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MULTIVARIATE FRACTAL INTERPOLATION FUNCTIONS: SOME APPROXIMATION ASPECTS AND AN ASSOCIATED FRACTAL INTERPOLATION OPERATOR*

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Abstract. In the classical (non-fractal) setting, the natural kinship between theories of interpolation and approximation is well explored. In contrast to this, in the context of fractal interpolation, the interrelation between interpolation and approximation is subtle, and this duality is relatively obscure. The notion of α-fractal functions provides a proper foundation for the approximation-theoretic facet of univariate fractal interpolation functions (FIFs). However, no comparable approximation-theoretic aspects of FIFs have been developed for functions of several variables. The current article intends to open the door for intriguing interactions between approximation theory and multivariate FIFs. To this end, in the first part of this article, we develop a general framework for constructing multivariate FIFs, which is amenable to provide a multivariate analogue of an α-fractal function. Multivariate α-fractal functions provide a parameterized family of fractal approximants associated with a given multivariate continuous function. Some elementary aspects of the multivariate fractal (not necessarily linear) interpolation operator that sends a continuous function defined on a hyperrectangle to its fractal analogue are studied. As in the univariate setting, the notion of α-fractal functions serves as a basis for fractalizing various results in multivariate approximation theory, including that of multivariate splines. For our part, we provide some approximation classes of multivariate fractal functions and prove a few results on the constrained fractal approximation of real-valued continuous functions of several variables.

Key words. multivariate fractal approximation, constrained approximation, fractal operator, nonlinear operator, Schauder basis, Müntz theorem

AMS subject classifications. 28A80, 41A05, 41A30.

1. Introduction.

The first systematic study of interpolation of univariate data with continuous functions whose graphs are self-referential sets (fractals)—popularly known as fractal interpolation—has its origin in Barnsley's fundamental paper on fractal functions and interpolation [1]. During the past three decades, various questions concerning fractal interpolation have led to numerous generalizations of the original setting by Barnsley; for a lucid exposition, we refer the reader to the book [17]. One of the features of fractal interpolation that distinguishes it from various standard interpolation techniques is its ability to produce both smooth and nonsmooth interpolants. It is worth noting that there are only a very few methods that produce nonsmooth interpolating functions, another popular method being subdivision schemes [10]. Recently, attempts have been made to link subdivision schemes and fractal interpolation [11, 14]. The Hausdorff dimension of the graph of a fractal interpolant provides an additional index to measure the complexity of a signal, for instance, EEG signals [9]. Furthermore, smooth fractal interpolation supplements and subsumes the theory of splines and Hermite interpolation [7, 18, 27].

Classical theories of interpolation and approximation often appear as two sides of the same coin; the results about the one frequently imply results about the other. Indeed, at a basic level, both are, in essence, one and the same. This duality between interpolation and approximation seems to be more subtle in the fractal setting. Fruitful interactions between the notion of univariate FIFs and classical approximation theory took place via a suitable subclass of fractal interpolation functions. This subclass was brought out by Barnsley himself; see, for instance, [1], especially his remarks in Example 2 on page 309 there. The subclass

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mentioned above was later named α-fractal functions to reflect the vectorial parameter α that influences the Hausdorff dimension of the graph of a FIF. Substantial extensions of this theme have been carried out by Navascués and her coworkers [21, 22, 23, 38]. It is our opinion that the concept of α-fractal functions assisted fractal interpolation to find applications in other fields of mathematics that belong, in a broad sense, to the topic of approximation of functions in various function classes, for instance, in the theory of bases and frames [24, 25]. Another interesting theoretical and practical ramification of these findings is the fact that fractal functions can be used for constrained approximation [36]. Recent years have witnessed a renewed level of interest in fractal interpolation, in particular, in the study of α-fractal functions.

Parallel with, or perhaps even prior to, the investigations on approximation-theoretic aspects of univariate FIFs through the notion of α-fractal functions, attempts have been made to study multivariate analogues of fractal interpolation, especially the bivariate FIFs or fractal surfaces. The study of multivariate FIFs, even in the bivariate case, is more complex, and approaches are less obvious; see, for instance, [4, 6, 8, 12, 15, 16, 19, 41, 42]. In most cases, the construction is confined to the case wherein some suitable restriction on the interpolation points is imposed or maps in the Iterated Function System (IFS) use equal scaling factors. In [31] the authors give a more general framework to construct bivariate fractal interpolation functions for data on rectangular grids. Our interest in [31] is attributed to the fact that the formalism therein can be easily adapted to obtain bivariate α-fractal functions, an interlude to the study of approximation-theoretic aspects of bivariate fractal interpolation; see the recent works reported in [39, 40].

In contrast to the univariate and bivariate theory of FIFs, higher-dimensional analogues are scarce in the literature. Hardin and Massopust [13, 17] constructed fractal interpolation functions from a polygonal set $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ using suitable triangulations of D. The construction of FIFs at arbitrary interpolation points placed on rectangular grids of $\mathbb{R}^n$ is undertaken in [3]. Both these constructions are based on the concept of recurrent IFSs [2, 17], and ensuring continuity of such multivariate fractal functions presents geometric complications beyond those which arise for similar univariate fractal functions or univariate vector-valued fractal functions.

The principal aim of this article is to initiate an interaction between multivariate FIFs and multivariate approximation theory and thereby expose some interesting approximation-theoretic considerations of multivariate FIFs. Having gained some experience with fractal approximation theory of univariate and bivariate functions, one could easily anticipate that the development of a multivariate analogue of α-fractal functions could be the first and foremost step to accomplish this. However, the impediment is that a general framework to construct multivariate FIFs that is appropriate to provide a notion of multivariate α-fractal functions is unavailable. The constructions of multivariate FIFs indicated in the previous paragraph do not seem to lend themselves to the α-fractal function formalism of multivariate FIFs.

In the first part of this contribution, we overcome this obstacle by developing a general framework for constructing multivariate FIFs. The corresponding problem for the bivariate case was treated in [31], to which the first part of the article may be considered a sequel. However, our interest is in the multivariate analogue of α-fractal functions because it provides a vehicle to interact with approximation theory. Our theory has been designed to establish a rigorous definition of a multivariate α-fractal function. Using this multivariate α-fractal function as the proper foundation, the second part of the article attempts to develop some approximation-theoretic aspects of multivariate FIFs. The present contribution represents, however, only the beginning of the study of multivariate fractal approximation theory, and many questions are left untouched. For instance, multivariate fractal splines and multivariate

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fractal functions in $\mathcal{L}^p$-spaces will appear elsewhere. Let us also remark that another standard approach to the construction of multivariate α-fractal functions is through the tensor product of univariate fractal functions. For instance, in [26] the authors study special classes of bivariate fractal functions given as the tensor product of univariate α-fractal functions and hint at the multidimensional case. To the best of our knowledge, the self-referentiality of the constructed multivariate approximants is not evident in this approach. As mentioned earlier, Hardin and Massopust [13] investigated the construction of multivariate fractal interpolation functions on domains other than hyperrectangles using triangulations of the domain and corresponding labeling maps. Their techniques do not appear to be suitable for the construction of α-fractal functions. A general framework for constructing multivariate fractal interpolation functions, in particular the extension of the α-fractal function formalism to domains other than hyperrectangles, appears to be more challenging, and this still remains open.

2. Rudimentary facts.

2.1. Multivariate fractal interpolation functions. As mentioned in the introductory section, a special class of multivariate FIFs, the so-called multivariate α-fractal functions, is the subject of the current study. However, we decided to include a brief overview of a general theory of multivariate FIFs due to the fact that 1) it not only forms the requisite background material for the present study but may also be of independent interest, and 2) we were unable to find an explicit treatment for multivariate FIFs. The focus in [31] is on the bivariate case, although it does include a straightforward extension to higher dimensions. Therefore, this section not only acts as a precursor to the current study but also as a realization that the treatment in [31] can be extended almost verbatim to the multivariate case.

Let $n \geq 2$ be a natural number. Consider a data set

$$\Delta = \big\{(x_{1,i_1}, x_{2,i_2}, \ldots, x_{n,i_n}, y_{i_1 i_2 \ldots i_n}) : \; i_k = 0, 1, \ldots, N_k, k = 1, \ldots, n\big\}$$

such that

$$a_k = x_{k,0} < x_{k,1} < \cdots < x_{k,N_k} = b_k$$

for each $k = 1, 2, \ldots, n$, $n \geq 2$. For $k = 1, 2, \ldots, n$, set $I_k = [a_k, b_k]$. To simplify the notation, for $m \in \mathbb{N}$, we write

$$\begin{aligned} \Sigma_m &= \{1, 2, \ldots, m\}, \qquad & \Sigma_{m,0} &= \{0, 1, \ldots m\}, \\ \partial\Sigma_{m,0} &= \{0, m\}, \qquad & \operatorname{int}\Sigma_{m,0} &= \{1, 2, \ldots, m-1\}. \end{aligned}$$

