new vertex $\bar{v} := \bigcup_{v_i \in \mathcal{V}_s} v_i$. Removing all edges connecting the vertices $v_i, v_k \in \mathcal{V}_s$, we obtain the *contraction* of $\mathcal{G}_{\mathrm{IJ}}$ with respect to $\mathcal{V}_s$, which is the graph $\bar{\mathcal{G}}_{\mathrm{IJ}} := \{\bar{\mathcal{V}}_{\mathrm{I}}, \bar{\mathcal{E}}_{\mathrm{J}}\}$ with $\bar{\mathcal{V}}_{\mathrm{I}} := (\mathcal{V}_{\mathrm{I}}\backslash\mathcal{V}_s) \bigcup \{\bar{v}\}$ and $\bar{\mathcal{E}}_{\mathrm{J}} := \mathcal{E}_{\mathrm{J}} \backslash \{e_j \,|\, e_j = (v_i, v_k) \,|\, v_i, v_k \in \mathcal{V}_s\}$. Note that $\bar{\mathcal{G}}$ might have multiple edges and self loops even if $\mathcal{G}$ is simple. The associated identification matrix is given by $\mathbf{1}^T\Pi$, where $\mathbf{1} = [1, \ldots, 1] \in \mathbb{R}^{|\mathcal{V}_s|}$ and $\Pi \in \mathbb{R}^{|\mathcal{V}_s| \times |\mathcal{V}_s|}$ is a permutation such that $[\mathbf{1}^T\Pi]_i = 1$ if and only if $v_i \in \mathcal{V}_s$.

In the following, we assume that $\mathcal{G}_{\text{IJ}}$ is numbered such that

$$\mathcal{E}_{\text{IJ}} = \mathcal{E}_{\text{IJ},1} \cup \ldots \cup \mathcal{E}_{\text{IJ},K} \cup \mathcal{E}_{\text{IJ}}^{isolated}, \qquad \mathcal{V}_{\text{IJ}} = \mathcal{V}_{\text{IJ},1} \cup \ldots \cup \mathcal{V}_{\text{IJ},K},$$

where $\mathcal{G}_{\text{IJ},k} = \{\mathcal{E}_{\text{IJ},k}, \mathcal{V}_{\text{IJ},k}\}$, $k = 1, \ldots, K$, are the connected components of $\mathcal{G}_{\text{IJ}}$. These are ordered such that $\mathcal{G}_{\text{IJ},k}$ corresponds to proper subgraphs for $k = 1, \ldots, \hat{k}_1$, to subsets with loose edges for $k = \hat{k}_1 + 1, \ldots, \hat{k}$, and to isolated vertices for $k = \hat{k} + 1, \ldots, K$. Accordingly, we denote by $\mathcal{G}_{\text{IJ},k}^{\text{outer}}$, $\mathcal{G}_{\text{IJ},k}^{\text{inner}}$ the corresponding subgraphs of $\mathcal{G}_{\text{IJ},k}$. Then, the connection matrix is given as

$$A_{\text{IJ}} = \begin{bmatrix} A_{\text{IJ},1} & & & \\ & \ddots & & \\ & & A_{\text{IJ},\hat{k}} & \\ & & & 0 \end{bmatrix} \qquad \text{with} \quad A_{\text{IJ},k} = \begin{bmatrix} A_{\text{IJ},k}^{\text{inner}} & A_{\text{IJ},k}^{\text{loose}} \end{bmatrix}, \tag{2.1}$$

where $A_{\text{IJ},k}^{\text{inner}}$ is the incidence matrix of $\mathcal{G}_{\text{IJ},k}^{\text{inner}}$ and $A_{\text{IJ},k}^{\text{loose}}$ denotes the connection matrix of $\{\mathcal{V}_{\text{IJ},k}, \mathcal{E}_{\text{IJ},k}^{\text{loose}}\}$. The rows and columns of the zero block correspond to the isolated vertices and edges, respectively. For each $\mathcal{G}_{\text{IJ},k}$, we number $\mathcal{G}_{\text{IJ},k}^{\text{outer}}$ such that $\mathcal{V}_{\text{IJ},k}^{\text{outer}} = \mathcal{V}_{\text{IJ},k} \cup \mathcal{V}_{\text{IJ},k}^{c}$ and the incidence matrix $A_{\text{IJ},k}^{\text{outer}}$ is given by $A_{\text{IJ},k}^{\text{outer}} = [A_{\text{IJ},k}^T, (A_{\text{IJ},k}^c)^T]^T$, where $A_{\text{IJ},k}^c$ describes the connection structure of $\{\mathcal{V}_{\text{IJ},k}^c, \mathcal{E}_{\text{IJ},k}\}$.

Considering the substructures introduced above, we define the *reduced vertex set* $\mathcal{V}_{\text{IJ},1} := \bigcup_{k=1}^{K} \mathcal{V}_{\text{IJ},k,1}$ with the *ground node* $\mathcal{V}_{\text{IJ},2} := \bigcup_{k=1}^{K} \mathcal{V}_{\text{IJ},k,2}$ and the selection matrices

$$\Gamma_{\text{IJ}} = [\Gamma_{\text{IJ},1}, \Gamma_{\text{IJ},2}]\,, \tag{2.2a}$$

the *spanning tree* $\mathcal{T}_{\text{IJ},1} := \bigcup_{k=1}^{K} \mathcal{T}_{\text{IJ},k,1}$ with the *chord set* $\mathcal{T}_{\text{IJ},2} := \bigcup_{k=1}^{K} \mathcal{T}_{\text{IJ},k,2}$ and the selection matrices

$$\Pi_{\text{IJ}} = [\Pi_{\text{IJ},1}, \Pi_{\text{IJ},2}]\,, \tag{2.2b}$$

the *fundamental cycles* $\mathcal{C}_{\text{IJ}} := \bigcup_{k=1}^{K} \mathcal{C}_{\text{IJ},k}$ and the crossing paths $\mathcal{P}_{\text{IJ}} := \bigcup_{k=1}^{K} \mathcal{P}_{\text{IJ},k}$ and the selection matrices

$$V_2 = [C_{\text{IJ}}, P_{\text{IJ}}, L_{\text{IJ}}]\,, \tag{2.2c}$$

where we denote by $\mathcal{V}_{\text{IJ},k,i}$, $\mathcal{T}_{\text{IJ},k,i}$, for $i = 1, 2$, $\mathcal{C}_{\text{IJ},k} = \{\mathcal{C}_{\text{IJ},k,1}, \ldots, \mathcal{C}_{\text{IJ},k,|\mathcal{C}_{\text{IJ},k}|}\}$ and $\mathcal{P}_{\text{IJ},k} = \{\mathcal{P}_{\text{IJ},k,1}, \ldots, \mathcal{P}_{\text{IJ},k,|\mathcal{P}_{\text{IJ},k}|}\}$ the respective structure in each component $\mathcal{G}_{\text{IJ},k}$ with the selection matrices given by $\Gamma_{\text{IJ},k,i}$, $\Pi_{\text{IJ},k,i}$, $C_{\text{IJ},k}$, $P_{\text{IJ},k}$ such that $\Gamma_{\text{IJ},i} = \operatorname{diag}(\Gamma_{\text{IJ},k,i})_k$, $\Pi_{\text{IJ},i} = \operatorname{diag}(\Pi_{\text{IJ},k,i})_k$, $P_{\text{IJ}} = \operatorname{diag}(P_{\text{IJ},k})_k$, $C_{\text{IJ}} = \operatorname{diag}(C_{\text{IJ},k})_k$, for $i = 1, 2$, $k = 1, \ldots, K$, and $L_{\text{IJ}} = [0, I_{|\mathcal{E}_{\text{IJ},k}^{\text{loose}}|}]^T$. For the connected components that are proper subgraphs, we further consider the *identification* $\bar{\mathcal{V}}_{\text{IJ}} = \bigcup_{k=1}^{\hat{k}} \bar{v}_{i_k}$ with $\bar{v}_{i_k} := v_{i_{k,0}} \cup \left(\bigcup_{v_i \in \mathcal{V}_{\text{I},k}} v_i\right)$ and the identification matrix

$$\mathbf{1}_{\text{IJ}} = \operatorname{diag}(\mathbf{1}_{|V_{\text{IJ},k}|})_{k=1,\ldots,\hat{k}}\,. \tag{2.2d}$$

As for a proper graph and its incidence matrix, cf., e.g., [8, pp. 23], we can interpret the fundamental subspaces of a submatrix A_{IJ} as substructures of the set $\mathcal{G}_{\text{IJ}}$.

LEMMA 2.1. *Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a simple, oriented graph with $\mathcal{V} = \bigcup_{I \in I_\mathcal{V}} \mathcal{V}_I$, $\mathcal{E} = \bigcup_{J \in J_\mathcal{E}} \mathcal{E}_J$. Consider a subset $\mathcal{G}_{IJ} := \{\mathcal{V}_I, \mathcal{E}_J\}$ with connection matrix A_{IJ}. Then, $\operatorname{rank}(A_{IJ}) = \sum_{k=1}^{\hat{k}} |\mathcal{V}_{IJ,k}| - \hat{k}$, where $\mathcal{G}_{IJ,k}$, $k = 1, \ldots, \hat{k}$, denotes the connected components of $\mathcal{G}_{IJ}$ that itself are subgraphs. For the matrices defined in (2.2), it holds that $\ker(A_{IJ}) = \operatorname{span}(V_2)$, $\operatorname{corange}(A_{IJ}) = \operatorname{span}(\Pi_{IJ,1})$, and $\operatorname{coker}(A_{IJ}) = \operatorname{span}(\mathbf{1}_{IJ})$, $\operatorname{range}(A_{IJ}) = \operatorname{span}(\Gamma_{IJ,1})$.*

The matrices $U := [\Gamma_{IJ,1}, \mathbf{1}_{IJ}]$ and $V := [\Pi_{IJ,1}, V_{IJ,2}]$ are nonsingular with

$$U^{-1} = \begin{bmatrix} U_2^- \\ \Gamma_{IJ,2}^T \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} V_2^- & 0 \\ \Pi_{IJ,2}^T & 0 \\ 0 & I_{|\mathcal{E}_{IJ,k}^{loose}|} \end{bmatrix},$$

where $U_2^- = \Gamma_{IJ,1}^T - \mathbf{1}_{IJ}\Gamma_{IJ,2}^T$ and $V_2^- = \Pi_{IJ,1}^T - \Pi_{IJ,1}^T[C_{IJ}, P_{IJ}]\Pi_{IJ,2}^T$.

Proof. From (2.1), we get that $\operatorname{rank}(A_{\mathrm{IJ}}) = \sum_{k=1}^{\hat{k}} \operatorname{rank}(A_{\mathrm{IJ},k})$. Noting that $\mathcal{G}_{\mathrm{IJ},k}^{\mathrm{outer}}$ is connected as $\mathcal{G}_{\mathrm{IJ},k}$ is connected and using the fact that $\mathcal{V}_{\mathrm{IJ}}^{\mathrm{outer}} = \mathcal{V}_{\mathrm{I}} \cup \mathcal{V}_{\mathrm{IJ}}^{\mathrm{c}}$, we have that $\operatorname{rank}(A_{\mathrm{IJ},k}^{\mathrm{outer}}) = |\mathcal{V}_{\mathrm{IJ},k}| + |\mathcal{V}_{\mathrm{IJ},k}^{\mathrm{c}}| - 1$, cf. [3, p. 23]. For $k = 1, \ldots, \hat{k}_1$, we have that $\mathcal{V}_{\mathrm{IJ},k}^{\mathrm{c}} = \varnothing$ implying that $\operatorname{rank}(A_{\mathrm{IJ},k}^{\mathrm{outer}}) = \operatorname{rank}(A_{\mathrm{IJ},k})$ with $\operatorname{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}| - 1$. For the indices $k = \hat{k}_1 + 1, \ldots, \hat{k}$, we have that $\mathcal{V}_{\mathrm{IJ},k}^{\mathrm{c}} \neq \varnothing$, implying that $\operatorname{rank}(A_{\mathrm{IJ},k}^{\mathrm{outer}}) > |\mathcal{V}_{\mathrm{IJ},k}|$. Thus, we can choose $|\mathcal{V}_{\mathrm{IJ},k}|$ linearly independent rows from $A_{\mathrm{IJ},k}^{\mathrm{outer}}$, and by selecting the block row associated with $A_{\mathrm{IJ},k}$, we get $\operatorname{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}|$. In conclusion, we have proven that

$$\operatorname{rank}(A_{\mathrm{IJ}}) = \sum_{k=1}^{\hat{k}_1} (|\mathcal{V}_{\mathrm{IJ},k}| - 1) + \sum_{k=\hat{k}_1+1}^{\hat{k}} |\mathcal{V}_{\mathrm{IJ},k}|,$$

i.e., $\operatorname{rank}(A_{\mathrm{IJ}}) = \sum_{k=1}^{\hat{k}} |\mathcal{V}_{\mathrm{IJ},k}| - \hat{k}_1$.

Now, we consider a connected component $\mathcal{G}_{\mathrm{IJ},k}$. With the given numbering, the fundamental cycle and crossing path matrices are given by

$$C_{\mathrm{IJ},k} = \begin{bmatrix} C_{\mathrm{IJ}}^{\mathrm{inner}} \\ 0 \\ 0 \end{bmatrix}, \quad P_{\mathrm{IJ},k} = \begin{bmatrix} * \\ \mathbf{1}_{|\mathcal{E}_{\mathrm{IJ},k}|-1} \\ I_{|\mathcal{E}_{\mathrm{IJ},k}|-1} \end{bmatrix}, \tag{2.3}$$

where $C_{\mathrm{IJ},k}^{\mathrm{inner}}$ denotes the fundamental cycle matrix of $\mathcal{G}_{\mathrm{IJ},k}^{\mathrm{inner}}$. From (2.1) and (2.3) it follows that

$$A_{\mathrm{IJ},k} C_{\mathrm{IJ},k} = A_{\mathrm{IJ},k}^{\mathrm{inner}} C_{\mathrm{IJ},k}^{\mathrm{inner}} = 0,$$

as the fundamental cycles of $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ lie in $\ker(A_{\mathrm{IJ}})$. Similarly, we get from (2.1) and (2.3) that

$$A_{\mathrm{IJ},k}^{\mathrm{outer}} P_{\mathrm{IJ},k} = \begin{bmatrix} 0 \\ A_{\mathrm{IJ},k}^{\mathrm{c}} P_{\mathrm{IJ},k} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{1}_{|\mathcal{E}_{\mathrm{IJ},k}|-1} \\ I_{|\mathcal{E}_{\mathrm{IJ},k}|-1} \end{bmatrix}$$

as the incidence matrix applied to a path matrix returns exactly the starting and end vertices of the path [6, p. 157]. Thus, $A_{\mathrm{IJ},k} P_{\mathrm{IJ},k} = 0$, implying that $\operatorname{span}([C_{\mathrm{IJ},k}, P_{\mathrm{IJ},k}]) \subset \ker(A_{\mathrm{IJ},k})$. Considering (2.3), we find that $\operatorname{rank}([C_{\mathrm{IJ},k}, P_{\mathrm{IJ},k}]) = \operatorname{rank}(C_{\mathrm{IJ},k}) + \operatorname{rank}(P_{\mathrm{IJ},k})$ with $\operatorname{rank}(C_{\mathrm{IJ},k}) = |\mathcal{E}_{\mathrm{IJ},k}^{\mathrm{inner}}| - |\mathcal{V}_{\mathrm{IJ},k}| + 1$ and $\operatorname{rank}(P_{\mathrm{IJ},k}) = |\mathcal{E}_{\mathrm{IJ},k}^{\mathrm{loose}}| - 1$. For $k = 1, \ldots, \hat{k}_1$, we thus get that

$$\operatorname{rank}([C_{\mathrm{IJ},k}, P_{\mathrm{IJ},k}]) = |\mathcal{E}_{\mathrm{IJ},k}| - |\mathcal{V}_{\mathrm{IJ},k}| + 1 = \dim(\ker(A_{\mathrm{IJ},k})),$$

while for $k = \hat{k}_1 + 1, \ldots, \hat{k}$, we get that

$$\operatorname{rank}([C_{\mathrm{IJ},k}, P_{\mathrm{IJ},k}]) = |\mathcal{E}^{\text{inner}}_{\mathrm{IJ},k}| - |\mathcal{V}_{\mathrm{IJ},k}| + 1 + |\mathcal{E}^{\text{loose}}_{\mathrm{IJ},k}| - 1 = |\mathcal{E}_{\mathrm{IJ},k}| - |\mathcal{V}_{\mathrm{IJ},k}| = \dim(\ker(A_{\mathrm{IJ},k})).$$

Hence, $\operatorname{span}([C_{\mathrm{IJ},k}, P_{\mathrm{IJ},k}]) = \ker(A_{\mathrm{IJ},k})$, and it follows that $\operatorname{span}(V_2) = \ker(A_{\mathrm{IJ}})$.

For the left nullspace, we note that $\mathcal{G}_{\mathrm{IJ},k}$ is a proper subgraph for $k = 1, \ldots, \hat{k}$, implying that every column of $A_{\mathrm{IJ},k}$ contains exactly the two nonzero entries $1, -1$. Hence, $\mathbf{1}^T_{|\mathcal{V}_{\mathrm{IJ},k}|} A_{\mathrm{IJ},k} = 0$. As $\operatorname{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}| - 1$, it follows that $\operatorname{span}(\mathbf{1}_{|\mathcal{V}_{\mathrm{IJ},k}|}) = \operatorname{coker}(A_{\mathrm{IJ},k})$, and thus $\operatorname{span}(\mathbf{1}_{|\mathcal{V}_{\mathrm{IJ}}|}) = \operatorname{coker}(A_{\mathrm{IJ}})$.

For $\operatorname{corange}(A_{\mathrm{IJ}})$, we note that $\operatorname{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}| - 1$, for $k = 1, \ldots, \hat{k}_1$, such that we can select $|\mathcal{V}_{\mathrm{IJ},k}| - 1$ linearly independent columns in $A_{\mathrm{IJ},k}$, i.e., there exists a permutation $\Pi_{\mathrm{IJ},1}, \Pi_{\mathrm{IJ},2} \in \mathbb{R}^{|\mathcal{E}_{\mathrm{IJ},k}| \times |\mathcal{E}_{\mathrm{IJ},k}|}$ such that $A_{\mathrm{IJ},k}\Pi_{\mathrm{IJ},k,1}$ has full rank. For $k = \hat{k}_1 + 1, \ldots, \hat{k}$, $\operatorname{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}|$, and there exists a permutation $\Pi_{\mathrm{IJ},1}, \Pi_{\mathrm{IJ},2} \in \mathbb{R}^{|\mathcal{E}_{\mathrm{IJ},k}| \times |\mathcal{E}_{\mathrm{IJ},k}|}$ such that $A_{\mathrm{IJ},k}\Pi_{\mathrm{IJ},k,1}$ has full rank, where $\Pi_{\mathrm{IJ},k,1}$ selects $|\mathcal{V}_{\mathrm{IJ},k}| - 1$ linearly independent columns associated with edges on a spanning tree of $\mathcal{G}^{\text{inner}}_{\mathrm{IJ},k}$ as well as the reference loose edge $e_{k_0} \in \mathcal{E}_{\mathrm{IJ},k}$.

