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MODULUS-BASED CIRCULANT AND SKEW-CIRCULANT SPLITTING ITERATION METHOD FOR THE LINEAR COMPLEMENTARITY PROBLEM WITH A TOEPLITZ MATRIX*

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Abstract. By reformulating the linear complementarity problem involving a positive definite Toeplitz matrix as an equivalent fixed-point system, we construct a modulus-based circulant and skew-circulant splitting (MCSCS) iteration method. We also analyze the convergence of the method and show that the new method is effective by providing some numerical results.

Key words. linear complementarity problem, Toeplitz matrix, modulus-based circulant and skew-circulant splitting

AMS subject classifications. 65F10, 65Y05, 65H10

1. Introduction.

Some elastic normal contact problems are modeled by linear complementarity problems (LCPs) with Toeplitz matrices. Vollebregt [23] presented the BCCG+FAI method to solve the contact problem, Belsky [7] developed a multigrid strategy for contact problems, and Zhao in [26] proposed a full multigrid strategy combined with an active set algorithm. Wu and Li [24] presented a preconditioned modulus-based matrix multisplitting block iteration method for solving the linear complementarity problem with a symmetric positive definite Toeplitz matrix.

The modulus-based matrix splitting method, firstly presented by Bai [3], has been extensively studied due to its low computational complexity [1, 2, 5, 6, 9, 25]. The basic idea of this method is to find the numerical solution of the LCP's equivalent system of nonlinear equations, which can be expressed as the following absolute value equation (AVE)

$$Ax + B|x| = b. \tag{1.1}$$

The AVE has become the focus of attention for many scholars in the recent years. Rohn [19] obtained sufficient conditions for the existence of a unique solution, Mangasarian [13] pointed out that the AVE is in the class of NP-hard problems, and he proposed relevant solutions of AVEs when the matrices satisfy $B = -I$ (see [14, 15]). In addition, Mangasarian [16] and Prokopyev [18] discussed the relationship between the AVE and the linear complementarity problem. If B is the zero matrix, $B = 0$, then Bai, Golub, and Ng [4] established the HSS iteration method by splitting the system matrix into a Hermitian matrix plus a skew-Hermitian matrix. Based on the HSS iteration method, Salkuyeh developed a Picard-HSS iteration method to solve the AVE in [20]. For solving Toeplitz linear systems, Ng [17] established the CSCS iteration method based on a so-called circulant and skew-circulant splitting. Several kinds of circulant matrices are used as preconditioners in PCG methods to solve Toeplitz linear systems; see [8, 11, 12, 21, 22]. For the AVE (1.1) where A is a Toeplitz matrix and $B = -I$, the Picard-CSCS iteration method and the nonlinear CSCS-like iteration method were proposed in [10].

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In this paper, by reformulating the linear complementarity problem as a class of AVEs, we construct a modulus-based CSCS iteration method inspired by the Picard-CSCS iteration method. The method is based on a splitting of the system matrix into a circulant matrix and a skew-circulant matrix and solving the fixed-point equations by the Fast Fourier Transform (FFT). The new method can save a lot of computational work, and the numerical examples also show that the new method is efficient.

The remainder of this paper is organized as follows. Section 2 gives some necessary notations and lemmas. In Section 3, we establish the modulus-based CSCS iteration methods. An analysis of the optimal parameters is presented in Section 4. Numerical experiments are reported in Section 5. Finally, the paper closes with some conclusions in Section 6.

2. Preliminaries. In this section, we briefly review some necessary notations, definitions, and lemmas. A matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ is a Toeplitz matrix if it has the form

$$A = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{2-n} & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & \cdots & a_1 & a_0 & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix},$$

i.e., the matrix elements satisfy $a_{ij} = a_{i-j}$.

When B is the zero matrix and A is a Toeplitz matrix, the AVE (1.1) is reduced to the Toeplitz linear systems $Ax = b$. Ng proposed the CSCS methods to solve Toeplitz linear systems in [17], and the relevant background is provided next.