We denote by I_{k,i_k} a subinterval of I_k determined by the partition $\{x_{k,0}, x_{k,1}, \ldots, x_{k,N_k}\}$, with $I_{k,i_k} = [x_{k,i_k-1}, x_{k,i_k}]$, for $i_k \in \Sigma_{N_k}$. For any $i_k \in \Sigma_{N_k}$, let $u_{k,i_k} : I_k \to I_{k,i_k}$ be an affine map satisfying

$$\begin{cases} u_{k,i_k}(x_{k,0}) = x_{k,i_k-1} \quad \text{and} \quad u_{k,i_k}(x_{k,N_k}) = x_{k,i_k}, & \text{if } i_k \text{ is odd}, \\ u_{k,i_k}(x_{k,0}) = x_{k,i_k} \quad \text{and} \quad u_{k,i_k}(x_{k,N_k}) = x_{k,i_k-1}, & \text{if } i_k \text{ is even}, \end{cases} \tag{2.1}$$

$$\big|u_{k,i_k}(x) - u_{k,i_k}(x')\big| \leq \alpha_{k,i_k}|x - x'|, \qquad \forall x, x' \in I_k,$$

where $0 \leq \alpha_{k,i_k} < 1$ is a constant. Using the definition of u_{k,i_k}, it is easy to verify that

$$u_{k,i_k}^{-1}(x_{k,i_k}) = u_{k,i_k+1}^{-1}(x_{k,i_k}), \qquad \forall i_k \in \operatorname{int}\Sigma_{N_k,0}. \tag{2.2}$$

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Let $\tau : \mathbb{N} \times \{0, N_1, N_2, \ldots, N_n\} \to \mathbb{N}$ be defined by

$$\begin{cases} \tau(i,0) = i-1 \;\text{ and }\; \tau(i,N_k) = i, & \text{if } i \text{ is odd}, \\ \tau(i,0) = i \quad\;\; \text{ and }\; \tau(i,N_k) = i-1, & \text{if } i \text{ is even}. \end{cases}$$

Using the above notation we see that

$$u_{k,i_k}(x_{k,j_k}) = x_{k,\tau(i_k,j_k)}, \qquad \forall i_k \in \Sigma_{N_k}, j_k \in \partial\Sigma_{N_k,0}, k \in \Sigma_n.$$

Let $K := \left(\prod_{k=1}^{n} I_k\right) \times \mathbb{R}$. For each $(i_1, i_2, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$, let the function $F_{i_1 i_2 \ldots i_n} : K \to \mathbb{R}$ be continuous satisfying the following conditions:

$$F_{i_1 i_2 \ldots i_n}(x_{1,j_1}, x_{2,j_2}, \ldots, x_{n,j_n}, y_{j_1 j_2 \ldots j_n}) = y_{\tau(i_1,j_1)\tau(i_2,j_2)\ldots\tau(i_n,j_n)} \tag{2.3}$$

for all $(j_1, j_2, \ldots, j_n) \in \prod_{k=1}^{n} \partial\Sigma_{N_k,0}$ and

$$\left|F_{i_1 i_2 \ldots i_n}(x_1, x_2, \ldots, x_n, y) - F_{i_1 i_2 \ldots i_n}(x_1, x_2, \ldots, x_n, y')\right| \le \gamma_{i_1 i_2 \ldots i_n}|y - y'|, \tag{2.4}$$

for all $(x_1, x_2, \ldots, x_n) \in \prod_{k=1}^{n} I_k$ and $y, y' \in \mathbb{R}$, where $0 \le \gamma_{i_1 i_2 \ldots i_n} < 1$ is a constant.

Finally, for each $(i_1, i_2, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$, we define $W_{i_1 i_2 \ldots i_n} : K \to K$ by

$$\begin{aligned} &W_{i_1 i_2 \ldots i_n}(x_1, x_2, \ldots, x_n, y) \\ &\qquad := \big(u_{1,i_1}(x_1), u_{2,i_2}(x_2), \ldots, u_{n,i_n}(x_n), F_{i_1 i_2 \ldots i_n}(x_1, x_2, \ldots, x_n, y)\big), \end{aligned} \tag{2.5}$$

and we consider the Iterated Function System (IFS)

$$\Big\{K, W_{i_1 i_2 \ldots i_n} : (i_1, i_2, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}\Big\}.$$

For the definition of an IFS and its role in the theory of univariate fractal interpolation function, the interested reader may consult [1]. The following theorem is a multivariate analogue of the construction of univariate FIFs, originally appearing in [1] with its bivariate extension studied in [31]. The proof of the theorem is relegated to the appendix at the referees' behest.

THEOREM 2.1. *Let*

$$\Delta = \big\{(x_{1,i_1}, x_{2,i_2}, \ldots, x_{n,i_n}, y_{i_1 i_2 \ldots i_n}) : i_k = 0, 1, \ldots, N_k, k = 1, \ldots, n\big\}$$

be a prescribed multivariate data set and

$$\big\{K, W_{i_1 i_2 \ldots i_n} : (i_1, i_2, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}\big\}$$

be the IFS associated to it, as defined above. Assume that for each $(i_1, i_2, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$*, the map* $F_{i_1 i_2 \ldots i_n}$ *satisfy the following matching conditions:*

For all $i_k \in \operatorname{int} \Sigma_{N_k,0}$, $1 \le k \le n$, *and* $x_k^* = u_{k,i_k}^{-1}(x_{k,i_k}) = u_{k,i_k+1}^{-1}(x_{k,i_k})$,

$$\begin{aligned} &F_{i_1 \ldots i_k \ldots i_n}(x_1, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_n, y) \\ &\qquad = F_{i_1 \ldots i_k+1 \ldots i_n}(x_1, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_n, y), \end{aligned} \tag{2.6}$$

where $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \prod_{j=1, j\ne k}^{n} I_j$ *and* $y \in \mathbb{R}$. *Then there exists a unique continuous function* $\tilde{f} : \prod_{k=1}^{n} I_k \to \mathbb{R}$ *such that*

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1. *$\tilde{f}$ interpolates the given multivariate data. That is,*

$$\tilde{f}(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) = y_{i_1 i_2 \dots i_n}, \qquad \forall (i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k,0}.$$

2. *The graph of $\tilde{f}$ defined by*

$$G = \Big\{ \big(x_1, x_2, \dots, x_n, \tilde{f}(x_1, x_2, \dots, x_n)\big) : (x_1, x_2, \dots, x_n) \in \prod_{k=1}^{n} I_k \Big\}$$

is self-referential in the following sense: G is the union of transformed copies of itself given by

$$G = \bigcup_{(i_1,i_2,\dots,i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}} W_{i_1 i_2 \dots i_n}(G).$$

DEFINITION 2.2. *The function $\tilde{f}$ appearing in the previous theorem is termed a multivariate FIF.*

REMARK 2.3. As mentioned in the introductory section, in contrast to the univariate and bivariate settings, very few studies have addressed multivariate fractal interpolation; [3] and [13] are worth mentioning. In [3] the authors construct multivariate fractal interpolation functions on the hypercube $[0,1]^n$. However, the approach in these references is primarily based on a recurrent IFS. The construction in [13] is based on a recurrent IFS and uses the technique of triangulation of the domain. Also, the authors deal with a special choice, namely affine maps, for the functions $v_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, which play the role of $F_{i_1 \dots i_n}$ in our notation. We deal with more general functions (not necessarily affine) satisfying the required boundary conditions. More importantly, the constructions in [3, 13] do not assist us for an α-fractal function formalism and related approximation-theoretic aspects of the multivariate FIFs that form the main focus of the current paper.

2.2. Some elementary notions from functional analysis. In this section, we will review some fundamental concepts from functional analysis and the perturbation theory of operators, which will be useful in the upcoming sections. Recall that a sequence $(x_m)_{m\in\mathbb{N}}$ in a normed linear space $\mathbb{X}$ is called a Schauder basis for $\mathbb{X}$ if for each $x \in \mathbb{X}$, there exists a unique sequence $(x_m)_{m\in\mathbb{N}}$ of scalars such that $x = \sum_{m\in\mathbb{N}} c_m x_m$. The following two definitions which are fundamental in the perturbation theory of operators can be found in [30] and the references thereat. Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

DEFINITION 2.4. *Let A, B be two operators between $\mathbb{X}$ and $\mathbb{Y}$. Then A is called relatively bounded with respect to B (or simply B-bounded) if there exist nonnegative constants a, b such that*

$$\|A(x)\| \le a\|x\| + b\|B(x)\|, \qquad \forall x \in \mathbb{X}. \tag{2.7}$$

The infimum of all such values of b is called the B-bound of A.

DEFINITION 2.5. *Let $A, B : \mathbb{X} \to \mathbb{Y}$ be two operators between $\mathbb{X}$ and $\mathbb{Y}$. Then A is called relatively Lipschitz with respect to B (or simply B-Lipschitz) if*

$$\|A(x) - A(y)\| \le a\|x - y\| + b\|B(x) - B(y)\|, \qquad \forall x, y \in \mathbb{X}, \tag{2.8}$$

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for some nonnegative constants a and b. The infimum of all such values of b is called the B-Lipschitz constant of A.

REMARK 2.6. Let b_0 be the infimum of all values of b satisfying (2.7) or (2.8). Then (2.7) or (2.8) may not hold with $b = b_0$ because a may tend to infinity as b approaches b_0.