Similarly, for $k = 1, \ldots, \hat{k}_1$, we can select $|\mathcal{V}_{\mathrm{IJ},k}| - 1$ linearly independent rows in $A_{\mathrm{IJ},k}$, i.e., there exists a permutation $\Gamma_{\mathrm{IJ},k,1}, \Gamma_{\mathrm{IJ},k,2}$ such that $\Gamma^T_{\mathrm{IJ},1} A_{\mathrm{IJ}}$ has full rank. For $k = \hat{k}_1 + 1, \ldots, \hat{k}$, $\operatorname{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}|$, implying that $\Gamma_{\mathrm{IJ},k,1} = I_{|\mathcal{V}_{\mathrm{IJ},k}|}$.

If $\ker(A_{\mathrm{IJ}}) = \operatorname{span}(V_2)$, $\operatorname{corange}(A_{\mathrm{IJ}}) = \operatorname{span}(\Pi_{\mathrm{IJ},1})$ and $\operatorname{coker}(A_{\mathrm{IJ}}) = \operatorname{span}(\mathbf{1}_{\mathrm{IJ}})$, $\operatorname{range}(A_{\mathrm{IJ}}) = \operatorname{span}(\Gamma_{\mathrm{IJ},1})$, then the matrices $U := [\Gamma_{\mathrm{IJ},1}, \mathbf{1}_{\mathrm{IJ}}]$ and $V := [\Pi_{\mathrm{IJ},1}, V_2]$ are nonsingular. To verify the representation of U^{-1}, V^{-1}, we verify the properties of the inverse. For convenience, we drop the index IJ of the subset $\mathcal{G}_{\mathrm{IJ}}$. First, we note that

$$\begin{bmatrix}\Gamma_1 & \mathbf{1}\end{bmatrix}\begin{bmatrix}\Gamma_1^T - \mathbf{1}\Gamma_2^T \\ \Gamma_2^T\end{bmatrix} = \Gamma_1\Gamma_1^T + (\mathbf{1} - \Gamma_1\mathbf{1}_{\mathrm{IJ}})\Gamma_2^T = \Gamma_1\Gamma_1^T + \Gamma_2\mathbf{1}\Gamma_2^T.$$

Noting that $[\Gamma_1\mathbf{1}]_i = 1$, $v_i \in \mathcal{V}_{\mathrm{IJ},1}$ and $[\Gamma_1\mathbf{1}]_i = 0$, $v_i \in \mathcal{V}_{\mathrm{IJ},2}$, we have that $\mathbf{1} - \Gamma_1\mathbf{1} = \Gamma_2$ such that

$$\begin{bmatrix}\Gamma_1 & \mathbf{1}\end{bmatrix}\begin{bmatrix}\Gamma_1^T - \mathbf{1}\Gamma_2^T \\ \Gamma_2^T\end{bmatrix} = \Gamma_1\Gamma_1^T + \Gamma_2\Gamma_2^T = I_{|V_{\mathrm{IJ}}|}.$$

On the other hand, we have that

$$\begin{bmatrix}\Gamma_1^T - \mathbf{1}\Gamma_2^T \\ \Gamma_2^T\end{bmatrix}\begin{bmatrix}\Gamma_1 & \mathbf{1}\end{bmatrix} = \begin{bmatrix} I_{|\mathcal{V}_{red}|-1} & (\Gamma_1^T - \mathbf{1}\Gamma_2^T)\mathbf{1} \\ 0 & \Gamma_2^T\mathbf{1}\end{bmatrix}.$$

In $\Gamma_1^T - \mathbf{1}\Gamma_2^T$, every row contains exactly the two non-zero entries $1, -1$ such that it holds that $(\Gamma_1^T - \mathbf{1}\Gamma_2^T)\mathbf{1} = 0$. With $\Gamma_2^T\mathbf{1}_{\mathrm{IJ}} = I_{|\mathcal{V}_{\mathrm{IJ},2}|}$, we thus get $[\Gamma_1, \mathbf{1}]^{-1}[\Gamma_1, \mathbf{1}] = I_{|\mathcal{V}_{\mathrm{IJ}}|}$.

As $\mathcal{E}(\mathcal{T}) \cap \mathcal{E}^{\text{loose}}_{\mathrm{IJ}} = \varnothing$, where $\mathcal{E}(\mathcal{T})$ denotes the edges of a spanning tree $\mathcal{T}$, we have that

$$\begin{bmatrix}\Pi_1 & V_2\end{bmatrix} = \begin{bmatrix}\tilde{\Pi}_1 & C & P & 0 \\ 0 & 0 & 0 & I_{|\mathcal{E}^{\text{loose}}_{\mathrm{IJ},k}|}\end{bmatrix}.$$

Thus, it suffices to show that

$$\begin{bmatrix}\tilde{\Pi}_1 & [C, P]\end{bmatrix}^{-1} = \begin{bmatrix}\Pi_1^T - \Pi_1^T[C, P]\Pi_2^T \\ \Pi_2^T\end{bmatrix}.$$

We show the assertion by verifying the properties of the inverse. First, using the identity $I_{|\mathcal{V}_{\mathrm{IJ}}|} - \Pi_1\Pi_1^T = \Pi_2\Pi_2^T$, we get that

$$\begin{aligned}\begin{bmatrix}\Pi_1 & [C, P]\end{bmatrix}\begin{bmatrix}\Pi_1^T - \Pi_1^T[C, P]\Pi_2^T \\ \Pi_2^T\end{bmatrix} &= \Pi_1\Pi_1^T - \Pi_1\Pi_1^T[C, P]\Pi_2^T + [C, P]\Pi_2^T \\ &= \Pi_1\Pi_1^T + \Pi_2\Pi_2^T[C, P]\Pi_2^T.\end{aligned}$$

The matrix $\Pi_2\Pi_2^T$ is a projection onto the edges of the chord set $\mathcal{T}_{\mathrm{IJ},2}$. As the fundamental cycles and crossing paths $\mathcal{C}_{k_l}, \mathcal{P}_{k_m}$ contain exactly one edge $e_{k_l}, e_{k_m} \in \mathcal{T}_{\mathrm{IJ},2}$, respectively, we have that $\Pi_2\Pi_2^T[C,P] = \Pi_2$. Hence,

$$\begin{bmatrix}\Pi_1 & C & P\end{bmatrix}\begin{bmatrix}\Pi_1^T - \Pi_1^T[C,P]\Pi_2^T \\ \Pi_2^T\end{bmatrix} = \Pi_1\Pi_1^T + \Pi_2\Pi_2^T = I_{|\mathcal{E}_{\mathrm{IJ}}|}.$$

On the other hand, we have that

$$\begin{bmatrix}\Pi_1^T - \Pi_1^T[C,P]\Pi_2^T \\ \Pi_2^T\end{bmatrix}\begin{bmatrix}\Pi_1 & [C,P]\end{bmatrix} = \begin{bmatrix}I_{|\mathcal{E}(\mathcal{T}_{\mathrm{IJ}})|} & \Pi_1^T[C,P](I_{n_{\mathrm{J}}-n_{\mathrm{I}}+1} - \Pi_2^T[C,P]) \\ 0 & \Pi_2^T[C,P]\end{bmatrix}.$$

Again, as every fundamental cycle and crossing path contains exactly one chord, we have that $\Pi_2^T[C,P] = I_{n_{\mathrm{J}}-n_{\mathrm{I}}+1}$. Then, it follows that $[\Pi_1,[C,P]]^{-1}[\Pi_1,[C,P]] = I_{|\mathcal{E}_{\mathrm{IJ}}|}$. ∎

Hence, the fundamental cycles, crossing paths, and loose edges span the right nullspace $\ker(A_{\mathrm{IJ}})$, while the edges in the spanning tree build a basis of $\mathrm{corange}(A_{\mathrm{IJ}})$. The identification of connected components with a ground node spans the left nullspace $\mathrm{coker}(A_{\mathrm{IJ}})$, while the vertices of the reduced vertex set correspond to a basis of $\mathrm{corange}(A_{\mathrm{IJ}})$.

Now, we equip the vertices and edges of $\mathcal{G}$ with potentials and flows, respectively. To each vertex $v_i \in \mathcal{V}$, we assign a potential p_i, and all potentials are collected into a vector $p = [p_i]_{i=1,\ldots,|\mathcal{V}|}$. Similarly, to each edge $e_j \in \mathcal{G}$, we assign a *flow* q_j and set $q = [q_j]_{j=1,\ldots,|\mathcal{E}|}$. The flow is directed: a flow q_j is called *positive*, i.e., $q_j > 0$, if q_j agrees with the direction of its associated edge e_j. If $q_j \neq 0$ is opposed to the direction of edge e_j, then q_J is called *negative*, i.e., $q_j < 0$. If $\mathcal{G} = \{\mathcal{V},\mathcal{E}\}$ with $\mathcal{V} = \bigcup_{I\in I_{\mathcal{V}}} \mathcal{V}_{\mathrm{I}}$, $\mathcal{E} = \bigcup_{J\in I_{\mathcal{E}}} \mathcal{E}_{\mathrm{J}}$, we partition the flows and potential accordingly and write $q_{\mathrm{J},j} \in \mathcal{E}_{\mathrm{J}}$ and $p_{\mathrm{I},i} \in \mathcal{V}_{\mathrm{I}}$.

The flow and potential satisfy the following fundamental relations that generalize Kirchhoff's circuit laws, i.e.,

$$\Gamma_1^T A q = 0, \tag{2.4a}$$

$$V_2^T A^T p = 0. \tag{2.4b}$$

The equations (2.4) allow us to give a physical interpretation of Lemma 2.1. The fundamental cycles, crossing paths, and loose edges correspond to structures on which the potential difference vanishes, while the spanning tree selects a structure on which the potential difference is well defined. The reduced vertex set consists of those vertices on which the potential is fixed in relation to the reference value given by the ground node. Thus, the identification of the reduced vertex set with its ground node subsumes all vertices on which the potential is not fixed, like in an isolated vertex or a subgraph without connection to a ground node.

We transform the flow and potential with respect to these substructures by setting

$$\tilde{q} := [\Pi_{\mathrm{IJ},1}, V_{\mathrm{IJ},2}]^{-1}\, q, \qquad \tilde{p} := [\Gamma_{\mathrm{IJ},1}, \mathbf{1}_{\mathrm{IJ}}]^{-1}\, p \tag{2.5}$$

such that

$$\begin{aligned}\tilde{q}_1 &= (\Pi_{\mathrm{IJ},1}^T - \Pi_{\mathrm{IJ},1}^T[C_{\mathrm{IJ}}, P_{\mathrm{IJ}}]\Pi_{\mathrm{IJ},2}^T)q, & \qquad \tilde{p}_1 &= (\Gamma_{\mathrm{IJ},1}^T - \mathbf{1}_{\mathrm{IJ}}\Gamma_{\mathrm{IJ},2}^T)p,\\ \tilde{q}_2 &= \Pi_{\mathrm{IJ},2}^T q, & \tilde{p}_1 &= \Gamma_{\mathrm{IJ},2}^T p.\end{aligned}$$

The flows q_2 belong to edges on the *chord set* $\mathcal{T}_{\mathrm{IJ},2}$ while q_1 denote the difference between a branch flow $q_{1,j} \in \mathcal{T}_{\mathrm{IJ},1}$ and the chord flows $q_{2,l} \in \mathcal{T}_{\mathrm{IJ},2}$ of those fundamental cycles and crossing paths containing $q_{1,j}$. Similarly, the potentials p_2 belong to the ground nodes $\mathcal{V}_{\mathrm{IJ},2}$, while p_1 denote the difference between a potential $p_{1,j} \in \mathcal{V}_{\mathrm{IJ},1}$ and its associated ground node $p_{2,j} \in \mathcal{V}_{\mathrm{IJ},2}$.

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Now, we think of the flow as information running through the network. To describe the structure of the subset $\mathcal{G}_{\mathrm{IJ}}$ on this informational level, for $v_i \in \mathcal{V}_{\mathrm{I}}$, we partition the set of incident edges $\mathcal{E}_{inc}(v_i)$ into those along which v_i *receives* and *sends* information, respectively, i.e., we set

$$\mathcal{E}_{inc,s}(v_i) := \{e_j \in \mathcal{E}_{inc}(v_i) \mid A_{ij}\,\mathrm{sgn}(q_j(t)) > 0,\ \text{i.e., } q_j \text{ starts in } v_i\},$$
$$\mathcal{E}_{inc,e}(v_i) := \{e_j \in \mathcal{E}_{inc}(v_i) \mid A_{ij}\,\mathrm{sgn}(q_j(t)) < 0,\ \text{i.e., } q_j \text{ ends in } v_i\}.$$

Defining the *flow matrix*

$$B_{\mathrm{IJ},i\ell} = \begin{cases} \sum_{e_j \in \mathcal{E}_{inc,s}(v_i)\cap\mathcal{E}_{\mathrm{J}}} |q_j|, & i = \ell, \\ -|q_j|, & e_j \in \mathcal{E}_{inc,e}(v_i) \cap \mathcal{E}_{inc}(v_\ell) \cap \mathcal{E}_{\mathrm{J}},\ i \neq \ell, \\ 0, & \text{else}, \end{cases}$$

the information flow in $\mathcal{G}_{\mathrm{IJ}}$ is graphically represented by the *flow graph* $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}} := \mathcal{G}(B_{\mathrm{IJ}}^T)$. The graph $\mathcal{G}(A)$ of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as $\mathcal{G}(A) = \{\{v_1, \ldots, v_n\}, \{(v_i, v_j)|\ a_{ij} \neq 0\}\}$, i.e., whenever the ij-th entry is nonzero, there is an edge from the vertex v_i to v_j [20, p. 528]. Hence,

$$\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}} = \{\mathcal{V}_{\mathrm{I}}, \mathcal{E}_{\mathrm{IJ}}^{\mathrm{flow}}\} \text{ with}$$
$$\mathcal{E}_{\mathrm{IJ}}^{\mathrm{flow}} := \{e_j := (v_\ell, v_i) \mid \mathcal{E}_{inc,e}(v_i) \cap \mathcal{E}_{inc}(v_\ell) \neq \varnothing, v_\ell \neq v_i \ \vee\ \mathcal{E}_{inc,s}(v_i) \neq \varnothing, v_\ell = v_i\}\,.$$

Basically, the graph $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}}$ has the same connection structure as the set $\mathcal{G}_{\mathrm{IJ}}$ except that at vertices $v_i \in \mathcal{V}_{\mathrm{I}}$ sending a non-zero mass flow into $\mathcal{G}_{\mathrm{IJ}}$, the flow graph $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}}$ has self loops and that edges $e_j \in \mathcal{E}_{\mathrm{J}}$ equipped with a zero mass flow $q_j = 0$ are absent in $\mathcal{E}_{\mathrm{IJ}}^{\mathrm{flow}}$. The orientation of $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}}$ is determined by the direction of the mass flows, i.e., $e_j \in \mathcal{E}_{\mathrm{IJ}}^{\mathrm{flow}}$ is directed from v_ℓ to v_i if v_i *receives* a mass flow from v_ℓ, cf. Figure 2.1.

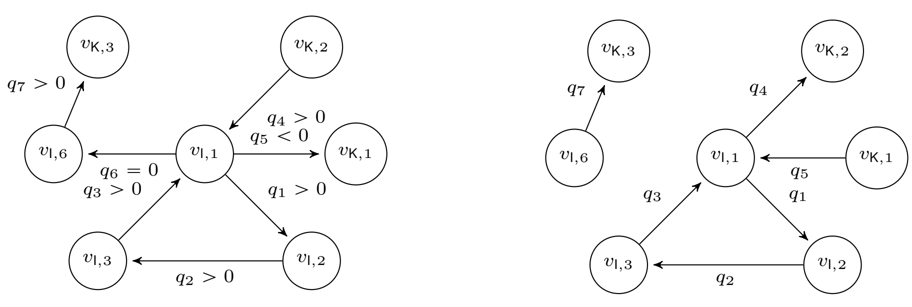

FIG. 2.1. *A graph (left) and its flow graph (right).*

The connectivity of $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}}$ on this informational level is described by the concept of *strong connectivity*. We assume that $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}}$ is composed of *strongly connected components* $\mathcal{G}_{flow,\mathrm{IJ},k} := \{\mathcal{E}_{flow,\mathrm{IJ},k}, \mathcal{V}_{flow,\mathrm{IJ},k}\}$, i.e., every pair of vertices $v_i, v_k \in \mathcal{V}_{flow,\mathrm{IJ},k}$ is connected by a directed path from v_i to v_k and a directed path from v_k to v_i, cf. [20, p. 528]. For each $\mathcal{G}_{flow,\mathrm{IJ},k}$, with $k = 1, \ldots, K$, we denote the interior subgraph by $\mathcal{G}_{flow,\mathrm{IJ},k}^{\mathrm{inner}} = \{\mathcal{E}_{flow,\mathrm{IJ},k}^{\mathrm{inner}}, \mathcal{V}_{flow,\mathrm{IJ},k}^{\mathrm{inner}}\}$.

The flow matrix B_{IJ} is nonsingular, if in every strongly connected component $\mathcal{G}_{\mathrm{IJ},k}$ of $\mathcal{G}_{\mathrm{IJ}}$ there exists at least one vertex $v_{\hat{i}}$ sending a nonzero flow into $\mathcal{G}_{\mathrm{IJ}} \backslash \mathcal{G}_{\mathrm{IJ},k}$.

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LEMMA 2.2. *Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a simple, oriented graph with $\mathcal{V} = \bigcup_{I \in I_\mathcal{V}} \mathcal{V}_I$ and $\mathcal{E} = \bigcup_{J \in J_\mathcal{E}} \mathcal{E}_J$. Consider a subset $\mathcal{G}_{IJ} := \{\mathcal{V}_I, \mathcal{E}_J\}$ with connection matrix A_{IJ}, and let $\mathcal{G}_{flow,IJ,k} := \{\mathcal{E}_{flow,IJ,k}, \mathcal{V}_{flow,IJ,k}\}$, $k = 1, \ldots, K$, denote the strongly connected components of $\mathcal{G}_{IJ}^{flow}$ with interior subgraphs $\mathcal{G}_{flow,IJ,k}^{inner} = \{\mathcal{E}_{flow,IJ,k}^{inner}, \mathcal{V}_{flow,IJ,k}^{inner}\}$. Then,*

$$B_{IJ} = \frac{1}{2} A_{IJ} \operatorname{diag}(q_{J,j})_j \big(\operatorname{diag}(\operatorname{sgn}(q_{J,j}))_j A_{IJ}^T + |A_{IJ}^T| \big).$$

If $\sum_{e_j \in \mathcal{E}_{inc,s}(v_i) \cap \mathcal{E}_J} |q_j| > 0$ for $v_i \in \mathcal{V}_I$ and

$$\sum_{e_j \in \mathcal{E}_{inc,s}(v_i) \cap (\mathcal{E}_J \backslash \mathcal{E}_{IJ,flow,k}^{inner})} |q_j| \geqslant 0, \tag{2.6}$$

for $k = 1, \ldots, K$, and there exists $v_{\hat{i}} \in \mathcal{V}_{flow,IJ,k}$ such that (2.6) is strictly satisfied, then B_{IJ} is nonsingular.