A Toeplitz matrix A has a circulant and skew-circulant splitting $A = C + S$, where

$$C = \frac{1}{2}\begin{bmatrix} a_0 & a_{-1}+a_{n-1} & \cdots & a_{2-n}+a_2 & a_{1-n}+a_1 \\ a_1+a_{1-n} & a_0 & a_{-1}+a_{n-1} & \cdots & a_{2-n}+a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2}+a_{-2} & \cdots & \cdots & a_0 & a_{-1}+a_{n-1} \\ a_{n-1}+a_{-1} & a_{n-2}+a_{-2} & \cdots & a_1+a_{1-n} & a_0 \end{bmatrix} \tag{2.1}$$

and

$$S = \frac{1}{2}\begin{bmatrix} a_0 & a_{-1}-a_{n-1} & \cdots & a_{2-n}-a_2 & a_{1-n}-a_1 \\ a_1-a_{1-n} & a_0 & a_{-1}-a_{n-1} & \cdots & a_{2-n}-a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2}-a_{-2} & \cdots & \cdots & a_0 & a_{-1}-a_{n-1} \\ a_{n-1}-a_{-1} & a_{n-2}-a_{-2} & \cdots & a_1-a_{1-n} & a_0 \end{bmatrix}. \tag{2.2}$$

Here C is a circulant matrix, and it can be diagonalized by the Fourier matrix F, i.e.,

$$FCF^H = \Lambda_C,$$

where the diagonal entries in the diagonal matrix Λ_C are the eigenvalues of the circulant matrix C and

$$F = (F)_{i,j} = \frac{1}{\sqrt{n}} e^{\frac{2\pi i j k}{n}}, \qquad 0 \le j,k \le n-1.$$

S is a skew-circulant matrix, and it also can be diagonalized in a similar way, i.e.,

$$\hat{F}S\hat{F}^H = \Lambda_S,$$

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where $\hat{F} = F\Omega$, $\Omega = \operatorname{diag}\left(1, e^{-\frac{\pi i}{n}}, \dots, e^{-\frac{(n-1)\pi i}{n}}\right)$, and the diagonal matrix Λ_S consists of the eigenvalues of the skew-circulant matrix.

Let $\lambda_k\,(k = 1, 2, \dots, n)$ and $\mu_k\,(k = 1, 2, \dots, n)$ be the eigenvalues of the circulant and the skew-circulant matrix, respectively. If the real parts of the eigenvalues of $\lambda_k(\mu_k)$ $(k = 1, 2, \dots, n)$ are positive, then it holds that the circulant matrix C (the skew-circulant matrix) is positive definite. With this approach, Ng extended the CSCS method in [17]. Specifically, given an initial guess x^0, for $k = 0, 1, 2, \dots$ until $\left\{x^k\right\}$ converges, compute

$$\begin{aligned}(\sigma I + C)\, x^{k+\frac{1}{2}} &= (\sigma I - S)\, x^k + b\\ (\sigma I + S)\, x^{k+1} &= (\sigma I - C)\, x^{k+\frac{1}{2}} + b,\end{aligned}$$

where σ is a positive constant.

Besides, the CSCS methods converge to the unique solution when the circulant and the skew-circulant splitting matrices are positive definite. Ng presented also the optimal parameters taken from reference [4].

Gu et al. [10] presented the Picard CSCS iteration method to solve the AVE (1.1) when B is the identity matrix and A is a Toeplitz matrix. That is, given an initial guess x^0 and a positive integer sequence $\{l_k\}_{k=0}^{\infty}$. When $k = 0, 1, 2, \dots$, for $l = 0, 1, \dots, l_k$, set $x^{k+1} = x^{k,l+1}$ until $\left\{x^{k+1}\right\}$ converges, where the iterates $x^{k,l+1}$ are computed by

$$\begin{aligned}(\sigma I + C)\, x^{k,l+\frac{1}{2}} &= (\sigma I - S)\, x^{k,l} + \left|x^k\right| + b,\\ (\sigma I + S)\, x^{k,l+1} &= (\sigma I - C)\, x^{k,l+\frac{1}{2}} + \left|x^k\right| + b.\end{aligned}$$

Here σ is a positive constant.