LEMMA 2.7 ([30, Proposition 5.1]). *Let $\mathbb{X}$ be a Banach space, $A : \mathbb{X} \to \mathbb{X}$ be a Lipschitz operator. Suppose that $|Id - A| < 1$, where Id is the identity operator on $\mathbb{X}$ and $|Id - A|$ is the Lipschitz constant of $Id - A$ defined by*

$$|Id - A| = \sup_{x \neq y} \frac{\|(Id - A)(x) - (Id - A)(y)\|}{\|x - y\|}.$$

Then $A^{-1} : \mathbb{X} \to \mathbb{X}$ is Lipschitz, and the Lipschitz constant satisfies

$$|A^{-1}| \leq \frac{1}{1 - |Id - A|}.$$

LEMMA 2.8 ([5, Lemma 1]). *Let $A : \mathbb{X} \to \mathbb{X}$ be a linear operator on a Banach space $\mathbb{X}$ such that*

$$\|A(x) - x\| \leq \lambda_1 \|x\| + \lambda_2 \|A(x)\|, \qquad \forall x \in \mathbb{X},$$

for some λ_1 and $\lambda_2 \in [0, 1)$. Then A is bounded, invertible, and possesses a bounded inverse. Further,

$$\begin{aligned} \frac{1-\lambda_1}{1+\lambda_2}\|x\| \leq \|A(x)\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|x\|, \\ \frac{1-\lambda_2}{1+\lambda_1}\|x\| \leq \|A^{-1}(x)\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|x\|, \end{aligned} \qquad \forall x \in \mathbb{X}.$$

DEFINITION 2.9. *Let $\mathbb{X}$ be a normed linear space and $A \subset \mathbb{X}$. A is said to be a fundamental set in $\mathbb{X}$ if $\overline{span(A)} = \mathbb{X}$.*

3. A parameterized family of multivariate fractal functions and the associated fractal operator. Influenced by the notion of univariate α-fractal functions and their role in approximation-theoretical aspects of FIFs [21, 22, 23], we shall develop a special class of multivariate FIFs, which we call multivariate α-fractal functions.

Let $n \in \mathbb{N}$, $n \geq 2$ be fixed, and $I_k = [a_k, b_k] \subset \mathbb{R}$ be a compact interval for $k = 1, 2, \dots, n$. Consider the n-dimensional hyperrectangle $\prod_{k=1}^n I_k$ and the space $\mathcal{C}\big(\prod_{k=1}^n I_k\big)$ endowed with the uniform norm. Let a function $f \in \mathcal{C}\big(\prod_{k=1}^n I_k\big)$ be fixed but arbitrary. We shall refer to this function as the *seed function* or *germ function*.

3.1. Multivariate α-fractal functions. Here we obtain a parameterized family of fractal functions associated with a prescribed germ function f by using the idea of multivariate fractal interpolation enunciated in the previous section.

With a slight abuse of notation, consider the set

$$\Delta = \Big\{(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) \in \prod_{k=1}^n I_k \subset \mathbb{R}^n : i_k \in \Sigma_{N_k,0},\ k \in \Sigma_n\Big\},$$

where $a_k = x_{k,0} < x_{k,1} < \cdots < x_{k,N_k} = b_k$ for each $k \in \Sigma_n := \{1, 2, \dots, n\}$. Note that $\{x_{k,0}, x_{k,1}, \dots, x_{k,N_k}\}$ forms a partition of the interval $[a_k, b_k]$ with the aid of which Δ

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determines a partition of the hyperrectangle. Let us sample the germ function f at the points in Δ and consider the data set

$$\Big\{\big(x_{1,i_1},\dots,x_{n,i_n},f(x_{1,i_1},\dots,x_{n,i_n})\big)\in\prod_{k=1}^{n} I_k\times\mathbb{R}: i_k\in\Sigma_{N_k,0},\ k\in\Sigma_n\Big\}.$$

We shall use the notation Δ to denote the above data set as well.

Suppose that the affine map $u_{k,i_k}: I_k\to I_{k,i_k}$ is defined as follows:

$$u_{k,i_k}(x)=a_{k,i_k}x+b_{k,i_k},\qquad i_k\in\Sigma_{N_k}, k\in\Sigma_n,$$

where a_{k,i_k} and b_{k,i_k} are chosen such that the contractive maps u_{k,i_k} satisfy (2.1) and (2.2). Choose a function $b\in\mathcal{C}\big(\prod_{k=1}^n I_k\big)$ such that for all $(j_1,j_2,\dots,j_n)\in\prod_{k=1}^n\partial\Sigma_{N_k,0}$,

$$b(x_{1,j_1},x_{2,j_2},\dots,x_{n,j_n})=f(x_{1,j_1},x_{2,j_2},\dots,x_{n,j_n}). \tag{3.1}$$

Consider a continuous map $\alpha:\prod_{k=1}^n I_k\to\mathbb{R}$ such that

$$\|\alpha\|_\infty:=\sup_{X\in\prod_{k=1}^n I_k}\big|\alpha(X)\big|<1.$$

As in the univariate counterpart, b is called the base function and α is called the scaling function. Define

$$\begin{aligned}
&F_{i_1i_2\dots i_n}\big(x_1,x_2,\dots,x_n,y\big)\\
&\quad=f\big(u_{1,i_1}(x_1),u_{2,i_2}(x_2),\dots,u_{n,i_n}(x_n)\big)\\
&\qquad+\alpha\big(u_{1,i_1}(x_1),u_{2,i_2}(x_2),\dots,u_{n,i_n}(x_n)\big)\big(y-b(x_1,x_2,\dots,x_n)\big).
\end{aligned}\tag{3.2}$$

For $(i_1,i_2,\dots,i_n)\in\prod_{k=1}^n\Sigma_{N_k}$ and $(j_1,j_2,\dots,j_n)\in\prod_{k=1}^n\Sigma_{N_k,0}$, we have

$$\begin{aligned}
&F_{i_1i_2\dots i_n}\big(x_{1,j_1},\dots,x_{n,j_n},f(x_{1,j_1},\dots,x_{n,j_n})\big)\\
&\quad=f\big(u_{1,i_1}(x_{1,j_1}),u_{2,i_2}(x_{2,j_2}),\dots,u_{n,i_n}(x_{n,j_n})\big)\\
&\quad=f\big(x_{1,\tau(i_1,j_1)},\dots,x_{n,\tau(i_n,j_n)}\big),
\end{aligned}$$

verifying that (2.3) holds. Using the condition $\|\alpha\|_\infty<1$, one can easily see that the contractivity condition prescribed in (2.4) is satisfied. Let $i_k\in\operatorname{int}\Sigma_{N_k,0}$, $1\le k\le n$, and $x_k^*=u_{k,i_k}^{-1}(x_{k,i_k})=u_{k,i_k+1}^{-1}(x_{k,i_k})$. For any $y\in\mathbb{R}$,

$$\begin{aligned}
&F_{i_1\dots i_{k-1}i_ki_{k+1}\dots i_n}(x_1,\dots,x_{k-1},x_k^*,x_{k+1},\dots,x_n,y)\\
&=f\big(u_{1,i_1}(x_1),u_{2,i_2}(x_2),\dots,u_{k-1,i_{k-1}}(x_{k-1}),u_{k,i_k}(x_k^*),u_{k+1,i_{k+1}}(x_{k+1}),\dots,u_{n,i_n}(x_n)\big)\\
&\quad+\alpha\big(u_{1,i_1}(x_1),u_{2,i_2}(x_2),\dots,u_{k-1,i_{k-1}}(x_{k-1}),u_{k,i_k}(x_k^*),u_{k+1,i_{k+1}}(x_{k+1}),\dots,u_{n,i_n}(x_n)\big)\\
&\qquad\times\big(y-b(x_1,x_2,\dots,x_n)\big)\\
&=f\big(u_{1,i_1}(x_1),u_{2,i_2}(x_2),\dots,u_{k-1,i_{k-1}}(x_{k-1}),u_{k,i_k+1}(x_k^*),u_{k+1,i_{k+1}}(x_{k+1}),\dots,u_{n,i_n}(x_n)\big)\\
&\quad+\alpha\big(u_{1,i_1}(x_1),u_{2,i_2}(x_2),\dots,u_{k-1,i_{k-1}}(x_{k-1}),u_{k,i_k+1}(x_k^*),u_{k+1,i_{k+1}}(x_{k+1}),\dots,u_{n,i_n}(x_n)\big)\\
&\qquad\times\big(y-b(x_1,x_2,\dots,x_n)\big)\\
&=F_{i_1\dots i_{k-1}i_k+1i_{k+1}\dots i_n}(x_1,\dots,x_{k-1},x_k^*,x_{k+1},\dots,x_n,y).
\end{aligned}$$

Therefore, the functions $F_{i_1i_2\dots i_n}$ in (3.2) satisfy the conditions prescribed in (2.3)–(2.4) and the matching condition given in (2.6). Ergo, by Theorem 2.1, there exists a unique fractal

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interpolation function, which we shall denote by $f^{\alpha}_{\Delta,b} : \prod_{k=1}^{n} I_k \to \mathbb{R}$, such that it satisfies the self-referential functional equation

$$\begin{aligned} f^{\alpha}_{\Delta,b}&(x_1, x_2, \dots, x_n) \\ &= f(x_1, x_2, \dots, x_n) \\ &\quad + \alpha(x_1, x_2, \dots, x_n)\Big((f^{\alpha}_{\Delta,b} - b)\big(u^{-1}_{1,i_1}(x_1), u^{-1}_{2,i_2}(x_2), \dots, u^{-1}_{n,i_n}(x_n)\big) \Big), \end{aligned} \tag{3.3}$$

for all $(x_1, x_2, \dots, x_n) \in \prod_{k=1}^{n} I_{k,i_k}$ and $(i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$. With the notation

$$X = (x_1, x_2, \dots, x_n),$$
$$u^{-1}_{i_1 i_2 \dots i_n}(X) = \big(u^{-1}_{1,i_1}(x_1), u^{-1}_{2,i_2}(x_2), \dots, u^{-1}_{n,i_n}(x_n)\big),$$

we may write the functional equation for the fractal function $f^{\alpha}_{\Delta,b}$ as follows: For $X \in \prod_{k=1}^{n} I_{k,i_k}$ and $(i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$,

$$f^{\alpha}_{\Delta,b}(X) = f(X) + \alpha(X)(f^{\alpha}_{\Delta,b} - b)\big(u^{-1}_{i_1 i_2 \dots i_n}(X)\big). \tag{3.4}$$

It is worth to mention that

$$f^{\alpha}_{\Delta,b}(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) = f(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}), \qquad \forall i_k \in \Sigma_{N_k,0},\ k \in \Sigma_n.$$

Now we have the multivariate analogue of α-fractal functions studied in [21].