Proof. To prove the representation of B_{IJ}, we set

$$\tilde{A}_{\mathrm{IJ}} := A_{\mathrm{IJ}} \operatorname{diag}(q_{J,j})_j \big(\operatorname{diag}(\operatorname{sgn}(q_{J,j}))_j A_{\mathrm{IJ}}^T + |A_{\mathrm{IJ}}^T| \big)$$

and note that

$$B_{\mathrm{IJ},i\ell} = \sum_{e_j \in \mathcal{E}_J} |q_j| A_{\mathrm{IJ},ij} \big(A_{\mathrm{IJ},\ell j} + |A_{\mathrm{IJ},\ell j}| \operatorname{sgn}(q_j) \big).$$

For $v_i, v_\ell \in I_\mathcal{V}$ and $j \in J_\mathcal{E}$, the entries of the incidence matrix A satisfy

$$A_{ij} A_{\ell j} = \begin{cases} 1, & e_j \in \mathcal{E}_{inc}(v_i),\ i = \ell, \\ -1, & e_j \in \mathcal{E}_{inc}(v_i) \cap \mathcal{E}_{inc}(v_\ell),\ i \neq \ell, \\ 0, & \text{else}, \end{cases}$$

$$A_{ij} |A_{\ell j}| = \begin{cases} A_{ij}, & e_j \in \mathcal{E}_{inc}(v_i) \cap \mathcal{E}_{inc}(v_\ell), \\ 0, & \text{else}, \end{cases}$$

and together with the definition of $\mathcal{E}_{inc,s}(v_i), \mathcal{E}_{inc,e}(v_i)$, we verify that $2B_{\mathrm{IJ}} = \tilde{A}_{\mathrm{IJ}}$.

If $\mathcal{G}_{flow,\mathrm{IJ}}$ is composed of K strongly connected components $\mathcal{G}_{flow,\mathrm{IJ},k} := \{\mathcal{V}_{flow,\mathrm{IJ},k},\ \mathcal{E}_{flow,\mathrm{IJ},k}\}$, then it follows that B_{IJ} is congruent to a block upper triangular matrix with irreducible diagonal blocks $B_{\mathrm{IJ},kk}$, $k = 1, \ldots, K$, i.e., there exists a permutation Π such that $\Pi^T B_{\mathrm{IJ}} \Pi = [B_{\mathrm{IJ},kl}]_{kl}$ with $B_{\mathrm{IJ},kl} = 0$, $k > l$, and $B_{\mathrm{IJ},kk}$ irreducible, cf. [20]. We show that under the given conditions, each $B_{\mathrm{IJ},kk}$ is irreducibly diagonally dominant and hence nonsingular for $k = 1, \ldots, K$, cf., e.g., [5, p. 67], [11, p. 403]. Thus, B_{IJ} is nonsingular.

The i-th column of $B_{\mathrm{IJ},kk}$ is given by

$$[B_{\mathrm{IJ},kk} e_i]_\ell = \begin{cases} 2 \sum_{e_j \in \mathcal{E}_{inc,s}(v_i) \cap \mathcal{E}_\mathrm{J}} |q_j|, & i = l, \\ -2|q_j|, & e_j \in \mathcal{E}_{inc,s}(v_i) \cap \mathcal{E}_{inc}(v_\ell) \cap \mathcal{E}_\mathrm{J},\ i \neq \ell, \\ 0, & \text{else}, \end{cases}$$

for $\ell = 1, \ldots, |\mathcal{V}_{flow,\mathrm{IJ},k}|$, and noting that

$$\bigcup_{v_\ell \in V_{\mathrm{IJ},flow,k} \backslash \{v_i\}} \Big(\mathcal{E}_{inc,s}(v_i) \cap \mathcal{E}_{inc}(v_\ell) \cap \mathcal{E}_\mathrm{J} \Big) = \mathcal{E}_{inc,s}(v_i) \cap \mathcal{E}_{\mathrm{IJ},flow,k}^{\mathrm{inner}},$$

for $v_i \in \mathcal{V}_{\mathrm{IJ},flow,k}$, the i-th column sum of $B_{\mathrm{IJ},kk}$ is given by

$$\sum_{v_\ell \in \mathcal{V}_{\mathrm{IJ},flow,k}} |B_{\mathrm{IJ},kk,i\ell}| = \sum_{e_j \in \mathcal{E}_{inc,s}(v_i) \cap (\mathcal{E}_\mathrm{J} \setminus \mathcal{E}^{\mathrm{inner}}_{\mathrm{IJ},flow,k})} |q_j|.$$

Hence, if condition (2.6) is satisfied for $v_i \in \mathcal{V}_{flow,\mathrm{IJ},k}$ and $k = 1, \ldots, K$, then $B_{\mathrm{IJ},kk}$ is diagonally dominant for $k = 1, \ldots, K$. For $k = 1, \ldots, K$, if there exists $v_{\hat{i}} \in \mathcal{V}_{flow,\mathrm{IJ},k}$ such that (2.6) is strictly satisfied, then, together with the irreducibility, it follows that $B_{\mathrm{IJ},kk}$ is irreducible diagonally dominant for $k = 1, \ldots, K$. Then, B_IJ is nonsingular. ∎

EXAMPLE 2.3. For the graph in Figure 2.1, the flow matrix is given by

$$B_\mathrm{IJ} = \begin{bmatrix} |q_1| + |q_4| & 0 & -|q_3| & 0 \\ -|q_1| & |q_2| & 0 & 0 \\ 0 & -|q_2| & |q_3| & 0 \\ 0 & 0 & 0 & |q_7| \end{bmatrix}.$$

The strongly connected components of $\mathcal{G}^{\mathrm{flow}}_{\mathrm{IJ}}$ are given by $\mathcal{E}^{\mathrm{inner}}_{flow,\mathrm{IJ},1} = \{\{v_{\mathrm{I},1}, v_{\mathrm{I},2}, v_{\mathrm{I},3}\}, \{e_1, e_2, e_3\}\}$ and $\mathcal{E}^{\mathrm{inner}}_{flow,\mathrm{IJ},2} = \{\{v_{\mathrm{I},4}\}\}$. The matrix B_IJ is irreducible as there exists no permutation that would transform this matrix into an upper triangular matrix, and we observe that B_IJ is nonsingular only if $|q_4|, |q_7| > 0$, i.e., only if the flows $|q_4|, |q_7|$ *start* at $v_{\mathrm{I},1}, v_{\mathrm{I},6}$. This agrees with the conditions of Lemma 2.2, claiming that

$$\sum_{e_j \in \mathcal{E}_{inc,s}(v_{\mathrm{I},1}) \cap (\mathcal{E}_\mathrm{J} \setminus \mathcal{E}^{\mathrm{inner}}_{\mathrm{IJ},flow,k})} |q_j| = |q_4|, \qquad \sum_{e_j \in \mathcal{E}_{inc,s}(v_{\mathrm{I},6}) \cap (\mathcal{E}_\mathrm{J} \setminus \mathcal{E}^{\mathrm{inner}}_{\mathrm{IJ},flow,k})} |q_j| = |q_7| > 0,$$

whereas $\sum_{e_j \in \mathcal{E}_{inc,s}(v_{\mathrm{I},i}) \cap (\mathcal{E}_\mathrm{J} \setminus \mathcal{E}^{\mathrm{inner}}_{\mathrm{IJ},flow,k})} |q_j| = 0$ for $i = 1, 2, 3$.

Besides these graph-theoretical results, we frequently use the following identity for the rank of a block matrix $A = [A_{ij}]_{i,j=1,2}$, cf., e.g., [11, p. 25]. If A_{11}, A_{22} are nonsingular, then

$$\mathrm{rank}(A) = \mathrm{rank}(A_{11}) + \mathcal{S}_{A_{11}}(A) = \mathrm{rank}(A_{22}) + \mathcal{S}_{A_{22}}(A), \tag{2.7}$$

where $\mathcal{S}_{A_{11}}(A) := A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $\mathcal{S}_{A_{22}}(A) = A_{11} - A_{12}A_{22}^{-1}A_{21}$ denotes the Schur complements.

3. A network model for incompressible flow networks. We consider a network

$$\mathcal{N} = \{Pi, Pu, De, Jc, Re\} \tag{3.1}$$

that is composed of pipes, pumps, demands, junctions, and reservoirs and that is filled by an incompressible fluid, for instance, water. The pipes $Pi := \{\mathrm{Pi}_1, \ldots, \mathrm{Pi}_{n_{\mathrm{Pi}}}\}$ and pumps $Pu := \{\mathrm{Pu}_1, \ldots, \mathrm{Pu}_{n_{\mathrm{Pu}}}\}$ are connected by junctions $Jc := \{\mathrm{Jc}_1, \ldots, \mathrm{Jc}_{n_{\mathrm{Jc}}}\}$ in which the mass flow of the fluid is split or merged. We distinguish between virtual connection points $Jc_0 := \{\mathrm{Jc}_{0,1}, \ldots, \mathrm{Jc}_{0,n_{\mathrm{Jc}_0}}\}$ and connection points $Jc_V := \{\mathrm{Jc}_{V,1}, \ldots, \mathrm{Jc}_{V,n_{\mathrm{Jc}_V}}\}$ possessing a volume $V_i > 0$. Those virtual connections point have a certain importance in the design of system simulation software since they allow us to connect standardized subcomponents without introducing additional volumes (and as a consequence additional thermal inertia). The connection to the environment is modeled by reservoirs $Re := \{\mathrm{Re}_1, \ldots, \mathrm{Re}_{n_{\mathrm{Re}}}\}$ and demand branches $De := \{\mathrm{De}_1, \ldots, \mathrm{De}_{n_{\mathrm{De}}}\}$ that impose predefined pressures enthalpies as well as mass and enthalpy flows onto the network. The number of each element in $\mathcal{N}$ is denoted by $n_{\mathrm{Pi}}, n_{\mathrm{Pu}}, n_{\mathrm{Jc}}, n_{\mathrm{De}}, n_{\mathrm{Re}}$, respectively, where $n_{\mathrm{Jc}} = n_{\mathrm{Jc}_0} + n_{\mathrm{Jc}_V}$, and we define $n := n_{\mathrm{Pi}} + n_{\mathrm{Pu}} + n_{\mathrm{Jc}} + n_{\mathrm{Re}} + n_{\mathrm{De}}$.

Given boundary conditions $\bar{p}_{\text{Re}} = [\bar{p}_{\text{Re}_i}]_i$, $\bar{h}_{\text{Re}} = [\bar{h}_{\text{Re}_i}]_i$, $i = 1, \ldots, n_{\text{Re}}$, and $\bar{q}_{\text{De}} = [\bar{q}_{\text{De}_j}]_j$, $\bar{H}_{\text{De}} = [\bar{H}_{\text{De}_j}]_j$, $j = 1, \ldots, n_{\text{De}}$, the task is to compute the mass and enthalpy flows $q_{\text{Pi}} = [q_{\text{Pi}_i}]_{i=1,\ldots,n_{\text{Pi}}}$, $q_{\text{Pu}} = [q_{\text{Pu}_i}]_{i=1,\ldots,n_{\text{Pu}}}$, $q_{\text{De}} = [q_{\text{De}_i}]_{i=1,\ldots,n_{\text{De}}}$ and $H_{\text{Pi}} = [H_{\text{Pi}_i}]_{i=1,\ldots,n_{\text{Pi}}}$, $H_{\text{Pu}} = [H_{\text{Pu}_i}]_{i=1,\ldots,n_{\text{Pu}}}$, $H_{\text{De}} = [H_{\text{De}_i}]_{i=1,\ldots,n_{\text{De}}}$ in the pipes, pumps, and demand branches as well as the pressures and specific enthalpies $p_{\text{Jc}} = [p_{\text{Jc}_i}]_{i=1,\ldots,n_{\text{Jc}}}$, $p_{\text{Re}} = [p_{\text{Re}_i}]_{i=1,\ldots,n_{\text{Re}}}$ and $h_{\text{Jc}} = [h_{\text{Jc}_i}]_{i=1,\ldots,n_{\text{Jc}}}$, $h_{\text{Re}} = [h_{\text{Re}_i}]_{i=1,\ldots,n_{\text{Re}}}$ in the junctions and reservoirs.

To set up the governing equations for the network, we consider the characteristic relation that every element imposes on the enthalpy flow and the specific enthalpy as well as on the mass flow and pressure.

In a pipe Pi_j directed from v_{j_1} to v_{j_2}, the mass flow q_j is specified by the transient momentum equation

$$\dot{q}_{\text{Pi},j} = c_{1,j}\Delta p_j + c_{2,j}(h_{j_1}) sgn(q_{\text{Pi},j}) q_{\text{Pi},j}^2 + c_{3,j} =: f_{\text{Pi},j}(q_{\text{Pi},j}, \Delta p_j) \tag{3.2a}$$

depending on the pressure difference $\Delta p_j = p_{j_1} - p_{j_2}$ between the adjacent nodes v_{j_1}, v_{j_2} and constants $c_{i,j}$ depending, e.g., on the pipe diameter, length, inclination angle, and physical properties. Including thermal effects, the density of the mass flow q_j typically depend on the specific enthalpy h_{j_1} in the originating vertex v_{j_1} leading to $c_{2,j} = c_{2,j}(h_{j_1})$. Then, $f_{\text{Pi},j} \in C(\Omega_{\text{Pi}_j} \times (-\infty, \infty), \mathbb{R})$, where $\Omega_{\text{Pi}_j} \subset (-\infty, \infty)$ denotes the domain of admissible mass flows in Pi_j. The enthalpy flow H_j in Pi_j agrees with the product of the mass flow q_j and the specific enthalpy h_{i_1}, i.e.,

$$H_j = \frac{q_{\text{Pi},j}}{2}((sgn(q_j) + 1)h_{j_1} - (sgn(q_j) - 1)h_{j_2}) =: f_{\text{Pi}*}(q_{\text{Pi},j}, h_{j_1}, h_{j_2}). \tag{3.2b}$$

Then, $f_{\text{Pi}*,j} \in C(\Omega_{\text{Pi}_j} \times (-\infty, \infty)^2, \mathbb{R})$ with

$$\text{D}_1 f_{\text{Pi}*,j}(q_{\text{Pi},j}, h_{j_1}, h_{j_2}) = \begin{cases} h_{j_1}, & q_{\text{Pi},j} > 0, \\ 0, & q_{\text{Pi},j} = 0, \\ h_{j_2}, & q_{\text{Pi},j} < 0. \end{cases}$$

In a pump Pu_j directed from v_{j_1} to v_{j_2}, the mass flow $q_{\text{Pu},j}$ is specified algebraically by the pressure drop $\Delta p_j = p_{j_1} - p_{j_2}$, i.e.,

$$p_{j_1} - p_{j_2} = f_{\text{Pu}_j}(q_{\text{Pu},j}). \tag{3.2c}$$

The function f_{Pu_j} is given by specialized pump models, cf., e.g., [9]. Without loss of generality, we assume that $f_{\text{Pu},j} \in C^1(\Omega_{\text{Pu}_j}, \mathbb{R})$, where $\Omega_{\text{Pu}_j} \subset (-\infty, \infty)$ denotes the domains of admissible mass flows in Pu_j. The enthalpy flow $H_{\text{Pu}*,j}$ in Pu_j is given by

$$\begin{aligned} H_{\text{Pu},j} &= \frac{q_{\text{Pu},j}}{2}((sgn(q_{\text{Pu},j}) + 1)h_{j_1} - (sgn(q_{\text{Pu},j}) - 1)h_{j_2}) + \delta h_j \\ &=: f_{\text{Pu}*}(q_{\text{Pu},j}, h_{j_1}, h_{j_2}), \end{aligned} \tag{3.2d}$$

with $f_{\text{Pu}*,j} \in C(\Omega_{\text{Pu}_j} \times (-\infty, \infty)^2, \mathbb{R})$. Here, δh_j is heat induced by the pump. For simplicity, in the following we assume that $\delta h_j = 0$.

Due to mass conservation, in a junction Jc_i, the amount of mass entering and leaving Jc_i is equal. Summarizing the indices of pipes and demand branches that are incident to Jc_i in the set $\hat{J}_i$, we thus get that

$$\sum_{j \in \hat{J}_i} q_j = 0. \tag{3.2e}$$

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Similarly, due to energy conservation, in a junction $\mathrm{Jc}_{V,i}$, the sum of all enthalpy fluxes H_j entering or leaving Jc_i equals the product of the volume V_i and the change of the specific enthalpy $h_{\mathrm{Jc}_V,i}$, i.e.,

$$\sum_{H_j \in \mathcal{E}_{inc}(\mathrm{Jc}_{V,i})} H_j = V_i \dot{h}_{\mathrm{Jc}_V,i}. \tag{3.2f}$$

In a virtual connection point $\mathrm{Jc}_{0,i}$, we have

$$\sum_{H_j \in \mathcal{E}_{inc}(\mathrm{Jc}_{0,i})} H_j = 0. \tag{3.2g}$$

In a demand branch De_j, the mass and enthalpy flow $q_{\mathrm{De},j}, H_{\mathrm{De},j}$ are specified by functions $\bar{q}_{\mathrm{De},j}, \bar{H}_{\mathrm{De},j} \in C^1(\mathcal{I}_{\mathrm{De}}, \mathbb{R})$, i.e.,

$$q_{\mathrm{De},j} = \bar{q}_{\mathrm{De},j}, \tag{3.2h}$$

$$H_{\mathrm{De},j} = \bar{H}_{\mathrm{De},j}. \tag{3.2i}$$

Similarly, in a reservoir Re_i, the pressure $p_{\mathrm{Re},i}$ and the specific enthalpy $h_{\mathrm{Re},i}$ are specified by functions $\bar{p}_{\mathrm{Re},i}, \bar{h}_{\mathrm{Re},i} \in C^1(\mathcal{I}_{\mathrm{Re}}, \mathbb{R})$, i.e.,

$$p_{\mathrm{Re},i} = \bar{p}_{\mathrm{Re},i}, \tag{3.2j}$$

$$h_{\mathrm{Re},i} = \bar{h}_{\mathrm{Re},i}. \tag{3.2k}$$

To include the connection structure of the network $\mathcal{N}$ and summarize the equations (3.2) for $\mathcal{N}$, we represent $\mathcal{N}$ as a graph $\mathcal{G}$. The pipes, pumps, and demand branches correspond to the edges of $\mathcal{G}$ while the junctions and reservoirs serve as vertices, i.e., we set

$$\mathcal{G} = \{\mathcal{V}, \mathcal{E}\} \quad \text{with} \quad \mathcal{E} = \{Pi, Pu, De\} \quad \text{and} \quad \mathcal{V} = \{Jc_0, Jc_V, Re\}. \tag{3.3}$$

We impose the following assumptions on the connection structure of $\mathcal{N}$.