In this paper, we consider the linear complementarity problem, denoted as LCP (q, A), finding a pair of feasible complementary solution w and $z \in \mathbb{R}^n$ such that

$$w = Az + q \geq 0, \qquad z \geq 0, \qquad z^T w = 0, \tag{2.3}$$

where $A \in \mathbb{R}^{n\times n}$ is a given Toeplitz matrix and $q = (q_1, q_2, \dots, q_n)^T \in \mathbb{R}^n$.

LEMMA 2.1 (See [3]). *Given $\alpha > 0$, and let I be the identity matrix. Then problem (2.3) is equivalent to the following fixed-point problem: find $x \in \mathbb{R}^n$ such that*

$$(\alpha I + A)\, x = (\alpha I - A)\, |x| - q. \tag{2.4}$$

1. *If x is a solution of (2.4), then*

$$w = \alpha\,(|x| - x)\,, \quad z = |x| + x$$

   defines the solution pair of the problem (2.3).
2. *If the vector pairs w and z solve problem (2.3), then $x = \frac{1}{2}\left(z - \frac{w}{\alpha}\right)$ solves the fixed-point problem (2.4).*

3. The modulus-based CSCS iteration method. Let $A = C + S$. We extend the modulus-based CSCS iteration method to solve the linear complementarity problem with a Toeplitz matrix.

METHOD 1. Modulus-based CSCS Iteration Method.

Step 1: Select an arbitrary initial vector $x^0 \in \mathbb{R}^n$, and set $k = 0$.

Step 2: Set $x^{k,0} = x^k$. For $l = 0, 1, 2, \ldots, l_k - 1$, compute

$$\begin{cases} (\alpha I + \sigma I + C)\, x^{k,l+\frac{1}{2}} = (\sigma I - S)\, x^{k,l} + (\alpha I - A)\left|x^k\right| - q, \\ (\alpha I + \sigma I + S)\, x^{k,l+1} = (\sigma I - C)\, x^{k,l+\frac{1}{2}} + (\alpha I - A)\left|x^k\right| - q, \end{cases} \tag{3.1}$$

where σ is a given positive constant, and set $x^{k+1} = x^{k,l_k}$.

Step 3: Compute $z^{k+1} = |x^{k+1}| + x^{k+1}$, $w^{k+1} = Az^{k+1} + q$.

Step 4: Compute RES $= \min\left(\left|z^{k+1}\right|, \left|w^{k+1}\right|\right)$. If RES $< \varepsilon$ then stop. Otherwise, set $k := k + 1$ and return to Step 2.

In Step 2, equation (3.1) can be solved rapidly by the FFT, and the numerical experiments in Section 5 show that Method 1 becomes efficient.

Let $A = C + S$ be a circulant and a skew-circulant splitting of the matrix A, and let α, σ be given positive constants. Then the iteration of Method 1 can be reformulated as follows:

$$\begin{aligned} x^{k,l+1} &= W(\alpha,\sigma)x^{k,l} + V(\alpha,\sigma)\left((\alpha I - A)\left|x^k\right| - q\right) \\ &= (W(\alpha,\sigma))^{l_k}\, x^{k,0} + \sum_{j=0}^{l_k-1} (W(\alpha,\sigma))^j\, V(\alpha,\sigma)\left((\alpha I - A)\left|x^k\right| - q\right), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} W(\alpha,\sigma) &= (\alpha I + \sigma I + S)^{-1}(\sigma I - C)(\alpha I + \sigma I + C)^{-1}(\sigma I - S), \\ V(\alpha,\sigma) &= (\alpha + 2\sigma)(\alpha I + \sigma I + S)^{-1}(\alpha I + \sigma I + C)^{-1}. \end{aligned}$$