DEFINITION 3.1. *The aforementioned continuous function $f^{\alpha}_{\Delta,b} : \prod_{k=1}^{n} I_k \to \mathbb{R}$ is referred to as the multivariate α-fractal interpolation function corresponding to the seed function f, associated with the scale function α, the partition Δ, and the base function b. It can be viewed as a fractal perturbation of the germ function f.*

REMARK 3.2. With different admissible choices of the parameters Δ, α, and b, in fact, we obtain a parameterized family of self-referential functions $\{f^{\alpha}_{\Delta,b}\}$, each of which interpolates the germ function at points in Δ.

Using the self-referential functional equation satisfied by $f^{\alpha}_{\Delta,b}$, it is straightforward to see the following inequality. For the univariate counterpart, we refer to [21].

PROPOSITION 3.3. *Let $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$ and the parameters α, Δ, and b be fixed as in the construction above. Then,*

$$\|f^{\alpha}_{\Delta,b} - f\|_{\infty} \le \|\alpha\|_{\infty} \|f^{\alpha}_{\Delta,b} - b\|_{\infty}. \tag{3.5}$$

Proposition 3.3 in conjunction with the triangle inequality yields the following upper bound for the uniform distance between a germ function f and its fractal counterpart $f^{\alpha}_{\Delta,b}$.

PROPOSITION 3.4. *Let $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$. Assume that the parameters α, Δ, and b are fixed as in the construction above. Then,*

$$\|f^{\alpha}_{\Delta,b} - f\|_{\infty} \le \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} \|f - b\|_{\infty}.$$

REMARK 3.5. Bearing Proposition 3.4 in mind, we have the following results that point to the approximation of the germ function $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$ with its fractal counterparts.

1. Let the base function b and the partition Δ in the construction of α-fractal interpolation functions corresponding to f be fixed. Assume that $(\alpha^m)_{m\in\mathbb{N}}$ is a sequence of scale functions such that $\|\alpha^m\|_{\infty} < 1$ for all $m \in \mathbb{N}$ and $\|\alpha^m\|_{\infty} \to 0$ as $m \to \infty$. Then, $\|f^{\alpha^m}_{\Delta,b} - f\|_{\infty} \to 0$ as $m \to \infty$.

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2. Let the scale function α and the partition Δ in the construction of α-fractal interpolation functions corresponding to f be fixed. Assume that $(b_m)_{m\in\mathbb{N}}$ is a sequence of base functions such that $\|f - b_m\|_\infty \to 0$ as $m \to \infty$. Then $\|f^\alpha_{\Delta,b_m} - f\|_\infty \to 0$ as $m \to \infty$.

Let us now illustrate the above construction of α-fractal functions by a few simple examples in the bivariate setting.

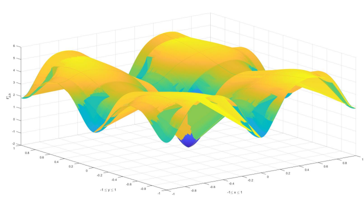
EXAMPLE 3.6. Consider the square, $[-1,1]\times[-1,1] \subset \mathbb{R}^2$. Let Δ be the following mesh partition of the square $\Delta = \{-1,-0.5,0,0.5,1\}\times\{-1,-0.5,0,0.5,1\}$. Let us take the bivariate germ function

$$f(x,y) = -20e^{-0.2\sqrt{0.5(x^2+y^2)}} - e^{0.5(\cos(2\pi x)+\cos(2\pi y))} + e + 20$$

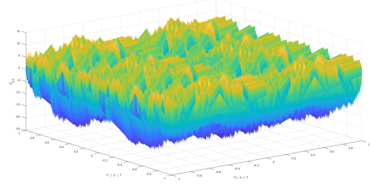
illustrated in Figure 3.1(a). Note that $f(x,y)$ is an Ackley function with specific choices of parameters involved; the general expression for Ackley's function in n dimension can be found in [20, Section 2.9]. It is worth mentioning that the Ackley's function is one of the important test functions considered in nonlinear optimization theory. Consider the base functions

1. $b_1(x,y) = \big(\sin(\pi x) + \cos(\frac{\pi y}{2}) + 1\big) f(x,y)$.
2. $b_2(x,y) = f(\sin(\frac{\pi x}{2}), \sin(\frac{\pi y}{2}))$.

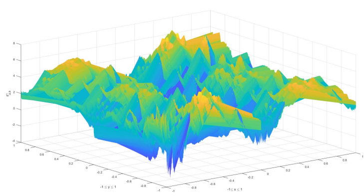
It can be easily verified that the above choices of b satisfy the conditions sought in (3.1).



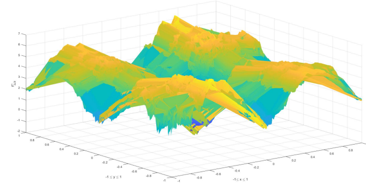
(a) Germ function f.



(b) $f^\alpha_{\Delta,b}$ with scale function $\alpha(x,y) = 0.9$ and $b = b_1$.



(c) $f^\alpha_{\Delta,b}$ with scale function $\alpha(x,y) = \dfrac{e^{xy}}{3}$ and $b = b_1$.



(d) $f^\alpha_{\Delta,b}$ with scale function $\alpha(x,y) = \dfrac{e^{xy}}{3}$ and $b = b_2$.

FIG. 3.1. *Fractal perturbation of Ackley's function with different choices of scaling and base function.*

Figures 3.1(b)–3.1(c) illustrate the surfaces corresponding to the fractal perturbations of f with the base function b_1 and the scaling function $\alpha : [-1,1]\times[-1,1] \to (-1,1)$ given by $\alpha(x,y) = 0.9$ and $\alpha(x,y) = \frac{e^{xy}}{3}$, respectively. Figure 3.1(d) is the graph of the fractal function of f associated with the base function b_2 and the scaling function $\alpha(x,y) = \frac{e^{xy}}{3}$.

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3.2. Fractal operator. Here we choose the base function b in the construction of the α-fractal function $f^{\alpha}_{\Delta,b}$ through an operator $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ as follows. Assume that $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ satisfies

$$L(f)(x_{1,j_1}, \dots, x_{n,j_n}) = f(x_{1,j_1}, \dots, x_{n,j_n}), \qquad \forall (j_1, \dots, j_n) \in \prod_{k=1}^{n} \partial\Sigma_{N_k,0}.$$

Such an operator will be referred to as an admissible operator. Take $b = L(f)$. In this case, we denote the α-fractal function $f^{\alpha}_{\Delta,b}$ corresponding to f by $f^{\alpha}_{\Delta,L}$. In contrast to the univariate case studied in the literature, here, in general, the operator L that defines the parameter map b is not necessarily linear. This will help the fractal operator to have access to the realm of nonlinear operator theory.

DEFINITION 3.7. *Fix a partition Δ, an operator $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$, and a scale function α as mentioned above. The operator*

$$\mathcal{F}^{\alpha}_{\Delta,L} : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k), \qquad \mathcal{F}^{\alpha}_{\Delta,L}(f) = f^{\alpha}_{\Delta,L},$$

is called the (multivariate) fractal operator.

REMARK 3.8. By the construction of an α-fractal function given in Section 3.1, it follows that the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ interpolates f at the points in the chosen partition Δ. Further, in view of Proposition 3.4, we have

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f) - f\|_\infty \le \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - L(f)\|_\infty. \tag{3.6}$$

Thus, for a suitable choice of the scale function α and/or the base function b, $\mathcal{F}^{\alpha}_{\Delta,L}(f)$ approximates f sufficiently well. Furthermore, if $L(f) = f$ or $\alpha = 0$, then $\mathcal{F}^{\alpha}_{\Delta,L}(f) = f$. In particular, if $L = Id$, the identity operator on $\mathcal{C}(\prod_{k=1}^{n} I_k)$, then the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} = Id$.

REMARK 3.9. As in the univariate setting [21], it can be proved that if the operator $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is a linear operator, then the corresponding fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is also linear. Furthermore, if L is a bounded linear operator, then it follows from (3.6) that the corresponding fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is also a bounded linear operator. In the remaining part of this section, we bring to light a few properties of the fractal operator beyond the familiar terrain of bounded linear operators.

PROPOSITION 3.10. *The fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is relatively bounded with respect to L with L-bound less than or equal to $\dfrac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}$.*

Proof. From (3.6) we have

$$\|f^{\alpha}_{\Delta,L} - f\|_\infty \le \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \big(\|f\|_\infty + \|L(f)\|_\infty\big).$$

Thus,

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f)\|_\infty = \|f^{\alpha}_{\Delta,L}\|_\infty \le \frac{1}{1 - \|\alpha\|_\infty} \|f\|_\infty + \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|L(f)\|_\infty,$$

completing the proof. ∎

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As a consequence of the previous proposition, we have the following result, which states that the (nonlinear) fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ shares some basic boundedness properties of the operator L. The proof is straightforward and hence omitted.

COROLLARY 3.11. *Consider the map $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ and the corresponding (multivariate) fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$.*

1. *If L is topologically bounded (that is, L maps bounded sets into bounded sets), then $\mathcal{F}^{\alpha}_{\Delta,L}$ is also topologically bounded.*
2. *If L is norm-bounded, that is,*
$$\rho(L) := \max\Big\{ \sup_{f\neq 0} \frac{\|L(f)\|_\infty}{\|f\|_\infty}, \|L(0)\|_\infty \Big\} < \infty,$$
then $\mathcal{F}^{\alpha}_{\Delta,L}$ is also norm-bounded.
3. *If L is quasibounded, that is,*
$$[L]_Q := \limsup_{\|f\|\to\infty} \frac{\|L(f)\|_\infty}{\|f\|_\infty} < \infty,$$
then $\mathcal{F}^{\alpha}_{\Delta,L}$ is quasibounded as well.