ASSUMPTIONS 3.1. *Consider a network $\mathcal{N}$ as in* (3.1).

(i) Two junctions are connected by at most one pipe or one pump. Each pipe, pump, and demand has an assigned direction.

(ii) The network is connected, i.e., every pair of junctions and/or reservoirs can be reached by a sequence of pipes and pumps.

(iii) Every junction is adjacent to at most one demand branch. Every reservoir is connected at most to one pipe or pump.

Under Assumptions 3.1, the graph $\mathcal{G}$ given in (3.3) is simple and connected, and the reservoirs are end vertices. Assigning a direction to each pipe, pump, and demand, $\mathcal{G}$ is oriented, allowing us to speak of a positive or negative mass flow. Note that the orientation of the pipes and pumps is arbitrary and only serves as a reference condition; it is not necessarily related to the true or expected direction of the fluid flow.

Representing the network as a simple, oriented graph, the structure of $\mathcal{N}$ is fully described by the incidence matrix A associated with $\mathcal{G}$. According to $\mathcal{G}$, we partition the incidence matrix as

$$A = \begin{bmatrix} A_{\mathrm{Jc}_V,\mathrm{Pi}} & A_{\mathrm{Jc}_V,\mathrm{Pu}} & A_{\mathrm{Jc}_V,\mathrm{De}} \\ A_{\mathrm{Jc}_0,\mathrm{Pi}} & A_{\mathrm{Jc}_0,\mathrm{Pu}} & A_{\mathrm{Jc}_0,\mathrm{De}} \\ A_{\mathrm{Re},\mathrm{Pi}} & A_{\mathrm{Re},\mathrm{Pu}} & A_{\mathrm{Re},\mathrm{De}} \end{bmatrix} = \begin{bmatrix} A_{\mathrm{Jc}} \\ A_{\mathrm{Re}} \end{bmatrix}$$

and summarize the flows, pressures, and pressure differences as

$$q = \begin{bmatrix} q_{\text{Pi}} \\ q_{\text{Pu}} \\ q_{\text{De}} \end{bmatrix}, \quad p = \begin{bmatrix} p_{\text{Jc}} \\ p_{\text{Re}} \end{bmatrix}, \quad \Delta p = \begin{bmatrix} \Delta p_{\text{Pi}} \\ \Delta p_{\text{Pu}} \\ \Delta p_{\text{De}} \end{bmatrix},$$

and the enthalpy fluxes, specific enthalpies, and their differences as

$$H = \begin{bmatrix} H_{\text{Pi}} \\ H_{\text{Pu}} \\ H_{\text{De}} \end{bmatrix}, \quad h = \begin{bmatrix} h_{\text{Jc}_V} \\ h_{\text{Jc}_0} \\ h_{\text{Re}} \end{bmatrix}, \quad \Delta h = \begin{bmatrix} \Delta h_{\text{Jc}_V} \\ \Delta h_{\text{Jc}_0} \\ \Delta h_{\text{Re}} \end{bmatrix},$$

where $h_{\text{Jc}_V}, h_{\text{Jc}_0}$ refer to the enthalpies associated with junctions of positive and zero volume, respectively. Furthermore, we consider the matrix

$$|A| = \big[|A_{ij}|\big]_{(i,j)\in\mathcal{V}\times\mathcal{E}}$$

containing the elementwise absolute values of the incidence matrix A and set

$$B_\star(q_*) = \frac{1}{2}\operatorname{diag}(q_*(t))\big(\operatorname{diag}(\operatorname{sgn}(q_*(t)))A^T_{\star,*} + |A^T_{\star,*}|\big),$$

for $\star = \text{Jc}_0, \text{Jc}_V, \text{Re}$, $* = \text{Pi}, \text{Pu}$.

By the definition of A, the pressure and enthalpy drops $\Delta p_j = p_{j_1} - p_{j_2}$, $\Delta h_j = h_{j_1} - h_{j_2}$ along a given edge $e_j = (v_{j_1}, v_{j_2})$ are given by $e_j^T A^T p = \Delta p_j$ and $e_j^T A^T h = \Delta h_j$. Setting $C_1 = \operatorname{diag}(c_{1,j})_j$, $C_2 = \operatorname{diag}(c_{2,j})_j$, $C_3 = [c_{3,j}]_j$, for $j = 1, \ldots, n_{\text{Pi}}$, we define the *pipe function*

$$\begin{aligned} f_{\text{Pi}}(q_{\text{Pi}}, p_{\text{Jc}}, p_{\text{Re}}, h_{\text{Jc}_0}, h_{\text{Jc}_V}, h_{\text{Re}}) := {} & C_1(A^T_{\text{Jc,Pi}} p_{\text{Jc}} + A^T_{\text{Re,Pi}} p_{\text{Re}}) \\ & + C_2(h_{\text{Jc}_V}, h_{\text{Jc}_0}, h_{\text{Re}}) \operatorname{diag}(|q_{\text{Pi},j}|)_j q_{\text{Pi}} + C_3, \end{aligned}$$

with $f_{\text{Pi}} \in C^1(\Omega_{\text{Pi}} \times \Omega_{\text{Jc}} \times \Omega_{\text{Re}}, \mathbb{R}^{n_{\text{Pi}}})$, where $\Omega_{\text{Pi}} = \bigtimes_{j=1}^{n_{\text{Pi}}} \Omega_{\text{Pi}_j}$, $\Omega_{\text{Jc}} = \bigtimes_{i=1}^{n_{\text{Jc}}} \Omega_{\text{Jc}_i}$, and $\Omega_{\text{Re}} = \bigtimes_{j=1}^{n_{\text{Re}}} \Omega_{\text{Re}_i}$ denote the domains of admissible mass flows and pressures in *Pi* and *Jc*, *Re*, respectively. Similarly, we summarize the pipe equation (3.2b) for the enthalpy flow H_{Pi} as

$$H_{\text{Pi}} = B_{\text{Jc}_0}(q_{\text{Pi}})h_{\text{Jc}_0} + B_{\text{Jc}_V}(q_{\text{Pi}})h_{\text{Jc}_V} + B_{\text{Re}}(q_{\text{Pi}})h_{\text{Re}} =: f_{\text{Pi}*}(q_{\text{Pi}}, h_{\text{Jc}_V}, h_{\text{Jc}_0}, h_{\text{Re}}), \tag{3.4a}$$

with $f_{\text{Pi}*} \in C^1(\Omega_{\text{Pi}} \times \Omega_{\text{Jc}*} \times \Omega_{\text{Re}*}, \mathbb{R}^{n_{\text{Pi}}})$, where $\Omega_{\text{Jc}*} = \bigtimes_{i=1}^{n_{\text{Jc}*}} \Omega_{\text{Jc}^*_i}$, $\Omega_{\text{Re}*} = \bigtimes_{j=1}^{n_{\text{Re}*}} \Omega_{\text{Re}^*_i}$ denote the domains of admissible enthalpies in *Jc**,*Re**, respectively.

For the pumps, the relation between mass flow and pressure drop is described by the *pump function*

$$f_{\text{Pu}} := [f_{\text{Pu},j}]_{j=1,\ldots,n_{\text{Pu}}},$$

where we assume that $f_{\text{Pu}} \in C^1(\Omega_{\text{Pu}}, \mathbb{R}^{n_{\text{Pu}}})$, cf., e.g., [9]. Then, we get the pump equation

$$A^T_{\text{Jc,Pu}} p_{\text{Jc}} + A^T_{\text{Re,Pu}} p_{\text{Re}} = f_{\text{Pu}}(q_{\text{Pu}}). \tag{3.4b}$$

Similarly, the pump equation (3.2d) for the enthalpy flows reads

$$\begin{aligned} H_{\text{Pu}} &= B_{\text{Jc}_0}(q_{\text{Pu}})h_{\text{Jc}_0} + B_{\text{Jc}_V}(q_{\text{Pu}})h_{\text{Jc}_V} + B_{\text{Re}}(q_{\text{Pu}})h_{\text{Re}} + \delta h \\ &=: f_{\text{Pu}*}(q_{\text{Pu}}, h_{\text{Jc}_V}, h_{\text{Jc}_0}, h_{\text{Re}}), \end{aligned} \tag{3.4c}$$

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with $f_{\text{Pu}*} \in C^1(\Omega_{\text{Pu}} \times \Omega_{\text{Jc}*} \times \Omega_{\text{Re}*}, \mathbb{R}^{n_{\text{Pi}}})$, where $\Omega_{\text{Pu}} = \times_{j=1}^{n_{\text{Pu}}} \Omega_{\text{Pu}_j}$ denotes the domain of admissible mass flow in *Pu* and $\delta h := [\delta h_j]_{j=1,\dots,n_{\text{Pu}}}$.

The sum of all mass flows entering or leaving a junction Jc_i is given by

$$e_i^T A q = \sum_{e_j \in \mathcal{E}_{inc}(\text{Jc}_i)} q_j$$

such that the junction equations (3.2e) can be summarized as

$$A_{\text{Jc,Pi}} q_{\text{Pi}} + A_{\text{Jc,Pu}} q_{\text{Pu}} + A_{\text{Jc,De}} q_{\text{De}} = 0. \tag{3.4d}$$

In the same manner, we summarize the junction equations (3.2f), (3.2g) as

$$A_{\text{Jc,Pi}} H_{\text{Pi}} + A_{\text{Jc,De}} H_{\text{De}} = V_{\text{Jc}} \dot{h}_{\text{Jc}}, \tag{3.4e}$$

$$A_{\text{Jc,Pi}} H_{\text{Pi}} + A_{\text{Jc,De}} H_{\text{De}} = 0. \tag{3.4f}$$

For the demand branches and reservoirs, we obtain the simple relations

$$q_{\text{De}} = \bar{q}_{\text{De}}, \tag{3.4g}$$

$$p_{\text{Re}} = \bar{p}_{\text{Re}}, \tag{3.4h}$$

$$H_{\text{De}} = \bar{H}_{\text{De}}, \tag{3.4i}$$

$$h_{\text{Re}} = \bar{h}_{\text{Re}}, \tag{3.4j}$$

where we assume that $\bar{H}_{\text{De}} \in C^1(\mathcal{I}_{\text{De}}, \mathbb{R})$, $\bar{h}_{\text{Re}} \in C^1(\mathcal{I}_{\text{Re}}, \mathbb{R})$ for $\mathcal{I}_{\text{De}} = \bigcap_{j=1}^{n_{\text{De}}} \mathcal{I}_{\text{De}_j}$ and $\mathcal{I}_{\text{Re}} = \bigcap_{j=1}^{n_{\text{Re}}} \mathcal{I}_{\text{Re}_i}$.

In conclusion, the dynamics of the network $\mathcal{N}$ is modeled by the differential-algebraic system (3.4). Each equation of (3.4) and each entry of the state has a direct physical counterpart in the network. We use this relation to find conditions when (3.4) is uniquely solvable and to reinterpret these conditions as conditions on the structure and the elements of the network $\mathcal{N}$.

As prerequisites for our analysis, we discuss the following substructures of the network $\mathcal{N}$. We consider the subset of junctions and pumps $\mathcal{G}_{\text{Jc,Pu}} := \{Jc, Pu\}$ with the connection matrix $A_{\text{Jc,Pu}}$. We assume that $\mathcal{G}_{\text{Jc,Pu}}$ is composed of K connected components $\mathcal{G}_{\text{Jc,Pu},k} = \{Jc_k, Pu_k\}$ that are numbered such that $\mathcal{G}_{\text{Jc,Pu},k}$ corresponds to proper subgraphs for $k = 1, \dots, \hat{k}$ and to subsets with loose edges for $k = \hat{k}+1, \dots, K$. The connection matrix is partitioned accordingly into $A_{\text{Jc,Pu}} = \text{diag}\,(A_{\text{Jc,Pu},k})_k$.

According to Section 2, we partition $\mathcal{G}_{\text{Jc,Pu}}$ into a *reduced vertex set* Jc_1 with *ground nodes* Jc_2 and into a *pump spanning tree* Pu_1 with *chord set* Pu_2. The associated selection matrices are given by, cf. (2.2a),

$$\Gamma = [\Gamma_1, \Gamma_2], \quad \Pi_{\text{Pu}} = [\Pi_{\text{Pu},1}, \Pi_{\text{Pu},2}].$$

We consider the *fundamental cycles* $\mathcal{C}_{\text{Jc,Pu}} = \bigcup_{k=1}^{K} \mathcal{C}_{\text{Jc,Pu},k}$, the *crossing paths* $\mathcal{P}_{\text{Jc,Pu}} = \bigcup_{k=1}^{K} \mathcal{P}_{\text{Jc,Pu},k}$, and the set of loose pumps Pu^{loose} with the selection matrix, cf. (2.2c),

$$V_2 := [C_{\text{Jc,Pu}},\ P_{\text{Jc,Pu}},\ L_{\text{Jc,Pu}}].$$

For $k = 1, \dots, \hat{k}$, we consider the componentwise vertex identification of Jc_k and set $\bar{Jc} := \{\bar{\text{Jc}}_k\}$, where $\bar{\text{Jc}}_k := \bigcup_{\text{Jc}_i \in Jc_k} \text{Jc}_i$. The associated identification matrix is given by, cf. (2.2d),

$$U_2 := \mathbf{1}_{\text{Jc,Pu}}.$$

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According to Lemma 2.1, $\operatorname{rank}(A_{\text{Jc,Pu}}) = n_{\text{Jc}} - \hat{k}$, where $\hat{k}$ denotes the number of connected components in $\mathcal{G}_{\text{Jc,Pu}}$ that itself are subgraphs. Furthermore, $\ker(A_{\text{Jc,Pu}}) = \operatorname{span}(V_2)$, $\operatorname{corange}(A_{\text{Jc,Pu}}) = \operatorname{span}(\Pi_{\text{Pu}_1})$ and $\operatorname{coker}(A_{\text{Jc,Pu}}) = \operatorname{span}(\mathbf{1}_{\text{Jc,Pu}})$, $\operatorname{range}(A_{\text{Jc,Pu}}) = \operatorname{span}(\Gamma_1)$. From these matrices, we define the transformations

$$U := [\Gamma_1, U_2], \quad V := [\Pi_{\text{Pu}_1}, V_2], \tag{3.5a}$$

of which the inverses are given by

$$U^{-1} = \left[(U_2^-)^T, \Gamma_2\right]^T, \quad V^{-1} = \left[(V_2^-)^T, \Pi_{\text{Pu}_2}\right]^T, \tag{3.5b}$$

where

$$\begin{aligned} U_2^- &= \operatorname{diag}(U_{2,k}^-)_{k=1,\dots,\hat{k}}, & U_{2,k}^- &= \Gamma_{1,k}^T - \mathbf{1}_{n_{\mathcal{V}_{\mathcal{G}_{\text{Jc,Pu}},k}}-1}\Gamma_{2,k}^T, \\ V_2^- &= [\operatorname{diag}(V_{2,k}^-)_{k=1,\dots,\hat{k}}, 0], & V_{2,k}^- &= \Pi_{\text{Pu},k,1}^T - \Pi_{\text{Pu},k,1}^T V_2 \Pi_{\text{Pu},k,2}^T. \end{aligned}$$

For the vertex identification $\bar{Jc}$, we define the set $\mathcal{G}_{\bar{Jc},Pi} := \{\bar{Jc}, Pi\}$ composed of L connected components $\mathcal{G}_{Jc,Pi,k} = \{\bar{Jc}_k, Pi_k\}$. We partition $\mathcal{G}_{\bar{Jc},Pi}$ into a *pipe spanning tree* $Pi_1 = \bigcup_{k=1}^{L} Pi_{k,1}$ with *pipe chord set* $Pi_2 = \bigcup_{k=1}^{L} Pi_{k,2}$ and denote the associated selection matrix by

$$\Pi_{\text{Pi}} = [\Pi_{\text{Pi},2}, \Pi_{\text{Pi},1}]\,. \tag{3.5c}$$

Then, $\operatorname{rank}(A_{\bar{\text{Jc}},\text{Pi}}) = \hat{k} - \hat{l}$, where $\hat{l}$ denotes the number of connected components in $\mathcal{G}_{\bar{\text{Jc}},\text{Pi}}$ that itself are subgraphs, and $\operatorname{corange}(A_{\bar{\text{Jc}},\text{Pi}}) = \operatorname{span}(\Pi_{\text{Pi}_1})$.

We partition and transform the variables according to these substructures and set

$$\begin{aligned} p_{\text{Jc}_1} &:= U_2^- p_{\text{Jc}}, & q_{\text{Pu}_1} &:= V_2^- q_{\text{Pu}}, & q_{\text{Pi}_1} &:= \Pi_{\text{Pi}_1}^T q_{\text{Pi}}, \\ p_{\text{Jc}_2} &:= \Gamma_2^T p_{\text{Jc}}, & q_{\text{Pu}_2} &:= \Pi_{\text{Pu}_2}^T q_{\text{Pu}}, & q_{\text{Pi}_2} &:= \Pi_{\text{Pi}_2}^T q_{\text{Pi}}. \end{aligned} \tag{3.6}$$

The mass flows q_{Pu_2}, q_{Pi_2} belong to pumps and pipes in the chord sets Pu_2, Pi_2 while the pressures p_{Jc_2} belong to the ground nodes Jc_2. The mass flows $q_{\text{Pi}_1}, q_{\text{Pu}_1}$ denote the difference between the branch flows in Pu_1, Pi_1 and the flows in the fundamental cycles and crossing paths containing the considered branch. The pressure $p_{\text{Jc}_1,i}$ denotes the pressure difference between a junction Jc_i in the reduced vertex set Jc_1 and the associated ground node.