THEOREM 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a positive definite Toeplitz matrix and α and σ be given positive constants. Let $A = C + S$ be a splitting, where C is real positive definite and S is positive definite defined by* (2.1) *and* (2.2)*, respectively. Then the iteration sequence $\left\{z^k\right\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ generated by Method 1 converges to the unique solution z^* of* LCP(q, A) *for any initial vector $x^0 \in \mathbb{R}^n$ and any sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, provided that $l = \liminf_{k\to\infty} l_k \geq N$, where N is a positive integer satisfying*

$$\|(W(\alpha,\sigma))\|_2^s < \frac{1-\zeta}{1+\zeta} \tag{3.3}$$

for any $s > N$, where $\zeta = \left\|(\alpha I + A)^{-1}(\alpha I - A)\right\|_2 < 1$.

Proof. Let z^*, w^* be a solution pair of LCP(q, A). Then from Lemma 2.1 we have that

$$x^* = \frac{1}{2}\left(z^* - \frac{1}{\alpha}w^*\right) \qquad (\alpha > 0)$$

satisfies the fixed-point equations (2.4). It can be rewritten as follows according to the matrix splitting $A = C + S$:

$$x^* = W(\alpha,\sigma)^{l_k}x^* + \sum_{j=0}^{l_k-1} W(\alpha,\sigma)^j V(\alpha,\sigma)\left((\alpha I - A)\left|x^*\right| - q\right). \tag{3.4}$$

Set $x^{k,0} = x^k$ and $x^{k,l+1} = x^{k+1}$. From (3.2) and (3.4), we have

$$\begin{aligned} x^{k+1} - x^* &= W(\alpha,\sigma)^{l_k} \left(x^k - x^*\right) \\ &\quad + \sum_{j=0}^{l_k-1} W(\alpha,\sigma)^j V(\alpha,\sigma)(\alpha I - A)\left(\left|x^k\right| - |x^*|\right). \end{aligned} \tag{3.5}$$

Let $\alpha I + A = B(\alpha,\sigma) - C(\alpha,\sigma)$, where

$$\begin{aligned} B(\alpha,\sigma) &= \frac{1}{\alpha+2\sigma}(\alpha I + \sigma I + C)(\alpha I + \sigma I + S), \\ C(\alpha,\sigma) &= \frac{1}{\alpha+2\sigma}(\sigma I - C)(\sigma I - S). \end{aligned}$$

Then $W(\alpha,\sigma) = B(\alpha,\sigma)^{-1}C(\alpha,\sigma)$ and $V(\alpha,\sigma) = B(\alpha,\sigma)^{-1}$. Because

$$\begin{aligned} \sum_{j=0}^{l_k-1} W(\alpha,\sigma)^j V(\alpha,\sigma) &= \left[I - W(\alpha,\sigma)^{l_k}\right]\left[I - W(\alpha,\sigma)\right]^{-1} V(\alpha,\sigma) \\ &= \left[I - W(\alpha,\sigma)^{l_k}\right]\left[I - B(\alpha,\sigma)^{-1}C(\alpha,\sigma)\right]^{-1} B(\alpha,\sigma)^{-1} \\ &= \left[I - W(\alpha,\sigma)^{l_k}\right](\alpha I + A)^{-1}, \end{aligned}$$

we can reformulate (3.5) as

$$\begin{aligned} &x^{k+1} - x^* \\ &\quad= W(\alpha,\sigma)^{l_k}\left(x^k - x^*\right) + \left[I - W(\alpha,\sigma)^{l_k}\right](\alpha I + A)^{-1}(\alpha I - A)\left(\left|x^k\right| - |x^*|\right) \\ &\quad= W(\alpha,\sigma)^{l_k}\left[\left(x^k - x^*\right) - (\alpha I + A)^{-1}(\alpha I - A)\left(\left|x^k\right| - |x^*|\right)\right] \\ &\qquad\quad + (\alpha I + A)^{-1}(\alpha I - A)\left(\left|x^k\right| - |x^*|\right). \end{aligned}$$