PROPOSITION 3.12. *The fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is relatively Lipschitz with respect to L, and its L-Lipschitz constant is less than or equal to $\dfrac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}$. In particular, if $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is a Lipschitz operator, then so is the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ with its Lipschitz constant $|\mathcal{F}^{\alpha}_{\Delta,L}| \leq \dfrac{1+\|\alpha\|_\infty |L|}{1-\|\alpha\|_\infty}$.*

Proof. Let f and g be in $\mathcal{C}(\prod_{k=1}^{n} I_k)$. Using the functional equations for the fractal functions $f^{\alpha}_{\Delta,L}$ and $g^{\alpha}_{\Delta,L}$ (see (3.3)) and some routine computations we have

$$\begin{aligned}\big\|\mathcal{F}^{\alpha}_{\Delta,L}(f) - \mathcal{F}^{\alpha}_{\Delta,L}(g)\big\|_\infty &= \|f^{\alpha}_{\Delta,L} - g^{\alpha}_{\Delta,L}\|_\infty \\ &\leq \frac{1}{1-\|\alpha\|_\infty}\|f-g\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|L(f)-L(g)\|_\infty,\end{aligned}$$

proving that $\mathcal{F}^{\alpha}_{\Delta,L}$ is relatively Lipschitz with respect to L. In particular, if L is Lipschitz with Lipschitz constant $|L|$, then the previous inequality yields

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f) - \mathcal{F}^{\alpha}_{\Delta,L}(g)\|_\infty \leq \frac{1}{1-\|\alpha\|_\infty}\|f-g\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}|L|\|f-g\|_\infty,$$

proving that $\mathcal{F}^{\alpha}_{\Delta,L}$ is a Lipschitz operator. ∎

PROPOSITION 3.13. *Assume that $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is a Lipschitz operator and that the scaling function α is such that $\|\alpha\|_\infty < \big(2+|L|\big)^{-1}$. Then the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is a Lipschitz isomorphism (surjective bilipschitz map), that is, $\mathcal{F}^{\alpha}_{\Delta,L}$ is a bijective Lipschitz operator and its inverse $(\mathcal{F}^{\alpha}_{\Delta,L})^{-1} : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ is also Lipschitz.*

Proof. Let $f, g \in \mathcal{C}(\prod_{k=1}^{n} I_k)$. Using the functional equations (see (3.3)) for the self-referential counterparts to f and g, we have

$$\begin{aligned}&\big|(Id - \mathcal{F}^{\alpha}_{\Delta,L})(f)(X) - (Id - \mathcal{F}^{\alpha}_{\Delta,L})(g)(X)\big| \\ &\quad = \Big| -\alpha(X)\left[\big(f^{\alpha}_{\Delta,b} - L(f)\big)\right]\big(u^{-1}_{i_1\dots i_n}(X)\big) + \alpha(X)\left[\big(g^{\alpha}_{\Delta,b} - L(g)\big)\right]\big(u^{-1}_{i_1\dots i_n}(X)\big)\Big| \\ &\quad \leq |\alpha(X)| \Big[\big|\big(f^{\alpha}_{\Delta,L} - g^{\alpha}_{\Delta,L}\big)\big(u^{-1}_{i_1\dots i_n}(X)\big)\big| + \big|\big(L(f)-L(g)\big)\big(u^{-1}_{i_1\dots i_n}(X)\big)\big| \Big],\end{aligned}$$

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for all $X \in \prod_{k=1}^{n} I_{k,i_k}$ and $(i_1, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$. Thus, we obtain

$$\begin{aligned} &\|(Id - \mathcal{F}^{\alpha}_{\Delta,L})(f) - (Id - \mathcal{F}^{\alpha}_{\Delta,L})(g)\|_{\infty} \\ &\qquad \leq \|\alpha\|_{\infty}\Big(\|L(f) - L(g)\|_{\infty} + \|\mathcal{F}^{\alpha}_{\Delta,L}(f) - \mathcal{F}^{\alpha}_{\Delta,L}(g)\|_{\infty}\Big). \end{aligned} \tag{3.7}$$

Since L is Lipschitz, by the previous proposition, the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is Lipschitz with

$$|\mathcal{F}^{\alpha}_{\Delta,L}| \leq \frac{1 + \|\alpha\|_{\infty}|L|}{1 - \|\alpha\|_{\infty}},$$

and hence from (3.7), we infer that

$$\begin{aligned} |Id - \mathcal{F}^{\alpha}_{\Delta,L}| &\leq \|\alpha\|_{\infty}\big(|L| + |\mathcal{F}^{\alpha}_{\Delta,L}|\big) \leq \|\alpha\|_{\infty}\Big(|L| + \frac{1 + \|\alpha\|_{\infty}|L|}{1 - \|\alpha\|_{\infty}}\Big) \\ &= \frac{\|\alpha\|_{\infty}\big(1 + |L|\big)}{1 - \|\alpha\|_{\infty}}. \end{aligned}$$

The assertion is now immediate by Lemma 2.7. ∎

4. On approximation aspects of multivariate α-fractal functions.

4.1. Schauder basis consisting of α-fractal functions. The theory of Schauder bases is an important tool in functional analysis, for instance, for solving partial differential equations and boundary value problems; see, for example [34] for background information. The question of existence of a Schauder basis for $\mathcal{C}([0,1]^n)$ being settled (see, for instance, [33]), the emphasis switches to searching for bases with some nice properties. In this section, the existence of a Schauder basis consisting of multivariate self-referential functions for the space $\mathcal{C}([0,1]^n)$ is attempted; the maps involved are obtained as suitable fractal perturbations of those belonging to a classical Schauder basis for the space $\mathcal{C}([0,1]^n)$.

PROPOSITION 4.1. *Let L be a bounded linear operator and $\|\alpha\|_{\infty} < \min\{1, \|L\|^{-1}\}$. Then the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is a topological isomorphism (i.e., a bijective bounded linear operator with bounded inverse).*

Proof. By virtue of Remark 3.9 it follows that $\mathcal{F}^{\alpha}_{\Delta,L}$ is a bounded linear operator whenever L is so. In view of Proposition 3.3 we have

$$\begin{aligned} \|\mathcal{F}^{\alpha}_{\Delta,L}(f) - f\|_{\infty} = \|f^{\alpha}_{\Delta,L} - f\|_{\infty} &\leq \|\alpha\|_{\infty}\|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty} \\ &\leq \|\alpha\|_{\infty}\|\mathcal{F}^{\alpha}_{\Delta,L}(f)\|_{\infty} + \|\alpha\|_{\infty}\|L\|\|f\|_{\infty}. \end{aligned}$$

The claim is immediate from Lemma 2.8. ∎

Let us mention here that a typical example for a Schauder basis for the space $\mathcal{C}([0,1])$ is the Faber-Schauder system or the orthonormal Franklin system. Since the space $\mathcal{C}\big([0,1]^n\big)$ can be thought of as the injective tensor product of n copies of $\mathcal{C}([0,1])$, the corresponding products of the Faber-Schauder bases in $\mathcal{C}([0,1])$ form a basis of $\mathcal{C}\big([0,1]^n\big)$ [33].

THEOREM 4.2. *The space $\mathcal{C}([0,1]^n)$ admits a Schauder basis consisting of multivariate fractal functions.*

Proof. Let $L : \mathcal{C}([0,1]^n) \to \mathcal{C}([0,1]^n)$ be a bounded linear operator and the scaling function $\alpha \in \mathcal{C}([0,1]^n)$ be such that $\|\alpha\|_{\infty} < \min\{1, \|L\|^{-1}\}$. With these assumptions, from the previous proposition, it follows that the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is a topological isomorphism. Since a topological isomorphism preserves Schauder bases, it follows that if $(f_m)_{m\in\mathbb{N}}$ is a Schauder basis of $\mathcal{C}([0,1]^n)$, then $\big((f_m)^{\alpha}_{\Delta,L}\big)_{m\in\mathbb{N}}$ is also a Schauder basis. ∎

4.2. Perturbed fractal approximation classes. Let $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $d = (d_1, \dots, d_n)$ be an n-tuple with nonnegative integer components. Let us use the notation

$$X^d = x_1^{d_1} \dots x_n^{d_n}, \qquad |d| = d_1 + \dots + d_n.$$

Then

$$p(X) = \sum_{|d| \le m} c_d X^d,$$

where $c_d \in \mathbb{R}$ are given constants, is called a polynomial in n variables of degree less than or equal to m. Let

$$P_{m,n} = \{p : p(X) = \sum_{|d| \le m} c_d X^d, c_d \in \mathbb{R}\}.$$

We write $P_{m,n}\big(\prod_{k=1}^n I_k\big)$ if the variable X is restricted to the domain $\prod_{k=1}^n I_k \subset \mathbb{R}^n$. Observe that $P_{m,n}$ is a finite-dimensional subspace of $\mathcal{C}\big(\prod_{k=1}^n I_k\big)$ and $\dim(P_{m,n}) = \binom{n+m}{m}$. We shall denote the space of all n-variate polynomials with real coefficients by P_n, and by $P_n\big(\prod_{k=1}^n I_k\big)$ if the variable X is restricted to the domain $\prod_{k=1}^n I_k \subset \mathbb{R}^n$.