We denote the associated connection matrices accordingly and set, for instance, $A_{\text{Jc}_1,\text{Pu}_1} := \Gamma_1^T A_{\text{Jc,Pu}} \Pi_{\text{Pu}_1}$. Furthermore, we consider the matrices

$$\begin{aligned} C &:= A_{\bar{\text{Jc}},\text{Pi}} C_2 A_{\bar{\text{Jc}},\text{Pi}}^T, \qquad D(q_{\text{Pu}}) := V_2^T\, \mathrm{D} f_{\text{Pu}}(q_{\text{Pu}}) V_2, \\ B(q_{\text{Pi}}, q_{\text{Pu}}) &:= A_{\text{Jc}_0,\text{Pi}} B_{\text{Jc}_0}(q_{\text{Pi}}) + A_{\text{Jc}_0,\text{Pu}} B_{\text{Jc}_0}(q_{\text{Pu}}), \end{aligned}$$

i.e., C is the Jacobian of the pipe function f_{Pi} with respect to the pressure p_{Jc_2} restricted to the contraction $\bar{Jc}$, B is the Jacobian of the pipe and the enthalpy function $f_{\text{Pi}*}, f_{\text{Pu}*}$ with respect to h_{Jc_0}, and D is the Jacobian of the pump function $f_{\text{Pu}*}$ with respect to the pump flows q_{Pu_2} restricted to the virtual connection points Jc_0. In order to give topological conditions when B, C are nonsingular, we consider the flow graph $\mathcal{G}^{\text{flow}}_{\text{Jc}_0,(\text{Pi,Pu})}$ of the subset $\mathcal{G}_{\text{Jc}_0,(\text{Pi,Pu})}$. As the directions of the mass flows may change with $t \in \mathcal{I}$, the flow graph is state-dependent in general, and we write $\mathcal{G}^{\text{flow}}_{\text{Jc}_0,(\text{Pi,Pu})}(q_{\text{Pi}}, q_{\text{Pu}})$. Accordingly, the sets $\mathcal{E}_{inc,s}(\text{Jc}_{0,i})$, $\mathcal{E}_{inc,e}(\text{Jc}_{0,i})$ are state-dependent, and we write $\mathcal{E}_{inc,s}(\text{Jc}_{0,i}; q_{\text{Pi}}, q_{\text{Pu}})$, $\mathcal{E}_{inc,e}(\text{Jc}_{0,i}; q_{\text{Pi}}, q_{\text{Pu}})$.

LEMMA 3.2. *Consider the network* (3.1) *with graph* $\mathcal{G}$ *and incidence matrix* A. *Consider the subsets* $\mathcal{G}_{Jc,Pu}, \mathcal{G}_{\bar{J}c,Pi}$ *with submatrices* $A_{Jc,Pu}, A_{\bar{J}c,Pi}$.

(i) If $n_{Re} > 0$, *then* $\mathrm{rank}(A_{\bar{J}c,Pi}) = \hat{k}$ *and* C *is nonsingular.*

(ii) For $q_{Pi} \in \Omega_{Pi}, q_{Pu} \in \Omega_{Pu}$, *let* $\mathcal{G}^{flow}_{Jc_0,(Pi,Pu);k}(q_{Pi}, q_{Pu})$, $k = 1, \dots, K$, *denote the strongly connected components in the flow graph* $\mathcal{G}^{flow}_{Jc_0,(Pi,Pu)}(q_{Pi}, q_{Pu})$. *For* $Jc_{0,i} \in \mathrm{Jc}_{0,k}$, $k = 1, \dots, K$, *if*

$$\sum_{e_j \in \mathcal{E}_{inc,s}(Jc_{0,i};q_{Pi},q_{Pu}) \cap \{\mathrm{Pi} \cup \mathrm{Pu}\}} |q_j| > 0,$$

(3.7)

$$\sum_{e_j \in \mathcal{E}_{inc,s}(Jc_{0,i};q_{Pi},q_{Pu}) \cap (\{\mathrm{Pi} \cup \mathrm{Pu}\} \setminus \mathcal{E}^{inner}_{Jc_0,(Pi,Pu),flow,k}(q_{Pi},q_{Pu}))} |q_j| \geqslant 0, \quad k = 1, \dots, K \geqslant 0,$$

and for every $k = 1, \dots, K$, *there exists* $\hat{J}c_{0,k} \in \mathrm{Jc}_{0,k}$ *such that* (3.7) *is strictly satisfied, then* $B(q_{Pi}, q_{Pu})$ *is nonsingular.*

Proof. (i) Neglecting the demand branches $\mathrm{De}_1, \dots, \mathrm{De}_{n_{\mathrm{De}}}$ in $\mathcal{G}_{\mathcal{N}}$, we obtain the subgraph $\mathcal{G}_{\mathcal{N}\setminus De} := \{\{Jc, Re\}, \{Pi, Pu\}\}$ whose incidence matrix is given by

$$A_{\mathcal{G}_{\mathcal{N}\setminus De}} = \begin{bmatrix} A_{\mathrm{Jc,Pi}} & A_{\mathrm{Jc,Pu}} \\ A_{\mathrm{Re,Pi}} & A_{\mathrm{Re,Pu}} \end{bmatrix}.$$

As $\mathcal{G}_{\mathcal{N}\setminus De}$ is a connected subgraph, it follows that $\mathrm{rank}(A_{\mathcal{G}_{\mathcal{N}\setminus De}}) = n_{\mathcal{V}} - 1$, cf. Lemma 2.1. For $[A_{\mathrm{Jc,Pi}}, A_{\mathrm{Jc,Pu}}]$, this implies that $\mathrm{rank}([A_{\mathrm{Jc,Pi}}, A_{\mathrm{Jc,Pu}}]) = n_{\mathrm{Jc}}$ if $n_{\mathrm{Re}} > 0$. Considering the transformations U, V defined in (3.5), we thus have that

$$\begin{aligned} n_{\mathrm{Jc}} &= \mathrm{rank}\left(U^T [A_{\mathrm{Jc,Pi}}, A_{\mathrm{Jc,Pu}}] V\right) = \mathrm{rank}\left(\begin{bmatrix} A_{\mathrm{Jc_1,Pi}} & A_{\mathrm{Jc_1,Pu_1}} & 0 \\ A_{\bar{\mathrm{J}}\mathrm{c,Pi}} & 0 & 0 \end{bmatrix}\right) \\ &= \mathrm{rank}(A_{\mathrm{Jc_1,Pu_1}}) + \mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}). \end{aligned}$$

Since $\mathrm{rank}(A_{\mathrm{Jc,Pu}}) = n_{\mathrm{Jc}} - \hat{k}$, where $\hat{k}$ denotes the number of connected components in $\mathcal{G}_{\mathrm{Jc,Pu}}$ that itself are subgraphs, cf. Lemma 2.1, it follows that $\mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}) = \hat{k}$.

Noting that $C_1 = \mathrm{diag}\,(c_{1,j})_{j=1,\dots,n_{\mathrm{Pi}_j}}$ is positive definite since $c_{1,j} > 0$, $j = 1, \dots, n_{\mathrm{Pi}_j}$, we can factor the matrix C according to $C = (A_{\bar{\mathrm{J}}\mathrm{c,Pi}}\sqrt{C_1})(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}\sqrt{C_1})^T$. As the matrix $\sqrt{C_1} = \mathrm{diag}\,(\sqrt{c_{1,j}})_{j=1,\dots,n_{\mathrm{Pi}_j}}$ is nonsingular, it follows that

$$\mathrm{rank}(C) = \mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}\sqrt{C_1}) = \mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}) = \hat{k},$$

implying that $C \in \mathbb{R}^{\hat{k} \times \hat{k}}$ is nonsingular.

(ii) Noting that

$$\begin{aligned} B(q_{\mathrm{Pi}}, q_{\mathrm{Pu}}) = \frac{1}{2} [A_{\mathrm{Jc_0,Pi}}, A_{\mathrm{Jc_0,Pu}}] &\begin{bmatrix} \mathrm{diag}\,(q_{\mathrm{Pi}}) & 0 \\ 0 & \mathrm{diag}\,(q_{\mathrm{Pu}}) \end{bmatrix} \\ &\left(\begin{bmatrix} \mathrm{diag}\,(\mathrm{sgn}(q_{\mathrm{Pi}})) & 0 \\ 0 & \mathrm{diag}\,(\mathrm{sgn}(q_{\mathrm{Pu}})) \end{bmatrix} [A_{\mathrm{Jc_0,Pi}}, A_{\mathrm{Jc_0,Pu}}]^T + [|A_{\mathrm{Jc_0,Pi}}|, |A_{\mathrm{Jc_0,Pu}}|]^T\right), \end{aligned}$$

we find that $B(q_{\mathrm{Pi}}, q_{\mathrm{Pu}})$ corresponds to the sum of the flow matrix $B_{\mathrm{Jc_0},\{Pi,Pu\}}$ of the subgraph $\{Jc_0, \{Pi, Pu\}\}$, i.e., $B = B_{\mathrm{Jc_0},\{Pi,Pu\}}$. Under the given assertions, we can apply Lemma 2.2 and find that $B(q_{\mathrm{Pi}}, q_{\mathrm{Pu}})$ is nonsingular. ∎

We call the set of virtual connection points Jc_0 *enthalpy reachable in* $t \in \mathcal{I}$, if the assertions of Lemma 3.2 (ii) are satisfied on $\Omega_{\mathrm{Pi}} \times \Omega_{\mathrm{Pu}}$.

4. Topological solvability conditions for the pressure and temperature model. To analyze the solvability of (3.4), we define the *network function* $F = [F_{pres}^T, F_{enth}^T, F_{bound}^T]^T \in C^1(\mathbb{D}, \mathbb{R}^{2n})$ with

$$F_1 := \begin{bmatrix} F_{\text{Pi}} \\ F_{\text{Jc}_V^*} \end{bmatrix}, \quad F_{pres} := \begin{bmatrix} F_{\text{Pu}} \\ F_{\text{Jc}} \end{bmatrix}, \quad F_{enth} := \begin{bmatrix} F_{\text{Jc}_0^*} \\ F_{\text{Pi}*} \\ F_{\text{Pu}*} \end{bmatrix}, \quad F_{bound} := \begin{bmatrix} F_{\text{De}} \\ F_{\text{Re}} \\ F_{\text{De}*} \\ F_{\text{Re}*} \end{bmatrix}, \tag{4.1}$$

for $x = [q^T, H^T, p^T, h^T]^T$, the domain of definition $\mathbb{D} := \mathcal{I} \times \Omega_x \times \dot{\Omega}_x$ with $\mathcal{I} = \mathcal{I}_{\text{De}} \cap \mathcal{I}_{\text{Re}}$, $\Omega_x := (\Omega_{\text{Pi}} \times \Omega_{\text{Pu}} \times \Omega_{\text{De}})^2 \times (\Omega_{\text{Jc}} \times \Omega_{\text{Re}})^2$, $\Omega_{\dot{x}} \subset \mathbb{R}^{2n}$, and

$$F_{\text{Pi}}(t, x, \dot{x}) = \dot{q}_{\text{Pi}} - f_{\text{Pi}}(q_{\text{Pi}}, p_{\text{Jc}}, p_{\text{Re}}, h_{\text{Jc}}, h_{\text{Re}}), \tag{4.2a}$$
$$F_{\text{Pu}}(t, x) = A_{\text{Jc,Pu}}^T p_{\text{Jc}} + A_{\text{Re,Pu}}^T p_{\text{Re}} - f_{\text{Pu}}(q_{\text{Pu}}), \tag{4.2b}$$
$$F_{\text{Jc}}(t, x) = A_{\text{Jc,Pi}} q_{\text{Pi}} + A_{\text{Jc,Pu}} q_{\text{Pu}} + A_{\text{Jc,De}} q_{\text{De}}, \tag{4.2c}$$
$$F_{\text{Jc}_V^*}(t, x, \dot{x}) = \hat{V} \dot{h}_{\text{Jc}_V} - A_{\text{Jc}_V,\text{Pi}} H_{\text{Pi}} - A_{\text{Jc}_V,\text{Pu}} H_{\text{Pu}} - A_{\text{Jc}_V,\text{De}} H_{\text{De}}, \tag{4.2d}$$
$$F_{\text{Jc}_0^*}(t, x) = -A_{\text{Jc,Pi}} H_{\text{Pi}} - A_{\text{Jc,Pu}} H_{\text{Pu}} - A_{\text{Jc,De}} H_{\text{De}}, \tag{4.2e}$$
$$F_{\text{Pi}*}(t, x) = H_{\text{Pi}} - f_{\text{Pi}*}(q_{\text{Pi}}, h_{\text{Jc}_V}, h_{\text{Jc}_0}, h_{\text{Re}}), \tag{4.2f}$$
$$F_{\text{Pu}*}(t, x) = H_{\text{Pu}} - f_{\text{Pu}*}(q_{\text{Pu}}, h_{\text{Jc}_V}, h_{\text{Jc}_0}, h_{\text{Re}}), \tag{4.2g}$$
$$F_{\text{De}}(t, x) = q_{\text{De}} - \bar{q}_{\text{De}}, \qquad F_{\text{Re}}(t, x) = p_{\text{Re}} - \bar{p}_{\text{Re}}, \tag{4.2h}$$
$$F_{\text{De}*}(t, x) = H_{\text{De}} - \bar{H}_{\text{De}}, \qquad F_{\text{Re}*}(t, x) = h_{\text{Re}} - \bar{h}_{\text{Re}}. \tag{4.2i}$$

To keep the smoothness assumptions on F as relaxed as possible, we partition the state into differential and algebraic variables x_d, x_a and set $x = [x_d{}^T, x_a{}^T]^T$ with

$$x_d = [q_{\text{Pi}}^T, h_{\text{Jc}_V}^T]^T, \quad x_a = [q_{\text{Pu}}^T, p_{\text{Jc}}^T, H_{\text{Pi}*}^T, H_{\text{Pu}*}^T, h_{\text{Jc}_0}^T, q_{\text{De}}^T, p_{\text{Re}}^T, q_{\text{De}*}^T, p_{\text{Re}*}^T]^T.$$

In addition to the network function, we define the following *surrogate network function* $\hat{F} = [\hat{F}_1^T, \hat{F}_2^T, F_{bound}^T]^T \in C^1(\mathbb{D}, \mathbb{R}^{2n})$ with

$$\hat{F}_1 := \begin{bmatrix} F_{\text{Pi}_2} \\ F_{\text{Jc}_V} \end{bmatrix}, \quad \hat{F}_{2,pres} := \begin{bmatrix} F_{\dot{\text{Jc}}} \\ F_{\text{Pu}} \\ F_{\text{Jc}} \end{bmatrix}, \quad \hat{F}_{2,enth} := \begin{bmatrix} F_{\text{Pi}*} \\ F_{\text{Pu}*} \\ F_{\text{Jc}_0^*} \end{bmatrix},$$

where $F_{\text{Jc}_V}, F_{\text{Pu}}, F_{\text{Jc}}, F_{\text{Pi}*}, F_{\text{Pu}*}, F_{\text{Jc}_0^*}$ are given as in (4.2) and (omitting arguments)

$$\begin{aligned} F_{\text{Pi}_2}(t, x, \dot{x}) &:= \Pi_{\text{Pi}_2}^T \dot{q}_{\text{Pi}} - \Pi_{\text{Pi}_2}^T f_{\text{Pi}}, \\ F_{\dot{\text{Jc}}}(t, x) &:= A_{\bar{\text{Jc}},\text{Pi}} f_{\text{Pi}} - A_{\bar{\text{Jc}},\text{De}} \dot{\bar{q}}_{\text{De}}. \end{aligned}$$

From the surrogate function, we define the set of *consistent initial values* by

$$\mathcal{C}_{IV} := \hat{F}_2^{-1}(0).$$

Using the concept of derivative arrays and the strangeness index as developed in [15, 16, 17, 18], we characterize the unique solvability of the DAE model (3.4).

THEOREM 4.1. *Let $\mathcal{N}$ be a network given by (3.1) that satisfies Assumptions 3.1, and let $F \in C^1(\mathbb{D}, \mathbb{R}^n)$ be the associated network function. If $n_{Re} > 0$ and, on $\mathcal{C}_{IV}$, the set* Jc_0 *is enthalpy reachable and the matrix $D(q_{Pu})$ is pointwise nonsingular, then the following assertions hold.*

1. *For every $(t_0, x_0) \in \mathcal{C}_{IV}$, there exists an interval $(t_0^-, t_0^+) \subset \mathcal{I}$ such that the initial value problem*

$$F(t, x, \dot{x}) = 0, \tag{4.3a}$$
$$x(t_0) = x_0, \tag{4.3b}$$

is uniquely solvable with $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$.

2. *For every $(t_0, x_0) \in \mathcal{C}_{IV}$, there exists an interval $(t_0^-, t_0^+) \subset \mathcal{I}$ such that a function $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$ solves (4.3) if and only if x solves the surrogate model*

$$\hat{F}(t, x, \dot{x}) = 0, \tag{4.4a}$$
$$x(t_0) = x_0. \tag{4.4b}$$

Proof. We structure our proof in the following way. First, we show that every solution of (4.4) solves (4.3). Using the transformations (3.5), we show that (4.4) can be decoupled into an explicit system, whose unique solvability is covered by classical ODE theory and the Implicit Function Theorem. Using the concept of derivative arrays and the strangeness index, we finally derive the surrogate model (4.4) and show that every solution of (4.3) solves (4.4).

To prove that every solution of (4.4) solves (4.3), let $x \in C^1(\hat{\mathcal{J}}, \mathbb{R}^{2n})$ solve (4.4) with $(t_0, x_0) \in \mathcal{C}_{IV}$. Using a nonsingular matrix $S \in \mathbb{R}^{2n \times 2n}$, we transform the states according to (2.5) and set

$$\tilde{x} := S^{-1}x = \left[q_{\mathrm{Pi}_2}^T, p_{\mathrm{Jc}_V}^T, q_{\mathrm{Pi}_1}^T, q_{\mathrm{Pu}_1}^T, q_{\mathrm{Pu}_2}^T, p_{\mathrm{Jc}_1}^T, p_{\mathrm{Jc}_2}^T, q_{\mathrm{De}}^T, p_{\mathrm{Re}}^T, H_{\mathrm{De}}^T, h_{\mathrm{Re}}^T\right]^T.$$

We transform the domain of definition accordingly and set $\tilde{\mathbb{D}} := \mathcal{I} \times \Omega_{\tilde{x}} \times \Omega_{\dot{\tilde{x}}}$ with $\Omega_{\tilde{x}} := S^{-1}\Omega_x$, $\Omega_{\dot{\tilde{x}}} := S^{-1}\Omega_{\dot{x}}$ and partition the state into $\tilde{x} = [\tilde{x}_d^T, \tilde{x}_a^T]^T$ with $\tilde{x}_d = q_{\mathrm{Pi}_2}$, $\tilde{x}_a = [p_{\mathrm{Jc}_2}^T, p_{\mathrm{Jc}_1}^T, q_{\mathrm{Pu}_2}^T, q_{\mathrm{Pu}_1}^T, q_{\mathrm{Pi}_1}^T, q_{\mathrm{De}}^T, p_{\mathrm{Re}}^T]^T$.