Thus,

$$\begin{aligned} \left\|x^{k+1} - x^*\right\|_2 &\le \left[\left\|W(\alpha,\sigma)^{l_k}\right\|_2 (1+\zeta) + \zeta\right]\left\|x^k - x^*\right\|_2 \\ &\le \left[\|W(\alpha,\sigma)\|_2^{l_k}(1+\zeta) + \zeta\right]\left\|x^k - x^*\right\|_2. \end{aligned}$$

Because $\zeta = \left\|(\alpha I + A)^{-1}(\alpha I - A)\right\|_2 < 1$ and since for any $s > N$ it holds that $\|W(\alpha,\sigma)\|_2^s < \frac{1-\zeta}{1+\zeta}$, we have

$$\|W(\alpha,\sigma)\|_2^{l_k}(1+\zeta) + \zeta < 1.$$

Thus, Method 1 is convergent. □

4. Optimal parameters. We now consider the choice of the parameters α and σ in the iteration (3.1): One problem concerns the optimal parameter α^* in $\min\left\|(\alpha I + A)^{-1}(\alpha I - A)\right\|_2$ and another one the optimal parameter σ^* that minimize the spectral radius $\rho(W(\alpha,\sigma))$ of the modulus-based CSCS iteration method (MCSCS). Let $\gamma_{\min}$ and $\gamma_{\max}$ be the lower and the upper bounds of the real parts of the eigenvalues of the positive definite matrix A. Then

$$\zeta = \left\|(\alpha I + A)^{-1}(\alpha I - A)\right\|_2 = \max_{\gamma_{\min} \le \gamma_j \le \gamma_{\max}} \left|\frac{\alpha - \gamma_j}{\alpha + \gamma_j}\right|.$$

According to [17], we have that

$$\alpha^* = \text{argmin}_\alpha \left\{ \max_{\gamma_j \in \gamma(A)} \left| \frac{\alpha - \gamma_j}{\alpha + \gamma_j} \right| \right\} = \sqrt{\gamma_{\min}\gamma_{\max}}$$

and

$$\zeta(\alpha^*) = \frac{\sqrt{\gamma_{\max}} - \sqrt{\gamma_{\min}}}{\sqrt{\gamma_{\max}} + \sqrt{\gamma_{\min}}} = \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1},$$

where $\kappa(A)$ is the spectral condition number of the matrix A.

For finding the optimal parameter σ^* that minimize the spectral radius $\rho(W(\alpha, \sigma))$, we consider the following two cases.

• Case I: The matrix A is symmetric.

If A is symmetric, then according to (2.1) and (2.2), C and S are symmetric positive definite. Since

$$\begin{aligned}\rho(W(\alpha,\sigma)) &= \rho\left((\sigma I - C)(\alpha I + \sigma I + C)^{-1}(\sigma I - S)(\alpha I + \sigma I + S)^{-1}\right) \\ &\leq \left\|(\sigma I - C)(\alpha I + \sigma I + C)^{-1}(\sigma I - S)(\alpha I + \sigma I + S)^{-1}\right\|_2 \\ &\leq \left\|(\sigma I - C)(\alpha I + \sigma I + C)^{-1}\right\|_2 \left\|(\sigma I - S)(\alpha I + \sigma I + S)^{-1}\right\|_2,\end{aligned}$$

it is easy to see that

$$\rho(W(\alpha,\sigma)) \leq \max_{\lambda_j \in \lambda(C)} \left| \frac{\sigma - \lambda_j}{\alpha + \sigma + \lambda_j} \right| \max_{\mu_j \in \mu(S)} \left| \frac{\sigma - \mu_j}{\alpha + \sigma + \mu_j} \right| = \sigma(M(\alpha,\sigma)) < 1. \tag{4.1}$$