Let Δ, L, and α be fixed as in the construction of the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$. Furthermore, we assume that L is a bounded linear operator so that the corresponding fractal operator $\mathcal{F}^\alpha_{\Delta,L} : \mathcal{C}\big(\prod_{k=1}^n I_k\big) \to \mathcal{C}\big(\prod_{k=1}^n I_k\big)$ is also a linear bounded operator. Consider the image of the subspace $P_{m,n}\big(\prod_{k=1}^n I_k\big)$ under the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$:

$$P^\alpha_{m,n}\big(\prod_{k=1}^n I_k\big) := \mathcal{F}^\alpha_{\Delta,L}\big(P_{m,n}\big(\prod_{k=1}^n I_k\big)\big).$$

The members of $P^\alpha_{m,n}(\prod_{k=1}^n I_k)$ are called the n-variate fractal polynomials of degree less than or equal to m. Similarly, the elements of the space

$$P^\alpha_n(\prod_{k=1}^n I_k) := \mathcal{F}^\alpha_{\Delta,L}\big(P_n\big(\prod_{k=1}^n I_k\big)\big)$$

are referred to as the n-variate fractal polynomials. That is,

DEFINITION 4.3. *A function $f \in \mathcal{C}\big(\prod_{k=1}^n I_k\big)$ is called an n-variate fractal polynomial if there exists an n-variate polynomial $p \in P_n\big(\prod_{k=1}^n I_k\big)$ such that $f = \mathcal{F}^\alpha_{\Delta,L}(p)$.*

The following result serves as a fractal counterpart to the multivariate polynomial approximation theorem.

THEOREM 4.4. *Let $f \in \mathcal{C}(\prod_{k=1}^n I_k)$ and $\epsilon > 0$.*

1. *Assume that a set Δ that determines a partition of the n-dimensional hyperrectangle $\prod_{k=1}^n I_k$ and an admissible bounded linear operator $L : \mathcal{C}(\prod_{k=1}^n I_k) \to \mathcal{C}(\prod_{k=1}^n I_k)$ in the construction of the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$ are fixed. Then there exist a non-null scale function $\alpha = \alpha(\epsilon)$ and a corresponding n-variate fractal polynomial $p^\alpha_{\Delta,L}$ such that $\|f - p^\alpha_{\Delta,L}\|_\infty < \epsilon$.*
2. *Assume that a set Δ that determines a partition of the n-dimensional hyperrectangle $\prod_{k=1}^n I_k$ and a non-null scale function α in the construction of the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$ are fixed. Then there exist an admissible bounded linear operator L and a corresponding n-variate fractal polynomial $p^\alpha_{\Delta,L}$ such that $\|f - p^\alpha_{\Delta,L}\|_\infty < \epsilon$.*

In particular, the set of all n-variate fractal polynomials with non-null scale vector is dense in $\mathcal{C}(\prod_{k=1}^n I_k)$ with respect to the $\|.\|_\infty$-norm and hence with respect to any $\mathcal{L}^p$-norm $\|.\|_p$, for $1 \le p < \infty$.

Proof. Let $f \in \mathcal{C}(\prod_{k=1}^n I_k)$ and $\epsilon > 0$. By the Stone-Weierstrass theorem there exists a multivariate polynomial p in n-variables such that

$$\|f - p\|_\infty < \frac{\epsilon}{2}.$$

Now, by virtue of Proposition 3.4,

$$\begin{aligned}\|f - p^\alpha_{\Delta,L}\|_\infty = \|f - p + p - p^\alpha_{\Delta,L}\|_\infty &\le \|f - p\|_\infty + \|p - p^\alpha_{\Delta,L}\|_\infty \\ &\le \frac{\epsilon}{2} + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|p - L(p)\|_\infty.\end{aligned}$$

Bearing the above inequality in mind, one can either (i) fix an admissible bounded linear operator L such that $Lp \neq p$ and choose a scale function $\alpha \neq 0$ such that

$$\|\alpha\|_\infty < \frac{\epsilon/2}{\epsilon/2 + \|p - L(p)\|_\infty},$$

or (ii) fix a scaling function $\alpha \neq 0$ satisfying $\|\alpha\|_\infty < 1$ and choose an admissible bounded linear operator $L : \mathcal{C}(\prod_{k=1}^n I_k) \to \mathcal{C}(\prod_{k=1}^n I_k)$, $L(p) \neq p$, such that

$$\|p - L(p)\|_\infty \le \frac{\epsilon(1 - \|\alpha\|_\infty)}{2\|\alpha\|_\infty}.$$

Then, with any of the above mentioned choice of parameters, we have

$$\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|p - L(p)\|_\infty < \frac{\epsilon}{2},$$

and consequently, $\|f - p^\alpha_{\Delta,L}\|_\infty < \epsilon$, establishing the claim. ∎

In fact, we have the following result:

THEOREM 4.5. *Let L be an admissible bounded linear operator and consider a fixed scaling function α such that $\|\alpha\|_\infty < \min\{1, \|L\|^{-1}\}$. Then the space of all n-variate fractal polynomials $P^\alpha_n(\prod_{k=1}^n I_k)$ is dense in $\mathcal{C}(\prod_{k=1}^n I_k)$.*

Proof. Recall that $P^\alpha_n(\prod_{k=1}^n I_k) := \mathcal{F}^\alpha_{\Delta,L}\big(P_n\big(\prod_{k=1}^n I_k\big)\big)$. By Proposition 4.1, it follows that the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$ is a topological isomorphism so that for any function $g \in \mathcal{C}(\prod_{k=1}^n I_k)$, there exists $f \in \mathcal{C}(\prod_{k=1}^n I_k)$ with $g = \mathcal{F}^\alpha_{\Delta,L}(f)$. By the density of $P_n(\prod_{k=1}^n I_k)$ in $\mathcal{C}(\prod_{k=1}^n I_k)$, there exists a sequence $(p_m)_{m\in\mathbb{N}}$ of n-variate polynomials in $P_n(\prod_{k=1}^n I_k)$ such that $f = \lim_{m\to\infty} p_m$. By the continuity of the fractal operator it follows that $g = \lim_{m\to\infty}(p_m)^\alpha_{\Delta,L}$, where $(p_m)^\alpha_{\Delta,L} = \mathcal{F}^\alpha_{\Delta,L}(p_m)$. ∎

Let Δ, α, and L be as prescribed in the definition of the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$, and further assume that L is linear. Let us recall that with these assumptions, the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$ is linear.

THEOREM 4.6. *Let Δ, α, and L be as prescribed in the definition of the fractal operator $\mathcal{F}^\alpha_{\Delta,L}$, and further assume that L is linear. Given any $f \in \mathcal{C}(\prod_{k=1}^n I_k)$, there exists a best approximant $p^\alpha_{\Delta,L}$ to f from the space $P^\alpha_{m,n}\big(\prod_{k=1}^n I_k\big)$, that is, there exists $p^\alpha_{\Delta,L}$, an n-variate fractal polynomial of degree less than or equal to m, such that*

$$\|f - p^\alpha_{\Delta,L}\|_\infty = \inf\Big\{\|f - q^\alpha_{\Delta,L}\|_\infty : q^\alpha_{\Delta,L} \in P^\alpha_{m,n}\big(\prod_{k=1}^n I_k\big)\Big\}.$$

Proof. As an immediate consequence of the finite dimensionality of $P_{m,n}\big(\prod_{k=1}^{n} I_k\big)$ and hence that of $P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big) := \mathcal{F}^{\alpha}_{\Delta,L}\big(P_{m,n}\big(\prod_{k=1}^{n} I_k\big)\big)$, the existence of a best approximant from $P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)$ to each $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$ follows. □

THEOREM 4.7. *For each $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$, the set $\mathcal{A}_{P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)}(f)$ of all best approximants to f from $P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)$ is a closed convex set. The associated metric projection, that is, the set-valued map $\Psi_{P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)} : \mathcal{C}(\prod_{k=1}^{n} I_k) \to P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)$ that maps each $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$ to the set of best approximants $\mathcal{A}_{P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)}(f)$ is upper semi-continuous, closed, and locally bounded.*

Proof. The proof follows exactly as for the univariate counterpart of this theorem (see [37, Theorem 2.4]) and hence is omitted. □

REMARK 4.8. Note that the space of all n-variate fractal polynomials $P^{\alpha}_{n}(\prod_{k=1}^{n} I_k)$ is equal to the union $\bigcup_{m\in\mathbb{N}} P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)$. As mentioned in the previous theorem, $P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)$ is finite dimensional and hence closed. This immediately implies that $P^{\alpha}_{n}(\prod_{k=1}^{n} I_k)$ is an F_{σ}-set. For the space $P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)$, being a proper subspace of $\mathcal{C}\big(\prod_{k=1}^{n} I_k\big)$, we have $\overline{P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)}^{0} = \Big(P^{\alpha}_{m,n}\big(\prod_{k=1}^{n} I_k\big)\Big)^{0} = \emptyset$. Consequently, $P^{\alpha}_{n}(\prod_{k=1}^{n} I_k)$ is of first category in $\mathcal{C}\big(\prod_{k=1}^{n} I_k\big)$ and hence topologically "small".

Let us introduce the following notation for the fractal minimax error:

$$\begin{aligned} E\big(f; P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)\big) &= \mathrm{dist}\big(f, P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)\big) \\ &:= \inf\Big\{\|f - p^{\alpha}_{\Delta,L}\|_{\infty} : p^{\alpha}_{\Delta,L} \in P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)\Big\}. \end{aligned}$$

The next theorem attempts to provide a Jackson-type estimate for the approximation with multivariate fractal polynomials.