For the initial value problem, we choose a nonsingular matrix $\tilde{S} \in \mathbb{R}^{2n \times 2n}$ and set $\tilde{F}(t, \tilde{x}, \dot{\tilde{x}}) := \tilde{S}^T F(t, S\tilde{x}, S\dot{\tilde{x}})$ such that $\tilde{F} := [\tilde{F}_1^T, \tilde{F}_{2,pres}^T, \tilde{F}_{2,enth}^T, F_{bound}^T]^T \in C^1(\tilde{\mathbb{D}}, \mathbb{R}^n)$ is given by

$$\tilde{F}_1 := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}_2} \\ \tilde{F}_{\mathrm{Jc}_V} \end{bmatrix}, \quad \tilde{F}_{2,pres} := \begin{bmatrix} \tilde{F}_{\mathrm{Pu}_1} \\ \tilde{F}_{\bar{\mathrm{Pu}}} \\ \tilde{F}_{\mathrm{Jc}_1} \\ \tilde{F}_{\bar{\mathrm{Jc}}} \\ \tilde{F}_{\dot{\bar{\mathrm{Jc}}}} \end{bmatrix}, \quad \tilde{F}_{2,enth} := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}*} \\ \tilde{F}_{\mathrm{Pu}*} \\ \tilde{F}_{\mathrm{Jc}_0^*} \\ \tilde{F}_{\tilde{\mathrm{Jc}}*} \end{bmatrix},$$

with

$$\tilde{F}_1 = \hat{F}_1 \circ (S^{-1} \times S^{-1}),$$
$$\tilde{F}_{\dot{\bar{\mathrm{Jc}}}} = F_{\dot{\bar{\mathrm{Jc}}}} \circ S^{-1}, \qquad \tilde{F}_{\mathrm{Pu}_1} = \Pi_{\mathrm{Pu}_1}^T F_{\mathrm{Pu}} \circ S^{-1},$$
$$\tilde{F}_{\mathrm{Pi}*} = F_{\mathrm{Pi}*} \circ S^{-1}, \qquad \tilde{F}_{\mathrm{Pu}*} = F_{\mathrm{Pu}*} \circ S^{-1},$$
$$\tilde{F}_{\tilde{\mathrm{Jc}}*} = (A_{\mathrm{Jc,Pi}} F_{\mathrm{Pi}*} + A_{\mathrm{Jc,Pu}} F_{\mathrm{Pu}*} + F_{\mathrm{Jc}*}) \circ S^{-1}, \qquad \tilde{F}_{\bar{\mathrm{Pu}}} = V_2^T F_{\mathrm{Pu}} \circ S^{-1},$$
$$\tilde{F}_{\mathrm{Jc}_1} = \Gamma_1^T F_{\mathrm{Jc}} \circ S^{-1}, \qquad \tilde{F}_{\bar{\mathrm{Jc}}} = U_2^T F_{\mathrm{Jc}} \circ S^{-1}.$$

Note that $\dot{q}_{\mathrm{Pi}_2} = \Pi_{\mathrm{Pi}_2}^T q_{\mathrm{Pi}}$ as Π_{Pi_2} is constant. Then, the transformation $\tilde{x} = S^{-1}x$ of the solution x solves

$$\tilde{F}(t, \tilde{x}, \dot{\tilde{x}}_d) = 0, \qquad \tilde{x}(t_0) = \tilde{x}_0. \tag{4.5}$$

Differentiating the mass balance $\tilde{F}_{\mathrm{Jc}}(t, \tilde{x}) = 0$ in (4.5) and noting that $A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1}$ is nonsingular, we find that $\tilde{x}$ also solves

$$\dot{q}_{\mathrm{Pi}_1} = -A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_2} \dot{q}_{\mathrm{Pi}_2} - A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{De}} \dot{q}_{\mathrm{De}}. \tag{4.6}$$

From the pipe and the demand equations $\tilde{F}_{\mathrm{Pi}_2}(t, \tilde{x}) = 0$, $\tilde{F}_{\mathrm{De}}(t, \tilde{x}) = 0$ in (4.5), we further find that

$$\dot{q}_{\mathrm{Pi}_2} = \Pi_{\mathrm{Pi}_2}^T f_{\mathrm{Pi}}\big(\Pi_{\mathrm{Pi}_1} q_{\mathrm{Pi}_1} + \Pi_{\mathrm{Pi}_2} q_{\mathrm{Pi}_2},\ A_{\mathrm{Jc}_1,\mathrm{Pi}}^T p_{\mathrm{Jc}_1} + A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}^T p_{\mathrm{Jc}_2} + A_{\mathrm{Re},\mathrm{Pi}}^T \bar{p}_{\mathrm{Re}}\big), \tag{4.7}$$

$$A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{De}} \dot{q}_{\mathrm{De}} = A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}} f_{\mathrm{Pi}}\big(\Pi_{\mathrm{Pi}_1} q_{\mathrm{Pi}_1} + \Pi_{\mathrm{Pi}_2} q_{\mathrm{Pi}_2},\ A_{\mathrm{Jc}_1,\mathrm{Pi}}^T p_{\mathrm{Jc}_1} + A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}^T p_{\mathrm{Jc}_2} + A_{\mathrm{Re},\mathrm{Pi}}^T \bar{p}_{\mathrm{Re}}\big) \dot{q}_{\mathrm{De}}. \tag{4.8}$$

Inserting (4.7), (4.8) into (4.6), it follows that $\tilde{x}$ solves the differential equation

$$\begin{aligned} 0 &= F_{\mathrm{Pi}_1}(t, \tilde{x}, \dot{\tilde{x}}) \\ &:= \dot{q}_{\mathrm{Pi}_1} - A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1} \Pi_{\mathrm{Pi}_1}^T f_{\mathrm{Pi}}\Big(\Pi_{\mathrm{Pi}_1} q_{\mathrm{Pi}_1} + \Pi_{\mathrm{Pi}_2} q_{\mathrm{Pi}_2}, \\ &\qquad\qquad A_{\mathrm{Jc}_1,\mathrm{Pi}}^T p_{\mathrm{Jc}_1} + A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}^T p_{\mathrm{Jc}_2} + A_{\mathrm{Re},\mathrm{Pi}}^T \bar{p}_{\mathrm{Re}}\Big). \end{aligned} \tag{4.9}$$

Replacing the equation $\tilde{F}_{\dot{\bar{\mathrm{J}}}\mathrm{c}}(t, \tilde{x}) = 0$ in (4.5) by (4.9), we find that the solution of (4.4) solves

$$\bar{F}(t, \tilde{x}, \dot{\tilde{x}}) = 0, \qquad \tilde{x}(t_0) = \tilde{x}_0,$$

where $\bar{F} = [\bar{F}_1^T, \bar{F}_{2,pres}^T, \bar{F}_{2,enth}^T, F_{bound}^T]^T$ is given by

$$\bar{F}_1 := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}_1} \\ \tilde{F}_{\mathrm{Pi}_2} \\ \tilde{F}_{\mathrm{Jc}_V} \end{bmatrix}, \quad \bar{F}_{2,pres} := \begin{bmatrix} \tilde{F}_{\mathrm{Pu}_1} \\ \tilde{F}_{\bar{\mathrm{P}}\mathrm{u}} \\ \tilde{F}_{\mathrm{Jc}_1} \\ \tilde{F}_{\bar{\mathrm{J}}\mathrm{c}} \end{bmatrix}, \quad \bar{F}_{2,enth} := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}*} \\ \tilde{F}_{\mathrm{Pu}*} \\ \tilde{F}_{\mathrm{Jc}_0^*} \end{bmatrix}.$$

Reversing the variable transformation and combining the pump and junction equations by V, U using a nonsingular transformation $\bar{S}$, we verify that x solves

$$\bar{S}^{-1}\bar{F}(t, S^{-1}x, S^{-1}\dot{x}) = F(t, x, \dot{x}).$$

Hence, if $x \in C^1(\hat{\mathcal{J}}, \mathbb{R}^{2n})$ solves (4.4) with $(t_0, x_0) \in \mathcal{C}_{IV}$, then x solves (4.3).

To prove that (4.4) possesses a unique solution for every $(t_0, x_0) \in \mathcal{C}_{IV}$, we decouple (4.4) using the transformations (3.5) into an explicit system to which we can apply classical ODE theory and the Implicit Function Theorem. Considering again the transformed system (4.5), we observe that the Jacobian $\partial_{\tilde{x}_a}\tilde{F}_2$ of (4.5) with respect to $\tilde{x}_a$ is given by

$$\partial_{\tilde{x}_a}\tilde{F}_2 = \begin{bmatrix} \partial_{\tilde{x}_a}\tilde{F}_{2,11} & * & * \\ 0 & \partial_{\tilde{x}_a}\tilde{F}_{2,12} & * \\ 0 & 0 & I_{2n_{\mathrm{De}}+2n_{\mathrm{Re}}} \end{bmatrix},$$

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where

$$\partial_{\tilde{x}_a}\tilde{F}_{2,11} = \begin{bmatrix} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}\,\mathrm{D}_2 f_{\mathrm{Pi}} A^T_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}} & A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}\,\mathrm{D}_2 f_{\mathrm{Pi}} A^T_{\mathrm{Jc}_1,\mathrm{Pi}} & 0 & 00 & A^T_{\mathrm{Jc}_1,\mathrm{Pu}_1} & 0 & 0 \\ 0 & 0 & I_{n_{\mathrm{Pi}}} & 0 & & & \\ 0 & 0 & 0 & I_{n_{\mathrm{Pu}}} & & & \end{bmatrix},$$

$$\partial_{\tilde{x}_a}\tilde{F}_{2,12} = \begin{bmatrix} B & A_{\mathrm{Jc},\mathrm{Pu}}\,\mathrm{D}_1 f_{\mathrm{Pu}*} V_2 & A_{\mathrm{Jc},\mathrm{Pu}}\,\mathrm{D}_1 f_{\mathrm{Pu}*}\Pi_{\mathrm{Pu}_1} & & A_{\mathrm{Jc},\mathrm{Pi}}\,\mathrm{D}_1 f_{\mathrm{Pi}*}\Pi_{\mathrm{Pi}_1} \\ 0 & -D & -V_2^T\,\mathrm{D} f_{\mathrm{Pu}}\Pi_{\mathrm{Pu}_1} & 0 & \\ 0 & 0 & A_{\mathrm{Jc}_1,\mathrm{Pu}_1} & A_{\mathrm{Jc}_1,\mathrm{Pi}_1} & \\ 0 & 0 & 0 & A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1} & \end{bmatrix}.$$

On $\mathcal{C}_{IV}$, the diagonal entries of $\partial_{\tilde{x}_a}\tilde{F}_{2,11}, \partial_{\tilde{x}_a}\tilde{F}_{2,12}$ are pointwise nonsingular such that $\partial_{\tilde{x}_a}\tilde{F}_2$ is pointwise nonsingular on $\mathcal{C}_{IV}$. For $(t_0, \tilde{x}_0)$ with $(t_0, S\tilde{x}_0) \in \mathcal{C}_{IV}$, we can thus solve the algebraic equation in (4.5) locally for $\tilde{x}_a$ as a function of $\tilde{x}_d$, cf. [21]. With $\tilde{F} \in C^1(\tilde{\mathbb{D}}, \mathbb{R}^n)$, there exist neighborhoods $\mathcal{I}_0 \times \mathcal{U}(q_{2,0}) \times \mathcal{U}(\tilde{x}_{a,0}) \subset \mathcal{I} \times \Omega_{\tilde{x}}$ and a function $g \in C^1(\mathcal{I}_0 \times \mathcal{U}(q_{2,0}), \mathcal{U}(\tilde{x}_{a,0}))$ such that $(t, \tilde{x})$ solves $\tilde{F}_2(t, \tilde{x}) = 0$ if and only if $\tilde{x}_a = g(t, \tilde{x}_d)$. Setting

$$f(t, x_d) := \tilde{F}_1(t, [\tilde{x}_d^T, g^T(t, \tilde{x}_d)]^T, \dot{x}_d) + \dot{x}_d,$$

it follows that a function $\tilde{x} \in C^1(\hat{\mathcal{J}}, \mathbb{R}^{2n})$ solves (4.5) and if and only if $\tilde{x}$ solves the explicit system

$$\dot{\tilde{x}}_d = f(t, \tilde{x}_d), \qquad \tilde{x}_d(t_0) = \tilde{x}_{d,0} \tag{4.10a}$$
$$\tilde{x}_a = g(t, x_d). \tag{4.10b}$$

As $g \in C^1(\mathcal{I}_0 \times \mathcal{U}(q_{2,0}), \mathcal{U}(\tilde{x}_{a,0}))$ and $\tilde{F}_1 \in C^1(\mathcal{I} \times \tilde{\Omega}_x \times \mathbb{R}^d, \mathbb{R}^d)$, the composition satisfies $f \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathbb{R}^{n_{\mathrm{Pi}}})$. Hence, for every initial value $(t_0, x_{d,0}) \in \mathcal{I}_0 \times \mathcal{U}(x_{d,0})$, (4.10a) has a unique, maximally extended solution $x_d \in C^2((t^-_{0,x_d}, t^+_{0,x_d}), \mathbb{R}^d)$, cf. [1]. Then, (4.10b) has a unique solution $\tilde{x}_a \in C^1(\mathcal{I}_{x_a}, \mathbb{R}^d)$, where $\mathcal{I}_{x_a} := \mathcal{I}_0 \cap (t^-_{0,x_d}, t^+_{0,x_d})$. In t_0, in particular, we have $\tilde{x}_a(t_0) = g(t_0, q_{\mathrm{Pi},2,0})$. Setting

$$\mathcal{C}_{sexp} := \{(t_0, \tilde{x}_d, \tilde{x}_a) \in \mathcal{I}_0 \times \mathcal{U}(q_{2,0}) \times \mathcal{U}(\tilde{x}_{a,0}) \,|\, q_{\mathrm{Pi},2,0} \in \mathcal{U}(q_{2,0}), \tilde{x}_a(t_0) = g(t_0, q_{\mathrm{Pi},2,0})\},$$

and $(t_0^-, t_0^+) := (t^-_{0,x_d}, t^+_{0,x_d}) \cap \mathcal{I}_{x_a}$, it follows that (4.10) is uniquely solvable for every $(t_0, \tilde{x}_0) \in \mathcal{C}_{sexp}$ with $x = [{x_d}^T, {x_a}^T]^T$ such that $x_d \in C^2(\mathcal{J}, \mathbb{R}^{n_{\mathrm{Pi}}})$, $x_a \in C^1(\mathcal{J}, \mathbb{R}^{n-n_{\mathrm{Pi}}})$. As a function $x \in C^1(\mathcal{J}, \mathbb{R}^n)$ solves the surrogate model (4.4) if and only if its transformation $\tilde{x} = S^{-1}x$ solves the explicit system (4.10) and noting that

$$\mathcal{C}_{IV} = \{(t_0, x_0) \in \mathcal{I}_0 \times \mathbb{D}_x \,|\, (t_0, S^{-1}x_0) \in \mathcal{C}_{sexp}\}$$

by the construction of (4.10), it follows that the surrogate model is uniquely solvable on $\mathcal{C}_{IV}$ with $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$.

Now, we show that every solution $x_d \in C^2(\mathcal{J}, \mathbb{R}^d)$, $x_a \in C^1(\mathcal{J}, \mathbb{R}^a)$, $\mathcal{J} \subset \mathcal{I}$, of (4.3) with $(t_0, x_0) \in \mathcal{C}_{IV}$ also solves the surrogate model (4.4) on $\mathcal{J}$. We consider the *derivative* *array* $\mathcal{F} := [F^T, \dot{F}^T]^T$ of size $\mu = 1$ with F given by (4.1) and

$$\dot{F}_{pres} := \frac{d}{dt} F_{pres} = \begin{bmatrix} \ddot{q}_{\text{Pi}} - \text{D}_1 f_{\text{Pi}} \dot{q}_{\text{Pi}} - \text{D}_2 f_{\text{Pi}} (A^T_{\text{Jc,Pi}} \dot{p}_{\text{Jc}} + A^T_{\text{Re,Pi}} \dot{p}_{\text{Re}}) \\ A^T_{\text{Jc,Pu}} \dot{p}_{\text{Jc}} + A^T_{\text{Re,Pu}} \dot{p}_{\text{Re}} - \text{D}_1 f_{\text{Pu}} \dot{q}_{\text{Pu}} \\ A_{\text{Jc,Pi}} \dot{q}_{\text{Pi}} + A_{\text{Jc,Pu}} \dot{q}_{\text{Pu}} + A_{\text{Jc,De}} \dot{q}_{\text{De}} \end{bmatrix},$$

$$\dot{F}_{enth} := \frac{d}{dt} F_{enth} = \begin{bmatrix} V_{\text{Jc}} \ddot{h}_{\text{Jc}} - A_{\text{Jc,Pi}} \dot{H}_{\text{Pi}} - A_{\text{Jc,Pu}} \dot{H}_{\text{Pu}} - A_{\text{Jc,De}} \dot{H}_{\text{De}} \\ -A_{\text{Jc,Pi}} \dot{H}_{\text{Pi}} - A_{\text{Jc,Pu}} \dot{H}_{\text{Pu}} - A_{\text{Jc,De}} \dot{H}_{\text{De}} \\ \dot{H}_{\text{Pi}} - \text{D}_1 f_{\text{Pi}*} \dot{q}_{\text{Pi}} - \text{D}_2 f_{\text{Pi}*} \dot{h} \\ \dot{H}_{\text{Pu}} - \text{D} f_{\text{Pu}} \dot{q}_{\text{Pu}} - \text{D}_1 f_{\text{Pu}} \dot{h} \end{bmatrix},$$

$$\dot{F}_{enth} := \frac{d}{dt} F_{enth} = \begin{bmatrix} \dot{q}_{\text{De}} - \dot{\bar{q}}_{\text{De}} \\ \dot{p}_{\text{Re}} - \dot{\bar{p}}_{\text{Re}} \\ \dot{H}_{\text{De}} - \dot{\bar{H}}_{\text{De}} \\ \dot{h}_{\text{Re}} - \dot{\bar{h}}_{\text{Re}} \end{bmatrix}.$$

We consider the *algebraic solution set* $\mathcal{F}^{-1}(0) = \{z \in \mathbb{R}^{6n+1} \mid \mathcal{F}(z) = 0\}$, i.e., the set of all vectors $z = (t, x, v, w)$ that satisfy $\mathcal{F}(z) = 0$ in the algebraic sense without a differential relation between the components and denote the set of initial values (t_0, x_0) that are part of a vector $(t_0, x_0, v_0, w_0) \in \mathcal{F}^{-1}(0)$ by

$$\mathcal{C}_1 := \{(t_0, x_0) \in \Omega_x \mid \exists (v_0, w_0) \in \Omega_x \times \mathbb{R}^n \colon (t_0, x_0, v_0, w_0) \in \mathcal{F}^{-1}(0)\}.$$

As every solution $x_d \in C^2(\mathcal{J}, \mathbb{R}^d)$, $x_a \in C^1(\mathcal{J}, \mathbb{R}^a)$, $\mathcal{J} \subset \mathcal{I}$, of (4.4) with $(t_0, x_0) \in \mathcal{C}_{IV}$ solves (4.3) and hence the derivative array satisfies $\mathcal{F}(t, x, \dot{x}, \ddot{x}) = 0$, it follows that $\mathcal{C}_{IV} \subset \mathcal{C}_1$. In particular, this implies that $\mathcal{F}^{-1}(0) \neq \emptyset$.