Let $\varphi_{\min}$ and $\varphi_{\max}$ be the lower and the upper bounds of the real parts of the eigenvalues of the matrices C or S. Then the optimal parameter σ^* can be estimated by

$$\max_{\varphi_{\min} \leq \varphi \leq \varphi_{\max}} \left(\frac{\sigma - \varphi}{\alpha + \sigma + \varphi} \right)^2,$$

which is an upper bound for $\sigma(M(\alpha,\sigma))$. By a simple computation, we have that

$$\max_{\varphi_{\min} \leq \varphi \leq \varphi_{\max}} \left(\frac{\sigma - \varphi}{\alpha + \sigma + \varphi} \right)^2 = \begin{cases} \left(\frac{\sigma - \varphi_{\min}}{\alpha + \sigma + \varphi_{\min}}\right)^2, & \sigma \geq \varphi_{\max}, \\ \left(\frac{\sigma - \varphi_{\max}}{\alpha + \sigma + \varphi_{\max}}\right)^2, & \sigma \leq \varphi_{\min}, \\ \max\{\left|\frac{\sigma - \varphi_{\min}}{\alpha + \sigma + \varphi_{\min}}\right|, \left|\frac{\sigma - \varphi_{\max}}{\alpha + \sigma + \varphi_{\max}}\right|\}, & \varphi_{\min} \leq \sigma \leq \varphi_{\max}. \end{cases}$$

If σ^* is optimal, then it must satisfy

$$\left(\frac{\sigma - \varphi_{\min}}{\alpha + \sigma + \varphi_{\min}} \right)^2 = \left(\frac{\sigma - \varphi_{\max}}{\alpha + \sigma + \varphi_{\max}} \right)^2.$$

When $\varphi_{\min} < \sigma < \varphi_{\max}$, the function

$$\left(\frac{\sigma - \varphi}{\alpha + \sigma + \varphi} \right)^2$$

is monotone with respect to the variable σ. Therefore, we have

$$\sigma^* = \begin{cases} \varphi_{\max}, & \sigma \geq \varphi_{\max}, \\ \varphi_{\min}, & \sigma \leq \varphi_{\min}, \\ \sqrt{\alpha \frac{\varphi_{\min} + \varphi_{\max}}{2} + \varphi_{\min}\varphi_{\max} + \frac{\alpha^2}{4}} - \frac{\alpha}{2}, & \varphi_{\min} \leq \sigma \leq \varphi_{\max}. \end{cases}$$

From the above analysis, we know that the upper bound of the convergent factor of the MCSCS iteration method is minimized when $\alpha = \alpha^*$ and $\sigma = \sigma^*$.

• Case II: The matrix A is nonsymmetric.

If the matrix A is nonsymmetric, then C and S need not be symmetric positive definite. According to the definition of (4.1),

$$\begin{aligned}\sigma(W(\alpha,\sigma)) &= \max_{\lambda_j\in\lambda(C)}\left|\frac{\sigma-\lambda_j}{\alpha+\sigma+\lambda_j}\right| \max_{\mu_j\in\mu(S)}\left|\frac{\sigma-\mu_j}{\alpha+\sigma+\mu_j}\right| \\ &= \max_{\lambda_j=\gamma_j+i\eta_j\in\lambda(C)}\left|\frac{\sigma-(\gamma_j+i\eta_j)}{\alpha+\sigma+\gamma_j+i\eta_j}\right| \max_{\mu_j=\zeta_j+i\xi_j\in\mu(S)}\left|\frac{\sigma-(\zeta_j+i\xi_j)}{\alpha+\sigma+\zeta_j+i\xi_j}\right| \\ &= \max_{\lambda_j=\gamma_j+i\eta_j\in\lambda(C)}\sqrt{\frac{(\sigma-\gamma_j)^2+\eta_j^2}{(\alpha+\sigma+\gamma_j)^2+\eta_j^2}} \max_{\mu_j=\zeta_j+i\xi_j\in\mu(S)}\sqrt{(\frac{(\sigma-\zeta_j)^2+\xi_j^2}{(\alpha+\sigma+\zeta_j)^2+\xi_j^2}}.\end{aligned}$$

Because of $\gamma_j > 0$ and $\zeta_j > 0$ and since α and σ are some positive constants, it is easy to see that $\sigma(M(\alpha,\sigma)) < 1$, and therefore $\rho(M(\alpha,\sigma)) < 1$.