THEOREM 4.9. *Let $L : \mathcal{C}(\prod_{k=1}^{n} I_k) \to \mathcal{C}(\prod_{k=1}^{n} I_k)$ be an admissible bounded linear operator. Then, for any $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$, we have*

$$E\big(f; P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)\big) \le \frac{5}{4}\sum_{k=1}^{n} \omega_k(f; \frac{1}{m}) + \frac{\|\alpha\|_{\infty}(1+\|L\|)}{1-\|\alpha\|_{\infty}}\|f\|_{\infty},$$

where the modulus of continuity of $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$ along the k-th variable is defined by

$$\begin{aligned} \omega_k(f;\epsilon) = \sup_{(x_1,x_2\ldots,x_n)\in\prod_{k=1}^{n} I_k} &\Big\{|f(x_1,\ldots,x_k,\ldots,x_n) - f(x_1,\ldots,x'_k,\ldots,x_n)| : \\ &a_k \le x'_k \le b_k,\ |x_k - x'_k| \le \epsilon\Big\}. \end{aligned}$$

Proof. Consider the multivariate Bernstein polynomials associated with $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$ defined by

$$\begin{aligned} B_m(f; x_1, x_2, \ldots, x_n) = \sum_{l_1=0}^{m}\cdots\sum_{l_n=0}^{m} f\Big(&x_{1,0} + \frac{l_1(x_{1,N_1} - x_{1,0})}{m}, \ldots, \\ &x_{n,0} + \frac{l_n(x_{n,N_n} - x_{n,0})}{m}\Big)\prod_{k=1}^{n} b_{l_k,m}(x_k), \end{aligned}$$

where $b_{l_k,m}(x_k) = \frac{1}{(x_{k,N_k}-x_{k,0})^m}\binom{m}{l_k}(x_k - x_{k,0})^{l_k}(x_{k,N_k} - x_k)^{m-l_k}$, $k = 1,\dots,n$. We have

$$\begin{aligned} E\big(f; P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)\big) = \text{dist}\big(f, P^{\alpha}_{m,n}(\prod_{k=1}^{n} I_k)\big) &\le \|f - (B_m(f))^{\alpha}_{\Delta,L}\|_\infty \\ &\le \|f - B_m(f)\|_\infty + \|B_m(f) - (B_m(f))^{\alpha}_{\Delta,L}\|_\infty \\ &\le \|f - B_m(f)\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|B_m(f) - L(B_m(f))\|_\infty. \end{aligned}$$

Since we have [32, Theorem 3.1]

$$\|f - B_m(f)\|_\infty \le \frac{5}{4}\sum_{k=1}^{n}\omega_k(f; \frac{1}{m})$$

and $\|B_m\| = 1$, the assertion follows. □

REMARK 4.10. Let us recall item 2. of Remark 3.5. Suppose that the scale function α and the partition Δ in the construction of the α-fractal interpolation functions corresponding to f are fixed. Assume that $(L_m)_{m\in\mathbb{N}}$ is a sequence of admissible operators that is strongly convergent to the identity operator, that is, $\|L_m(f) - f\|_\infty \to 0$ as $m \to \infty$ for each $f \in \mathcal{C}(\prod_{k=1}^{n} I_k)$. Then $\|f^{\alpha}_{\Delta,L_m} - f\|_\infty \to 0$ as $m \to \infty$. Consequently, the corresponding sequence of fractal operators $\big(\mathcal{F}^{\alpha}_{\Delta,L_m}\big)_{m\in\mathbb{N}}$ is strongly convergent to the identity operator. The Bernstein operator being one of the most attractive approximation operators, in place of $(L_m)_{m\in\mathbb{N}}$, one may consider the sequence of multivariate Bernstein (linear) operators $(B_m)_{m\in\mathbb{N}}$ for a hyperrectangle in $\mathbb{R}^n$. This provides a sequence of multivariate α-fractal functions $\big(f^{\alpha}_{\Delta,B_m}\big)_{m\in\mathbb{N}}$ and correspondingly a sequence of fractal operators $\big(\mathcal{F}^{\alpha}_{\Delta,B_m}\big)_{m\in\mathbb{N}}$ called the Bernstein fractal operators; the univariate counterpart to which is studied, for instance, in [35]. As indicated earlier, in particular, we have

$$\big\|\mathcal{F}^{\alpha}_{\Delta,B_m}(f) - f\big\|_\infty \to 0 \quad \text{as} \quad m \to \infty, \qquad \forall f \in \mathcal{C}(\prod_{k=1}^{n} I_k).$$

REMARK 4.11. Similar to the class of multivariate fractal polynomials, we can define fractal versions of multivariate trigonometric polynomials and rational functions using the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$, specifically, the bounded linear fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$, for convenience.

4.3. Multivariate fractal Müntz theorem. We shall give a fractal version of the multivariate Müntz theorem. Let us start with some preliminary notation and definitions; for a detailed exposure the reader can consult [29]. Consider the n-dimensional hypercube $[0,1]^n$. Following [29], let us introduce a few sets of powers involving n-variables $x_1, x_2, \dots, x_n$. For brevity, first let us consider $n = 3$. For the variables x_1, x_2, and x_3, $B_1^{(3)}$ is a set which is the null set or contains some of the three sequences $\{x_1^{a_i}\}$, $\{x_2^{b_i}\}$, $\{x_3^{c_i}\}$, where $a_i, b_i, c_i \in \mathbb{R}$ for $i \in \mathbb{N}$. The set $B_2^{(3)}$ is a null set or includes some of the three double sequences $\{x_1^{d_i}x_2^{e_j}\}$, $\{x_2^{f_i}x_3^{g_j}\}$, $\{x_3^{h_i}x_1^{k_j}\}$, where d_i, $e_j, \dots, k_j$ are real numbers for $i, j \in \mathbb{N}$. Similarly, $B_3^{(3)}$ is the null set or $\{x_1^{l_i}x_2^{m_j}x_3^{n_k}\}$, where l_i, m_j, n_k are in $\mathbb{R}$ for $i, j, k \in \mathbb{N}$. Assume that each sequence (a_i), $(b_j), \dots, (n_k)$ is positive and strictly monotonic increasing. Define the family $\mathcal{B}^{(3)} = B_1^{(3)} \cup B_2^{(3)} \cup B_3^{(3)}$. Similarly, in the general case, one can introduce the sets $B_1^{(n)}$, $B_2^{(n)}, \dots, B_n^{(n)}$, and $\mathcal{B}^{(n)} = B_1^{(n)} \cup B_2^{(n)} \cup \dots \cup B_n^{(n)}$.

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DEFINITION 4.12. *For the set $\mathcal{B}^{(n)}$ above, $\mathcal{M} = \{1\} \cup \mathcal{B}^{(n)}$ is called a Müntz set, and the linear span of the Müntz set is said to be a Müntz space. Elements in the Müntz space are called multivariate Müntz polynomials.*

DEFINITION 4.13. *The image $p^{\alpha}_{\Delta,L}$ of a multivariate Müntz polynomial p under the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is called a multivariate fractal Müntz polynomial.*

THEOREM 4.14 ([29, Proposition 1]). *The Müntz set $\{1\} \cup \mathcal{B}^{(n)}$ is fundamental in $\mathcal{C}([0,1]^n)$ if and only if*

1. *$\mathcal{B}^{(n)}_i$ consists of all possible i-tuple sequences involving i variables of $x_1, x_2, \dots, x_n$, of the form defined above.*
2. *In each i-tuple sequence of the previous item, for example, $\{x_1^{\lambda^1_{j(1)}}, \dots, x_i^{\lambda^i_{j(i)}}\}$, the exponents are such that $\sum_{j(l)=1}^{\infty} \frac{1}{\lambda^l_{j(l)}} = \infty$ for each $l = 1, \dots, i$, $i \in \Sigma_n$.*

THEOREM 4.15. *Let Δ be a fixed partition of the hypercube $[0,1]^n$ and the Müntz set $\mathcal{M} = \{1\} \cup \mathcal{B}^{(n)}$ be such that the two conditions of the previous theorem are satisfied. Suppose that $(\alpha^m)_{m\in\mathbb{N}}$ is a sequence of non-zero scaling functions in $\mathcal{C}(\prod_{k=1}^n I_k)$ such that $\|\alpha^m\|_\infty < 1$ for all $m \in \mathbb{N}$ and $\alpha^m \to 0$ as $m \to \infty$. Then the family of fractal Müntz polynomials $\{\mathcal{F}^{\alpha^m}_{\Delta,L}(\mathcal{M})\}_{m\in\mathbb{N}}$, where L is a fixed bounded linear admissible operator, is fundamental in $\mathcal{C}(\prod_{k=1}^n I_k)$.*

Proof. Let $f \in \mathcal{C}(\prod_{k=1}^n I_k)$ and $\epsilon > 0$ be given. By the previous theorem, there exists a Müntz polynomial $p \in \text{span}(\mathcal{M})$ such that

$$\|f - p\|_\infty < \frac{\epsilon}{2}.$$

Since $\alpha^m \to 0$ as $m \to \infty$, one can choose a nonzero scale function α^k such that $\|\alpha^k\|_\infty < \dfrac{\epsilon}{\epsilon + 2\|p - L(p)\|_\infty}$. Bearing (3.6) in mind, the triangle inequality yields

$$\begin{aligned}\|f - p^{\alpha^k}_{\Delta,L}\|_\infty &\le \|f - p\|_\infty + \|p - p^{\alpha^k}_{\Delta,L}\|_\infty \\ &< \frac{\epsilon}{2} + \frac{\|\alpha^k\|_\infty}{1 - \|\alpha^k\|_\infty}\|p - L(p)\|_\infty < \epsilon.\end{aligned}$$

Since L is linear, $p^{\alpha^k}_{\Delta,L} \in \text{span}\big(\mathcal{F}^{\alpha^k}_{\Delta,L}(\mathcal{M})\big)$, completing the proof. ∎

REMARK 4.16. The above theorem provides a fractal Müntz polynomial $p^{\alpha}_{\Delta,L}$ approximating a given multivariate continuous function up to a desired accuracy, keeping the bounded linear operator L fixed. There may be instances where one desires to keep the scale function fixed. In this regard, let us note that the family of fractal Müntz polynomials $\{\mathcal{F}^{\alpha}_{\Delta,B_m}(\mathcal{M})\}_{m\in\mathbb{N}}$, where B_m is the multivariate Bernstein operator, is fundamental in $\mathcal{C}(\prod_{k=1}^n I_k)$.