Considering the Jacobians $M(z) := \partial_{v,w} \mathcal{F}(z)$, $N(z) := \partial_x \mathcal{F}(z)$, with the argument $z = (t, x, v, w) \in \mathcal{F}^{-1}(0)$, we first show that $\partial_{v,w} M(z) = 0$, $\partial_{v,w} N(z) = 0$, implying that $M(z) = M(x)$ and $N(z) = N(x)$. Then, we prove that $M(x), N(x)$ satisfy the following rank assumptions for $(t, x) \in \mathcal{C}_{IV}$:

(i) $a := \operatorname{corank}(M(x)) = n_{\text{Pi}} + 2n_{\text{Pu}} + 2n_{\text{Jc}_V} + 3n_{\text{Jc}_0} + 2n_{\text{De}} + 2n_{\text{Re}} - \hat{k}$,
(ii) $\operatorname{rank}(Z_2^T N(z)) = a$, where $Z_2 \in \mathbb{R}^{n \times a}$ is a basis of $\operatorname{coker}(M(x))$,
(iii) $\operatorname{rank}(\partial_v F(z) T_2) = d$, where $T_2 \in \mathbb{R}^{a \times d}$ is a basis of $\ker(N)$ and $d := 2n - a$.

By [18, Theorem 4.11], it follows that every solution $x \in C^1(\hat{\mathcal{J}}, \mathbb{R}^{2n})$, $\hat{\mathcal{J}} \subset \mathcal{I}$, of (4.3) with $(t_0, x_0) \in \mathcal{C}_1$ solves the surrogate model (4.4) on $\hat{\mathcal{J}}$.

To verify the assumptions in items (i)–(iii), we transform the Jacobians M, N by nonsingular transformations constructed from the matrices U, V, Π_{Pi} defined in (3.5). For (i), we transform the Jacobian M by nonsingular transformations $\bar{\Pi}_M, \Pi_M \in \mathbb{R}^{4n \times 4n}$ such that

$$\bar{\Pi}_M^T M \Pi_M = \begin{bmatrix} I_{n_{\text{Pi}} + n_{\text{Jc}_V} + 2n_{\text{De}} + 2n_{\text{Re}}} & 0 & 0 & 0 \\ * & \tilde{M}_{22} & \tilde{M}_{24} & 0 \\ * & \tilde{M}_{32} & 0 & 0 \\ \tilde{M}_{41} & 0 & 0 & 0 \end{bmatrix},$$

where

$$\tilde{M}_{22} = \begin{bmatrix} A_{\mathrm{Jc}_1,\mathrm{Pu}_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -V_2^T \,\mathrm{D} f_{\mathrm{Pu}} V_1 & -D & 0 & 0 & 0 & 0 & 0 \\ -V_1^T \,\mathrm{D} f_{\mathrm{Pu}} V_1 & -V_1^T \,\mathrm{D} f_{\mathrm{Pu}} V_2 & A^T_{\mathrm{Jc}_1,\mathrm{Pu}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{D}_2 f_{\mathrm{Pi}} A^T_{\mathrm{Jc}_1,\mathrm{Pi}} & I_{\mathrm{Pi}} & 0 & 0 & 0 \\ -\mathrm{D} f_{\mathrm{Pu}*} V_1 & -\mathrm{D} f_{\mathrm{Pu}*} V_2 & 0 & 0 & I_{\mathrm{Pu}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Pi}} & 0 \\ 0 & 0 & 0 & 0 & -A_{\mathrm{Jc}_V,\mathrm{Pu}} & -A_{\mathrm{Jc}_V,\mathrm{Pi}} & V_{\mathrm{Jc}} \end{bmatrix},$$

$$\tilde{M}_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ V_1^T \dot{\mathrm{D}} F_{\mathrm{Pu}} & 0 & 0 & 0 & 0 \\ -\mathrm{D}_3 f_{\mathrm{Pi}} & 0 & \mathrm{D}_2 f_{\mathrm{Pi}} A^T_{\mathrm{Jc}_2,\mathrm{Pi}} & 0 & \\ -B_{\mathrm{Jc}_0}(q_{\mathrm{Pu}}) & 0 & 0 & 0 & \\ -B_{\mathrm{Jc}_0}(q_{\mathrm{Pi}}) & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{bmatrix},$$

$$\tilde{M}_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 & -A_{\mathrm{Jc}_0,\mathrm{Pu}} & -A_{\mathrm{Jc}_0,\mathrm{Pi}} & 0 \end{bmatrix},$$

$$\tilde{M}_{41} = \begin{bmatrix} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{De}} & 0 & 0 & 0 & A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the choice of U_1, V_1 and the assumption on D, the diagonal block $\tilde{M}_{22}(x)$ is nonsingular on $\mathcal{C}_{IV}$, implying that

$$\mathrm{rank}(M(x)) = n_{\mathrm{Pi}} + n_{\mathrm{Jc}_V} + 2n_{\mathrm{De}} + 2n_{\mathrm{Re}} + \mathrm{rank}(\tilde{M}_{22}) + \mathrm{rank}(\mathcal{S}_{11}(\bar{\Pi}_M^T M \Pi_M)(z))$$

on $\mathcal{C}_{IV}$, cf. (2.7), where the Schur complement is given by

$$\mathcal{S}_{11}(\bar{\Pi}_M^T M \Pi_M)(z) = -(\tilde{M}_{32}\tilde{M}_{22}^{-1}\tilde{M}_{24})(z) = \begin{bmatrix} B & 0 & 0 & 0 \end{bmatrix}.$$

As Jc_0 is enthalpy reachable on $\mathcal{C}_{IV}$, the matrix B is pointwise nonsingular on $\mathcal{C}_{IV}$ with $\mathrm{rank}(B(z)) = n_{\mathrm{Jc}_0}$, cf. Lemma 3.2, and it follows that

$$\mathrm{rank}(M(x)) = 3(n_{\mathrm{Pi}} + n_{\mathrm{Jc}_V}) + (n_{\mathrm{Pu}} + n_{\mathrm{Jc}_0} + n_{\mathrm{De}} + n_{\mathrm{Re}}) - \hat{k}$$

and $a = \mathrm{corank}(M(x))$ on $\mathcal{C}_{IV}$.

For (ii), we exploit the structure of $\bar{\Pi}_M^T M \Pi_M$ to construct a basis $Z_2 \in \mathbb{R}^{4n\times a}$ of $\mathrm{corange}(M(x))$. Setting

$$Z_2^T = \begin{bmatrix} -\tilde{M}_{41} & 0 & 0 & I_a \end{bmatrix} \bar{\Pi}_M^T, \tag{4.11}$$

we find that $\mathrm{span}(Z_2) = \mathrm{corange}(M(x))$ for every $x \in \mathcal{C}_{IV}$. Applying Z_2 and a suitable transformation $\Pi_N \in \mathbb{R}^{2n\times 2n}$ to the Jacobian N, we get that

$$\bar{\Pi}_N^T Z_2^T N \Pi_N(z) = \begin{bmatrix} I_{2n_{\mathrm{De}}+2n_{\mathrm{Re}}} & 0 & 0 \\ * & \tilde{N}_{22} & \tilde{N}_{23} \\ * & \tilde{N}_{32} & 0 \end{bmatrix},$$

where

$$\tilde{N}_{22} = \begin{bmatrix} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\mathrm{Jc}_1,\mathrm{Pi}_1} & A_{\mathrm{Jc}_1,\mathrm{Pu}_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & V_2^T \operatorname{D} f_{\mathrm{Pu}} V_1 & D & 0 & 0 & 0 & 0 \\ 0 & V_1^T \operatorname{D} f_{\mathrm{Pu}} V_1 & V_1^T \operatorname{D} f_{\mathrm{Pu}} V_2 & A_{\mathrm{Jc}_1,\mathrm{Pu}_1}^T & 0 & 0 & 0 \\ A_{\bar{\mathrm{Jc}},\mathrm{Pi}} \operatorname{D}_1 f_{\mathrm{Pi}} \Pi_1 & 0 & 0 & \tilde{\operatorname{D}}_2 f_{\mathrm{Pi}} & C & 0 & 0 \\ 0 & -\operatorname{D} f_{\mathrm{Pu}*} V_1 & -\operatorname{D} f_{\mathrm{Pu}*} V_2 & 0 & 0 & I_{\mathrm{Pu}} & 0 \\ -\operatorname{D} f_{\mathrm{Pi}*} \Pi_1 & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Pi}} \end{bmatrix},$$

$$\tilde{N}_{23} = \begin{bmatrix} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_2} & 0 & 0 \\ A_{\mathrm{Jc}_1,\mathrm{Pi}_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{\bar{\mathrm{Jc}},\mathrm{Pi}} \operatorname{D}_1 f_{\mathrm{Pi}} \Pi_2 & A_{\bar{\mathrm{Jc}},\mathrm{Pi}} \partial_{h_{\mathrm{Jc},V}} f_{\mathrm{Pi}} & A_{\bar{\mathrm{Jc}},\mathrm{Pi}} \operatorname{D}_3 f_{\mathrm{Pi}} \\ 0 & -B_{\mathrm{Jc}_V}(q_{\mathrm{Pu}}) & -B_{\mathrm{Jc}_0}(q_{\mathrm{Pu}}) \\ -\operatorname{D} f_{\mathrm{Pi}*} \Pi_2 & -B_{\mathrm{Jc}_V}(q_{\mathrm{Pi}}) & -B_{\mathrm{Jc}_0}(q_{\mathrm{Pi}}) \end{bmatrix},$$

$$\tilde{N}_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_{\mathrm{Jc}_0,\mathrm{Pu}} & -A_{\mathrm{Jc}_0,\mathrm{Pi}} \end{bmatrix}.$$

As $n_{\mathrm{Re}} > 0$, the matrix C is nonsingular, cf. Lemma 3.2. By the choice of $\Gamma_1, \mathrm{Pi}_{\mathrm{Pu}_1}$, and the assumptions on D, the diagonal block $\tilde{N}_{22}(z)$ is pointwise nonsingular on $\mathcal{C}_{IV}$ with $\operatorname{rank}(\tilde{N}_{22}(z)) = n_{\mathrm{Jc}} + 2n_{\mathrm{Pu}} + n_{\mathrm{Pi}} + \hat{k}$. Hence,

$$\operatorname{rank}(Z_2^T N) = 2(n_{\mathrm{De}} + n_{\mathrm{Re}}) + \operatorname{rank}(\tilde{N}_{22}) + \operatorname{rank}(\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23}).$$

Noting that $\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23} = [*, *, B^T]^T$, we have verified that $\operatorname{rank}(Z_2^T N(x)) = a$ on $\mathcal{C}_{IV}$.

For (iii), we exploit the structure of $Z_2^T N(x)\Pi_N$ and construct a basis $T_2 \in C(\mathbb{D}, \mathbb{R}^{n\times d})$ of $\ker(Z_2^T N(x))$, where $d = 2n - a = n_{\mathrm{Pi}} - n_{\mathrm{Jc}} + \hat{k}_2$. Choosing $X_3 \in C^1(\mathcal{I}\times\Omega_x, \mathbb{R}^{n_{\mathrm{Pi}}\times d})$ with $\operatorname{span}(X_3) = \ker(\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23})$ and setting

$$T_2 = \Pi_N^T \begin{bmatrix} 0 & -\tilde{N}_{22}^{-1}\tilde{N}_{23} & I_{n_{\mathrm{Pi}}} \end{bmatrix}^T X_3,$$

we have $\operatorname{span}(T_2(x)) = \ker(Z_2^T N(x))$ for every $x \in \mathcal{C}_{IV}$. Then, we find that

$$F_{\dot{q},\dot{p}} T_2 = \begin{bmatrix} (\Pi_{\mathrm{Pi}_2} - \Pi_{\mathrm{Pi}_1} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_2})^T & 0 & 0 & 0 \\ 0 & 0 & I_{n_{\mathrm{Jc}_V}} & 0 \end{bmatrix}^T.$$

Noting that

$$\Pi_{\mathrm{Pi}_2} - \Pi_{\mathrm{Pi}_1} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_2} = \Pi_{\mathrm{Pi}} \begin{bmatrix} -A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_2} \\ I_d \end{bmatrix},$$

it follows that $\operatorname{rank}((F_{\dot{x}} T_2)(z)) = d$ for $z = (t, x, v) \in \mathcal{F}^{-1}(0)$ with $(t, x) \in \mathcal{C}_{IV}$. Setting

$$Z_1 := \begin{bmatrix} \Pi_{\mathrm{Pi}_2}^T & 0 & 0 & 0 \\ 0 & I_{n_{\mathrm{Jc}_V}} & 0 & 0 \end{bmatrix}^T, \tag{4.12}$$

we have verified that $\operatorname{rank}((Z_1^T F_{\dot{x}} T_2)(z)) = d$ for $z = (t, x, v) \in \mathcal{F}^{-1}(0)$ with $(t, x) \in \mathcal{C}_{IV}$. Hence, the network model (4.3) satisfies the assumptions (i)–(iii), implying that every sufficiently smooth solution of (4.3) with $(t_0, x_0) \in \mathcal{C}_1$ solves the surrogate model

$$\begin{aligned} Z_1^T F(t, x, \dot{x}) &= 0, \qquad x(t_0) = x_0, \\ Z_2^T \mathcal{F}(t, x, \dot{x}, \ddot{x}) &= 0, \end{aligned}$$

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cf. [19, Theorem 4.11]. With Z_1, Z_2 given by (4.12), (4.11), we get $\hat{F}_1 := Z_1^T F$, $\hat{F}_2 := Z_2^T \mathcal{F}$.

As $f_{\mathrm{Pi}} \in C^1(\Omega_{\mathrm{Pi}} \times (-\infty, \infty)^{n_{\mathrm{Jc}}+n_{\mathrm{Re}}}, \mathbb{R}^{n_{\mathrm{Pi}}})$, every solution $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$ of (4.4) satisfies $x_d \in C^2(\mathcal{J}, \mathbb{R}^{n_{\mathrm{Pi}}})$, $x_a \in C^1(\mathcal{J}, \mathbb{R}^{n-n_{\mathrm{Pi}}})$, and, for $(t_0, x_0) \in \mathcal{C}_{IV}$, a function $x \in C^1(\mathcal{J}, \mathbb{R}^{2n})$ solves (4.3) if and only if x solves (4.4). ∎

Note that the smoothness of the algebraic components x_a depends on the smoothness of the pump function.

Translated as conditions on the network structure and its elements, the solvability conditions of Theorem 4.1 mean that a reservoir is required as reference value for the pressure p_{Jc} and the enthalpy h_{Jc_0} in the virtual connection points Jc_0. Furthermore, on fundamental cycles and crossing paths as well as in isolated pumps of $\mathcal{G}_{\mathrm{Jc,Pu}}$, the pumps must be able to adjust the mass flow to a given pressure difference. That is, because the transfer elements (the pipes and pumps) only specify the pressure difference, a reservoir is needed as reference value for the pressure p_{Jc}, thus, in every connected component there needs to be a reservoir. Similarly, in the virtual connection points Jc_0, the enthalpy h_{Jc_0} is computed by inserting the pipe and pump equations into the energy balance. Here as well, only the enthalpy difference is specified, so in order to obtain a unique solution, we need a reference value. As the enthalpy flow depends on the direction of the mass flow, these virtual connection points need to be *strongly connected* to a reservoir.

Usually, pumps return a pressure difference for a given mass flow. On structures of $\mathcal{G}_{\mathrm{Jc,Pu}}$ where the pressure difference vanishes, however, the pumps have to work the other way round, which, mathematically, is reflected by the nonsingularity condition on the matrix D. We illustrate this by an example.

EXAMPLE 4.2. We consider pumps $\mathrm{Pu}_1, \mathrm{Pu}_2, \mathrm{Pu}_3$ connected to a cycle that is connected to a demand De. The network model (4.3a) reads

$$\begin{aligned} p_{\mathrm{Jc},2} - p_{\mathrm{Jc},1} &= f_{\mathrm{Pu},1}(q_{\mathrm{Pu},1}), \quad q_{\mathrm{Pu},1} = q_{\mathrm{Pu},2}, \\ p_{\mathrm{Jc},3} - p_{\mathrm{Jc},2} &= f_{\mathrm{Pu},2}(q_{\mathrm{Pu},2}), \quad q_{\mathrm{Pu},2} = q_{\mathrm{Pu},3}, \\ p_{\mathrm{Jc},1} - p_{\mathrm{Jc},3} &= f_{\mathrm{Pu},3}(q_{\mathrm{Pu},3}), \quad q_{\mathrm{Pu},3} = q_{\mathrm{Pu},1} + q_{\mathrm{De}}, \quad q_{\mathrm{De}} = \bar{q}_{\mathrm{De}}. \end{aligned} \tag{4.13}$$

From the mass balances, we get that $\bar{q}_{\mathrm{De}} \equiv 0$ and $q_{\mathrm{Pu},1} = q_{\mathrm{Pu},2} = q_{\mathrm{Pu},3}$. In combination with the pump equations, it follows that

$$f_{\mathrm{Pu},1}(q_{\mathrm{Pu},1}) + f_{\mathrm{Pu},2}(q_{\mathrm{Pu},1}) + f_{\mathrm{Pu},3}(q_{\mathrm{Pu},1}) = 0.$$

Hence, the input $\bar{q}_{\mathrm{De}}$ is not freely choosable, and (4.13) is locally solvable for $q_{\mathrm{Pu},1,0} \in \mathbb{R}$ if and only if $\sum_{j=1}^{3} \mathrm{D} f_{\mathrm{Pu},j}(q_{\mathrm{Pu},1,0})$ is nonsingular. However, as the pump equations only specify the pressure difference, the DAE (4.13) will not be uniquely solvable unless the model is connected to a reservoir.