If the eigenvalues of the matrices C and S are contained in the rectangle $\Lambda = [\varphi_{\min},\varphi_{\max}] \times i[\eta_{\min},\eta_{\max}]$, then $\sigma(M(\alpha,\sigma))$ can be estimated by

$$\max_{\varphi+i\eta\in\Lambda}\frac{(\sigma-\varphi)^2+\eta^2}{(\alpha+\sigma+\varphi)^2+\eta^2}.$$

Similar to [17], we have the following theorem.

THEOREM 4.1. *The optimal parameter σ^* which minimizes $\sigma(W(\alpha,\sigma))$ is given by*

$$\sigma^* = \begin{cases} \sqrt{\alpha\frac{\varphi_{\min}+\varphi_{\max}}{2}+\varphi_{\min}\varphi_{\max}+\frac{\alpha^2}{4}-\eta_{max}^2}-\frac{\alpha}{2}, \\ \qquad\qquad\qquad \textit{if } \eta_{max} < \sqrt{\varphi_{\min}\varphi_{\max}+\frac{\alpha(\varphi_{\min}+\varphi_{\max})}{2}}, \\ \sqrt{\alpha\varphi_{\min}+\varphi_{\min}^2+\frac{\alpha^2}{4}+\eta_{max}^2}-\frac{\alpha}{2}, \\ \qquad\qquad\qquad \textit{if } \eta_{max} \geq \sqrt{\varphi_{\min}\varphi_{\max}+\frac{\alpha(\varphi_{\min}+\varphi_{\max})}{2}}. \end{cases}$$

5. Numerical results. In this section, we use an example to test the modulus-based CSCS iteration method (denoted by 'MCSCS'). In the tables we provide the iteration steps (denoted by 'iter'), the average iteration steps of (3.1) (denoted by 'itav'), and the CPU time (denoted by 'time'). In addition, let RES $\left(z^k\right)$ be defined as

$$\text{RES}\left(z^k\right) = \min\left(\left|Az^k+q\right|, \left|z^k\right|\right),$$

where z^k is the kth approximate solution of Method 1. We use n to denote the dimension of the system matrix A and α, σ to denote positive parameters. In the tests, numerical comparisons of the new method and the conjugate gradient method (denoted by 'CG') are presented.

EXAMPLE 5.1. Let the system matrix A of problem (2.3) be the positive definite Toeplitz matrix where the diagonal elements a_j are given as

$$a_j = (1+|j|)^{-p}, \qquad j = 0,\pm1,\pm2,\ldots,$$

where p is a given positive constant and $q = (1,-1,1,-1,\cdots)^T$.

We solve the following system, which is equivalent to equation (3.1), by using a CG method to compare with MCSCS :

$$(\alpha I + A)\,x^{k+1} = b^k.$$

Here $b^k = (\alpha I - A)\left|x^k\right| - q$.

The solution process uses the FFT; see [10] for more details. The initial vector is chosen to be $x^0 = (0, 0, \cdots, 0)^T \in \mathbb{R}^n$, and the iteration error satisfies RES$(z^k) < 10^{-6}$. We terminate when the inexact solution of (3.1) satisfies

$$\frac{\left\|b^k - (\alpha I + A)\, x^{k+1}\right\|_2}{\left\|b^k\right\|_2} < 10^{-6}. \tag{5.1}$$

The numerical results are listed in Tables 5.1–5.2.

TABLE 5.1
Numerical results of Example 5.1 of CG with the stopping criterion (5.1).

		$n = 262144$			$n = 524288$			$n = 1048576$		
p	α	iter	itav	time	iter	itav	time	iter	itav	time
0.9	4.6	122	8	76.05	134	8	148.722	146	8	495.643
	4.8	121	8	77.206	132	8	171.576	144	8	476.639
	5.2	113	8	75.032	130	8	143.776	142	8	470.88
	5.5	114	8	78.223	125	8	142.949	140	8	451.36
1.0	2.7	90	9	64.717	96	9	144.385	100	9	320.92
	3.0	90	8	63.047	96	8	127.957	100	8	316.936
	3.3	88	8	56.716	94	8	130.334	100	8	299.815
	3.5	88	8	58.645	94	7	114.271	98	7	281.903
	3.8	86	7	55.052	90	7	116.361	96	7	259.309
1.1	1.5	63	9	49.067	66	9	102.638	68	8	283.004
	2.1	67	8	53.221	71	8	106.317	74	7	247.819
	2.5	63	8	50.125	63	8	110.379	71	8	245.444
	2.7	67	8	46.175	69	8	82.146	69	8	253.714
	3.0	66	7	40.563	68	7	91.228	72	7	246.055

On the other hand, we can set a finite termination condition to get the inexact solution of (3.1); the value of ζ and $\rho(W(\sigma))$ can be estimated in the low-dimensional systems. For example, the value of the right-hand side of (3.3) is about 0.15 when $p = 1.1, \alpha = 2.8$, $\sigma = 2.5$, and the inexact solution can be obtained when $N = 2$. Therefore, the maximum number of inner iteration steps can be set as 2 (i.e., $l_k = 2, k = 0, 1, 2, \ldots$). MCSCS with finite termination conditions can reduce the unnecessary computation and accelerate the convergence process. The numerical results are listed in Tables 5.3.

From Tables 5.1–5.2, we can find that, under the conditions (5.1), MCSCS requires less computation time than the CG method. Specially, for example, when $n = 1048576$ and $p = 1.1$, for the optimal parameters $\alpha^* \approx 2.7$ and $\sigma^* \approx 2.4$, MCSCS shows its advantages.

From Table 5.3, it can be observed that MCSCS is superior to the CG method under the finite termination conditions with respect to computation time or iteration steps.

6. Conclusions. By transforming LCPs to a class of AVEs, we constructed MCSCS iteration methods to solve the LCP with a positive definite Toeplitz matrix and analyzed its convergence. Both theoretical analysis and numerical experiments show the effectiveness of the new method.

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TABLE 5.2
Numerical results of Example 5.1 of MCSCS with the stopping criterion (5.1).

			$n = 262144$			$n = 524288$			$n = 1048576$		
p	α	σ	iter	itav	time	iter	itav	time	iter	itav	time
0.9	4.6	6.5	82	4	54.183	88	4	129.271	93	5	411.93
		6.7	82	4	51.487	88	4	136.586	96	4	420.312
		6.9	77	4	40.019	91	4	143.678	96	4	420.182
		7.0	77	4	44.424	91	4	135.419	88	5	398.017
		7.4	80	4	43.09	89	4	149.953	94	4	417.241
1.0	3.3	3.5	70	4	36.652	73	4	106.967	78	4	272.349
		3.8	70	3	36.913	70	4	98.143	78	4	275.276
		4.1	68	4	39.87	65	4	89.15	73	4	250.218
		4.5	68	4	43.895	73	4	116.914	75	4	230.75
		4.8	68	4	40.882	68	4	101.994	74	4	264.198
1.1	2.7	1.5	50	4	32.988	50	4	84.848	58	3	230.019
		2.0	57	3	27.581	57	3	77.882	58	3	208.756
		2.4	55	3	27.744	55	3	66.089	58	3	177.931
		2.8	55	3	34.82	53	3	65.25	58	3	179.576
		3.0	55	3	28.562	53	3	66.416	59	3	203.848

TABLE 5.3
Numerical results of Example 5.1 under finite step termination.

			$n = 262144$			$n = 524288$			$n = 1048576$		
p	α	σ	iter	itav	time	iter	itav	time	iter	itav	time
0.9	4.6	6.7	109	3	35.241	119	3	93.666	127	3	298.612
		7.0	105	3	34.429	115	3	114.7	123	3	284.635
1.0	3.3	4.1	70	2	19.893	81	2	55.963	81	2	133.13
		4.5	76	2	18.876	81	2	47.253	79	2	145.22
1.1	2.7	2.0	60	2	18.449	58	2	47.053	67	2	126.98
		2.4	62	2	21.734	60	2	55.075	68	2	122.105

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