Similarly, coupling two pumps $\mathrm{Pu}_1, \mathrm{Pu}_2$ between two reservoirs $\mathrm{Re}_1, \mathrm{Re}_2$, we obtain the system

$$\begin{aligned} p_{\mathrm{Jc},1} - p_{\mathrm{Re},1} &= f_{\mathrm{Pu},1}(q_{\mathrm{Pu},1}), \quad q_{\mathrm{Pu},1} = q_{\mathrm{Pu},2}, \\ p_{\mathrm{Re},2} - p_{\mathrm{Jc},1} &= f_{\mathrm{Pu},2}(q_{\mathrm{Pu},2}), \end{aligned} \tag{4.14}$$

and observe that (4.14) is locally solvable if and only if $\sum_{j=1}^{2} \mathrm{D} f_{\mathrm{Pu},j}(q_{\mathrm{Pu},1,0})$ is nonsingular for $q_{\mathrm{Pu},1,0} \in \mathbb{R}$.

In order to avoid the verification whether the pump function satisfies this solvability condition, i.e., to avoid the test if D is nonsingular, the considered network can be restricted to

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those in which pumps are coupled to a cycle or to those where paths between two reservoirs do not occur.

LEMMA 4.3. *Let $\mathcal{N}$ be a network given by (3.1) that satisfies Assumptions 3.1. Let $F \in C^1(\mathbb{D}, \mathbb{R}^n)$ be the associated network function. If $n_{Re} > 0$, Jc_0 is enthalpy reachable, and $\ker(A_{Jc,Pu}) = \{0\}$, then the assertions of Theorem 4.1 are satisfied.*

Proof. If $\ker(A_{\mathrm{Jc,Pu}}) = \{0\}$, then V_2 is the empty matrix, and the solvability condition of Theorem 4.1 is automatically satisfied. ∎

On the structural level, the condition $\ker(A_{\mathrm{Jc,Pu}}) = \{0\}$ means that in every cycle of pumps and every path of pumps between two reservoirs, there is at least one pipe.

EXAMPLE 4.4. In Example 4.2, replacing, e.g., the pump Pu_3 by a pipe Pi_3, we obtain the network DAE

$$
\begin{aligned}
p_{\mathrm{Jc},2} - p_{\mathrm{Jc},1} &= f_{\mathrm{Pu},1}(q_{\mathrm{Pu},1}), & q_{\mathrm{Pu},1} &= q_{\mathrm{Pu},2}, \\
p_{\mathrm{Jc},3} - p_{\mathrm{Jc},2} &= f_{\mathrm{Pu},2}(q_{\mathrm{Pu},2}), & q_{\mathrm{Pu},2} &= q_{\mathrm{Pi},3}, \\
\dot{q}_{\mathrm{Pi},3} &= f_{\mathrm{Pi},3}(q_{\mathrm{Pi},3}, p_{\mathrm{Jc},1} - p_{\mathrm{Jc},3}), & q_{\mathrm{Pi},3} &= q_{\mathrm{Pu},1} + q_{\mathrm{De}}, \quad q_{\mathrm{De}} = \bar{q}_{\mathrm{De}}.
\end{aligned} \tag{4.15}
$$

The system (4.15) can be solved for $q_{\mathrm{Pu},1}, q_{\mathrm{Pu},2}, q_{\mathrm{Pi},3}$ and, e.g., $p_{\mathrm{Jc},1}, p_{\mathrm{Jc},2}$, in dependency of the reference pressure $p_{\mathrm{Jc},3}$ by simply evaluating the pump equations; there is no need to invert the pump functions. Similarly, in the second example, replacing, e.g., the pump Pu_2 by a pipe Pi_2, we obtain a solvable system.

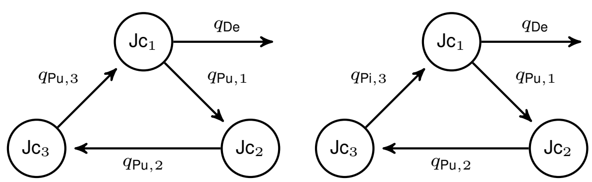

FIG. 4.1. *Pump constellations of Example 4.2 and Example 4.4 that trigger the solvability condition "D is nonsingular" (left) and constellations that avoid this condition (right).*

In conclusion, if pumps are present in the network, the solvability condition can be either imposed on the element level, claiming that D is pointwise nonsingular on $\mathcal{C}_{IV}$, or, in order to ensure that the model works for every pump specification, they can be imposed on the structural level. Depending on the desired modeling freedom, one can choose between these two options.

If the solvability conditions are satisfied and the network is plausible, the next step is to simulate the dynamics of $\mathcal{N}$. The DAE (4.3a) assembled by gluing together the element equations (3.2) using the incidence matrix, however, is not suitable for a numerical simulation as it contains hidden equations and does not reflect the number of differential and algebraic variables correctly.

While the pressure differences p_{Jc_1} associated with $\mathrm{range}(A_{\mathrm{Jc,Pu}})$ are uniquely specified from the pump equations, the pressures p_{Jc_2} in the ground nodes Jc_2 are associated with $\mathrm{coker}(A_{\mathrm{Jc,Pu}})$ and thus do not receive a pressure value from a pump. Instead, the pressures p_{Jc_2} are specified by the *hidden constraint*

$$A_{\bar{\mathrm{Jc}},\mathrm{Pi}} f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}, h_{\mathrm{Jc}_0}, h_{\mathrm{Jc}_V}, h_{\mathrm{Re}}) - A_{\bar{\mathrm{Jc}},\mathrm{De}} \dot{\bar{q}}_{\mathrm{De}} = 0$$

arising from inserting the pipe equation, i.e., a differential equation, into the mass balance in the junctions. Claiming that $A_{\bar{\mathrm{Jc}},\mathrm{Pi}} f_{\mathrm{Pi}} A^T_{\bar{\mathrm{Jc}},\mathrm{Pi}}$ is nonsingular, this equation uniquely specifies the pressure p_{Jc_2}. To compensate for the additional equations, the surrogate model (4.4a) specifies

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only the pipe flows on the chord set Pi_2 by a differential equation, while the mass flows in pipes on the spanning tree Pi_1 are given by the mass balance $F_{\text{Jc}}(t,x) = 0$, cf. (4.2c). We illustrate this again by an example.

EXAMPLE 4.5. We consider two pipes Pi_1, Pi_2 that are coupled by a junction Jc_1, cf. Figure 4.2. For simplicity, we assume that the pipes are connected to reservoirs Re_1, Re_2. Then, we obtain the network DAE

$$\dot{q}_{\text{Pi},1} = f_{\text{Pi},1}(q_{\text{Pi},1}, p_{\text{Re},1} - p_{\text{Jc},1}), \qquad q_{\text{Pi},1}(t_0) = q_{\text{Pi},1,0}, \tag{4.16a}$$

$$\dot{q}_{\text{Pi},2} = f_{\text{Pi},2}(q_{\text{Pi},2}, p_{\text{Jc},1} - p_{\text{Re},2}), \qquad q_{\text{Pi},2}(t_0) = q_{\text{Pi},2,0}, \tag{4.16b}$$

$$q_{\text{Pi},1} = q_{\text{Pi},2}. \tag{4.16c}$$

The pipes specify the mass flows differentially while the junction relates the flows algebraically. Consequently, only one mass flow evolves dynamically; the other one is fixed algebraically by the mass balance. In particular, only one initial value can be chosen. The pressure only occurs implicitly in the differential equations. Differentiating the algebraic equation and inserting the pipe equations for the derivatives of the mass flows, however, we discover the algebraic equation

$$f_{\text{Pi},1}(q_{\text{Pi},1}, \bar{p}_{\text{Re},1} - p_{\text{Jc},1}) = f_{\text{Pi},2}(q_{\text{Pi},2}, p_{\text{Jc},1} - \bar{p}_{\text{Re},2}). \tag{4.17}$$

As $\text{D}_2(f_{\text{Pi},2} - f_{\text{Pi},1}) = c_{1,1} + c_{1,2}$ is nonsingular, (4.17) can be solved for the pressure $p_{\text{Jc},1}$, and (4.16) is uniquely solvable. Hence, coupling two pipes by a junction, the network model (4.3a) contains a hidden algebraic equation that is needed to specify the pressure in the coupling junction. Also, (4.3a) does not correctly reflect the number of differential and algebraic variables as only one mass flow evolves dynamically. Thus, we consider the surrogate model

$$\begin{aligned} \dot{q}_{\text{Pi},1} &= f_{\text{Pi},1}(q_{\text{Pi},1}, p_{\text{Re},1} - p_{\text{Jc},1}), \qquad q_{\text{Pi},1}(t_0) = q_{\text{Pi},1,0}, \\ f_{\text{Pi},1}(q_{\text{Pi},1}, p_{\text{Re},1} - p_{\text{Jc},1}) &= f_{\text{Pi},2}(q_{\text{Pi},2}, p_{\text{Jc},1} - p_{\text{Re},2}), \\ q_{\text{Pi},1} &= q_{\text{Pi},2}, \end{aligned}$$

which corresponds to (4.4).

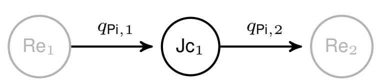

FIG. 4.2. *Network of Example 4.5.*

From the proof of Theorem 4.1, we observe that the solution of (4.3) can be computed from the explicit system (4.10). Exploiting the linearity and the triangular structure of J, we explicitly compute the function g. Using a nonsingular matrix $S \in \mathbb{R}^{2n\times 2n}$, we transform the states according to (3.6) and set

$$\tilde{x} := S^{-1}x := \left[q_{\text{Pi}_2}^T, p_{\text{Jc}_V}^T, q_{\text{Pi}_1}^T, q_{\text{Pu}_1}^T, q_{\text{Pu}_2}^T, p_{\text{Jc}_1}^T, p_{\text{Jc}_2}^T, q_{\text{De}}^T, p_{\text{Re}}^T, H_{\text{De}}^T, h_{\text{Re}}^T\right]^T.$$

We transform the domain of definition accordingly by setting $\tilde{\mathbb{D}} := \mathcal{I} \times \Omega_{\tilde{x}} \times \Omega_{\dot{\tilde{x}}}$ with $\Omega_{\tilde{x}} := S^{-1}\Omega_x$, $\Omega_{\dot{\tilde{x}}} := S^{-1}\Omega_{\dot{x}}$ and partition the state into $\tilde{x} = [\tilde{x}_d^T, \tilde{x}_a^T]^T$ with the vectors $\tilde{x}_a = [p_{\text{Jc}_2}^T, p_{\text{Jc}_1}^T, q_{\text{Pu}_2}^T, q_{\text{Pu}_1}^T, q_{\text{Pi}_1}^T, q_{\text{De}}^T, p_{\text{Re}}^T]^T$ and $\tilde{x}_d = q_{\text{Pi}_2}$.

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COROLLARY 4.6. *Let $\mathcal{N}$ be a network given by (3.1) that satisfies Assumptions 3.1. Let $F \in C^1(\mathbb{D}, \mathbb{R}^n)$ be the associated network function. If $n_{Re} > 0$, D is pointwise nonsingular on $\mathcal{C}_{IV}$, where $\operatorname{span}(V_2) = \ker(A_{Jc,Pu})$, and Jc_0 is enthalpy reachable, then a function $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$ solves (4.3a) if and only if its transformation $\tilde{x} = S^{-1}x$ solves the explicit system*

$$\dot{\tilde{x}}_d = f(t, \tilde{x}_d), \quad \tilde{x}_d(t_0) = x_{d,0}, \tag{4.18a}$$

$$\tilde{x}_a = g(t, \tilde{x}_d), \tag{4.18b}$$

where

$$\begin{aligned}
f_{Pi_2} &= f_{Pi_2}\big(g_{Pi}(q_{Pi_2}), A_{Jc,Pi}^T g_{Jc}(q_{Pi_2}) + A_{Re,Pi}^T \bar{p}_{Re}\big),\\
f_{Jc_V} &= V^{-1}\big(A_{Jc_V,Pi} f_{Pi}{*}(g_{Pi}, g_{Jc_V}, g_{Jc_0}, g_{Re}) + A_{Jc_V,Pu} f_{Pu}{*}(g_{Pi}, g_{Jc_V}, g_{Jc_0}, g_{Re})\\
&\quad + A_{Jc_V,De} g_{De}{*}\big),\\
g_{Pi}{*} &= f_{Pi}{*}(g_{Pi}(q_{Pi_2}), h_{Jc_V}, g_{Jc_0}(q_{Pi_2}, h_{Jc_V}), \bar{h}_{Re}),\\
g_{Pu}{*} &= f_{Pu}{*}(g_{Pu}(q_{Pi_2}), h_{Jc_V}, g_{Jc_0}(q_{Pi_2}, h_{Jc_V}), \bar{h}_{Re}),\\
g_{Jc_2}(q_{Pi_2}) &= -C^{-1} A_{\bar{J}c,Pi}\Big(C_2(h_{Jc_V}, g_{Jc_0}(q_{Pi_2}, h_{Jc_V}), \bar{h}_{Re}) \operatorname{diag}(g_{Pi,j}(q_{Pi_2})) g_{Pi}(q_{Pi_2})\\
&\quad + C_1 A_{Jc_1,Pi}^T g_{Jc_1}(q_{Pi_2}) + C_1 A_{Re,Pi}^T \bar{p}_{Re} + C_3\Big) - C^{-1} A_{\bar{J}c,De} \dot{\bar{q}}_{De},\\
g_{Jc_0}(q_{Pi_2}, h_{Jc_V}) &= -B^{-1}\big(A_{Jc,Pi} B_{Jc_V}(g_{Pi}(q_{Pi_2})) + A_{Jc,Pu} B_{Jc_V}(g_{Pu}(q_{Pi_2}))\big) h_{Jc_V}\\
&\quad - B^{-1}\big(A_{Jc,Pi} B_{Re}(q_{Pi})\\
&\quad + A_{Jc,Pu} B_{Re}(g_{Pu}(q_{Pi_2}))\big) \bar{h}_{Re} - B^{-1} A_{Jc,De} \bar{H}_{De},\\
g_{Jc_1}(q_{Pi_2}) &= -A_{Jc_1,\bar{P}u}^{-T} A_{\bar{J}c,\bar{P}u}^T p_{Jc_2} + A_{Jc_1,\bar{P}u}^{-T} \Pi_{Pu_1}^T f_{Pu}\big(g_{Pu}(q_{Pi_2})\big) - A_{Jc_1,\bar{P}u}^{-T} A_{Re,\bar{P}u}^T \bar{p}_{Re},\\
g_{Pu_2}(q_{Pi_2}) &= g_{Pu_2}(g_{Pu_1}(q_{Pi_2})),\\
g_{Pu_1}(q_{Pi_2}) &= -A_{Jc_1,Pu_1}^{-1} A_{Jc_1,Pi_1} g_{Pi_1}(q_{Pi_2}) - A_{Jc_1,Pu_1}^{-1} A_{Jc_1,Pi_2} q_{Pi_2} - A_{Jc_1,Pu_1}^{-1} A_{Jc_1,De} \bar{q}_{De},\\
g_{Pi_1}(q_{Pi_2}) &= -A_{\bar{J}c,Pi_1}^{-1} A_{\bar{J}c,Pi_2} q_{Pi_2} - A_{\bar{J}c,Pi_1}^{-1} A_{\bar{J}c,De} \bar{q}_{De},
\end{aligned}$$

with

$$\begin{aligned}
&g_{De} = \bar{q}_{De}, \quad g_{Re} = \bar{p}_{Re}, \quad g_{De}{*} = \bar{H}_{De}, \quad g_{Re}{*} = \bar{h}_{Re} \quad \text{and}\\
&g_{Pi}(q_{Pi_2}) := \Pi_{Pi_1} g_{Pi_1}(q_{Pi_2}) + \Pi_{Pi_2} q_{Pi_2}, \qquad g_{Pu}(q_{Pi_2}) := \Pi_{Pu_1} g_{Pu_1}(q_{Pi_2}) + V_2 g_{Pu_2}(q_{Pi_2}),\\
&g_{Jc}(q_{Pi_2}) := \Gamma_1 g_{Jc_1}(q_{Pi_2}) + U_2 g_{Jc_2}(q_{Pi_2}).
\end{aligned}$$

The function $g_{Pu_2} \in C^1(\mathcal{U}(q_{Pi_2,0}), \mathbb{R}^{n_{Pu}})$ is defined as solution of $F_{\bar{P}u}(t, \tilde{x}) = 0$.

Note that the algebraic equations (4.18b) can be solved from bottom to top such that the algebraic variables can be expressed as functions of the chord flows q_{Pi_2} and the input functions $\bar{q}_{\mathrm{De}}, \bar{p}_{\mathrm{Re}}, \bar{H}_{\mathrm{De}}, \bar{H}_{\mathrm{Re}}$.

REMARK 4.7. The solvability conditions of Theorem 4.1 are formulated on the connection structure and the elements of the network. This allows us to check the plausibility of the network in a preprocessing step using information about the incidence matrix A and the pump function f_{Pu}. If the solvability conditions are violated, the critical structures can be located in $\mathcal{N}$ and advice can be given how to modify the model to obtain a physically reasonable system.

The surrogate model (4.4a) can be assembled based on network information only. There is no need to compute (4.4a) from (4.3a) by symbolic or numerical manipulation as it is necessary for example in a general modeling language like Modelica. In a simulation, this saves computational time as the system-to-solve (4.4a) can be assembled directly from the network. Furthermore, the physical meaning of the equations and the states is preserved, i.e., in the DAE (4.4a), each equation and each variable still has a physical counterpart. Thus, errors in the simulation can be located in the network, allowing constructive error detection and handling.

REMARK 4.8. The assertions of Theorem 4.1 can be verified by showing that (4.3) has *regular* strangeness index $\mu = 1$. Therefore, we show that the rank assumptions (i)–(iii) are not only satisfied by the Jacobians M, N but also by the Jacobians $\tilde{M}(z) := \partial_{v,w}\tilde{\mathcal{F}}(z)$, $\tilde{N}(z) := \partial_x \tilde{\mathcal{F}}(z)$, where $\tilde{\mathcal{F}} = [F^T, \dot{F}^T, \ddot{F}^T]^T$ denotes the derivative array of size $\mu = 2$, cf. [19]. This, however, requires to restrict the interval $\mathcal{I}$ such that $\mathrm{sgn}(q) = const$ in order to provide the required smoothness of f_{Pi} as well as stricter smoothness assumptions on $f_{\mathrm{Pu}}, \bar{q}_{\mathrm{De}}, \bar{p}_{\mathrm{Re}}$.

5. Conclusion and outlook. This work provides a full analysis of a thermal fluid network, which is an extension of the well-studied water networks consisting of pipes solely. The analysis is based on a topological network approach, which allows us to impose conditions on the underlying network structure, represented by a graph. The provided topological solvability and index criteria in combination with efficient graph algorithm provide a powerful tool for the development of system simulation software. Anyhow, for the practical application it is important to extend those results to networks including valves and tanks, cf. the classification in [14], in order to be able to capture the whole cooling circuit. We mention, that further models for system simulation in automotive application (e.g., waste heat recovery, mobile air conditioning, lubrication systems), show up a similar network structure (with slightly modified equations). Therefore the presented analysis is representative for the latter mentioned ones.

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