REMARK 4.17. As the fractal approximants do not possess closed form expressions but enjoys implicit self-referential equations, the standard methods for convergence analysis/approximation error bounds such as a Taylor series analysis, the Cauchy remainder form, and the Peano kernel theorem may not easily be adapted. A look back at the arguments regarding error bounds should convince the reader that essentially we used the triangle inequality and known error bounds with classical approximants; see also [28]. These error bounds are not claimed to be sharp, and obtaining optimal error bounds has so far eluded us.

4.4. Some constrained approximation aspects. In this section, we provide some conditions on the scaling function α and the base function b such that the fractal functions $f^{\alpha}_{\Delta,b}$ can be constrained with respect to the germ function $f \in \mathcal{C}\big(\prod_{k=1}^n I_k\big)$. To this end, we begin with the following notation.

1. Let $X = (x_1, x_2, \dots, x_n) \in \prod_{k=1}^n I_k$,
$$m_* = \inf_{X \in \prod_{k=1}^n I_k} b(X), \qquad M^* = \sup_{X \in \prod_{k=1}^n I_k} b(X).$$
2. For $(i_1, \dots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}$, let
$$m_{i_1\dots i_n} = \inf_{X \in \prod_{k=1}^n I_k} f\big(u_{i_1\dots i_n}(X)\big), \qquad M_{i_1\dots i_n} = \sup_{X \in \prod_{k=1}^n I_k} f\big(u_{i_1\dots i_n}(X)\big).$$

THEOREM 4.18. *Let $f \in \mathcal{C}(\prod_{k=1}^n I_k)$ be such that $0 \le f(X) \le M$ for all $X \in \prod_{k=1}^n I_k$. Suppose that Δ is a partition of the hyperrectangle $\prod_{k=1}^n I_k$ and $b \in \mathcal{C}(\prod_{k=1}^n I_k)$ satisfies the condition in (3.1). Then the multivariate α-fractal function $f^\alpha_{\Delta,b}$ satisfies $0 \le f^\alpha_{\Delta,b}(x) \le M$ for all $X \in \prod_{k=1}^n I_k$, provided the scale function is so chosen that*
$$\max\Big\{-\frac{m_{i_1\dots i_n}}{M - m_*}, -\frac{M - M_{i_1\dots i_n}}{M^*}\Big\} \le \alpha(u_{i_1\dots i_n}(X)) \le \max\Big\{\frac{m_{i_1\dots i_n}}{M^*}, \frac{M - M_{i_1\dots i_n}}{M - m_*}\Big\},$$
for all $X \in \prod_{k=1}^n I_k$ and $(i_1, \dots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}$. In case $M^* = 0$, the terms containing M^* in the denominator can be dropped out in the above inequality. In particular, these constraints ensure that the fractal counterpart $f^\alpha_{\Delta,b}$ is nonnegative if the germ function f is so.

Proof. Recall from (3.4) that
$$f^\alpha_{\Delta,b}\big(u_{i_1\dots i_n}(X)\big) = \alpha(u_{i_1\dots i_n}(X)) f^\alpha_{\Delta,b}(X) + q_{i_1\dots i_n}(X),$$
for all $X \in \prod_{k=1}^n I_k$ and $(i_1, i_2, \dots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}$, where
$$q_{i_1\dots i_n}(X) = f(u_{i_1\dots i_n}(X)) - \alpha(u_{i_1\dots i_n}(X)) b(X).$$
For brevity, let
$$F_{i_1\dots i_n}(X, y) = \alpha(u_{i_1\dots i_n}(X)) y + q_{i_1\dots i_n}(X).$$
Since f is a nonnegative function and b satisfies (3.1), we have $M^* \ge 0$. Note that the hyperrectangle $\prod_{k=1}^n I_k$ is a finite union of transformed copies of itself, namely, $\prod_{k=1}^n I_k = \bigcup_{(i_1, i_2, \dots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}} u_{i_1 i_2, \dots i_n}\big(\prod_{k=1}^n I_k\big)$. Furthermore, the self-referential function $f^\alpha_{\Delta,b}$ is constructed iteratively by using the above functional equation, and it interpolates the germ function f at points in Δ. Consequently, to prove $0 \le f^\alpha(X) \le M$, for all $X \in \prod_{k=1}^n I_k$, it is sufficient to show that $0 \le F_{i_1\dots i_n}(X, y) \le M$ for all $(X, y) \in \Big(\prod_{k=1}^n I_k\Big) \times [0, M]$ and $(i_1, \dots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}$. We shall deal with it by considering the following two cases:

Case I. $0 \le \alpha(X) < 1$.
Let $(X, y) \in (\prod_{k=1}^n I_k) \times [0, M]$. We have
$$q_{i_1\dots i_n}(X) \le \alpha(u_{i_1\dots i_n}(X)) y + q_{i_1\dots i_n}(X) \le \alpha(u_{i_1\dots i_n}(X)) M + q_{i_1\dots i_n}(X).$$
Thus, $0 \le F_{i_1\dots i_n}(X, y) \le M$ holds if for all $X \in \prod_{k=1}^n I_k$ the following conditions are satisfied:

1. $f(u_{i_1\dots i_n}(X)) - \alpha(u_{i_1\dots i_n}(X)) b(X) \ge 0$.
2. $f(u_{i_1\dots i_n}(X)) - \alpha(u_{i_1\dots i_n}(X)) b(X) \le M\big(1 - \alpha(u_{i_1\dots i_n}(X))\big)$.

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We have, $f(u_{i_1\dots i_n}(X)) \geq m_{i_1\dots i_n}$ and $b(X) \leq M^*$. Choosing a continuous scale function $\alpha : \prod_{k=1}^n I_k \to (-1,1)$ such that $\alpha(u_{i_1\dots i_n}(X)) \leq \frac{m_{i_1\dots i_n}}{M^*}$, we get

$$\begin{aligned} f(u_{i_1\dots i_n}(X)) - \alpha(u_{i_1\dots i_n}(X))b(X) &\geq m_{i_1\dots i_n} - \alpha(u_{i_1\dots i_n}(X))b(X) \\ &\geq m_{i_1\dots i_n} - \frac{m_{i_1\dots i_n}}{M^*}M^* = 0. \end{aligned}$$

Also note that if $M^* = 0$, we have $f(u_{i_1\dots i_n}(X)) - \alpha(u_{i_1\dots i_n}(X))b(X) \geq 0$ for any choice of the scale function. On lines similar to the above, bearing in mind that $f(u_{i_1\dots i_n}(X)) \leq M_{i_1\dots i_n}$ and $b(X) \geq m^*$, the selection of a scale function satisfying $\alpha(u_{i_1\dots i_n}(X)) \leq \frac{M - M_{i_1\dots i_n}}{M - m^*}$ provides

$$f(u_{i_1\dots i_n}(X)) - \alpha(u_{i_1\dots i_n}(X))b(X) \leq M(1 - \alpha(u_{i_1\dots i_n}(X))).$$

Therefore, by choosing a continuous function $\alpha : \prod_{k=1}^n I_k \to (-1,1)$ satisfying the restraint $\alpha(u_{i_1\dots i_n}(X)) \leq \min\{\frac{m_{i_1\dots i_n}}{M^*}, \frac{M - M_{i_1\dots i_n}}{M - m^*}\}$, we get the desired inequality $0 \leq F_{i_1\dots i_n}(X,y) \leq M$.

Case II. $-1 < \alpha(X) \leq 0$.

On lines similar to the first case, we get $0 \leq F_{i_1\dots i_n}(X,y) \leq M$, provided the inequality $\max\{-\frac{m_{i_1\dots i_n}}{M - m_*}, -\frac{M - M_{i_1\dots i_n}}{M^*}\} \leq \alpha(u_{i_1\dots i_n}(X))$ holds. This completes the proof. □

REMARK 4.19. Let $f \in \mathcal{C}(\prod_{k=1}^n I_k)$ be such that $m \leq f(X) \leq 0$ for all $X \in \prod_{k=1}^n I_k$. As in the previous theorem, we can construct a fractal perturbation $f^\alpha_{\Delta,b}$ satisfying $m \leq f^\alpha_{\Delta,b}(X) \leq 0$ for all $X \in \prod_{k=1}^n I_k$. To achieve this, we apply the previous theorem to the function $\hat{f} = -f$ and the associated base function $\hat{b} = -b$. By choosing a continuous scale function $\alpha : \prod_{k=1}^n I_k \to \mathbb{R}$ such that

$$\max\Big\{-\frac{M_{i_1\dots i_n}}{m - M_*}, -\frac{m - m_{i_1\dots i_n}}{m^*}\Big\} \leq \alpha(u_{i_1\dots i_n}(X)) \leq \max\Big\{\frac{M_{i_1\dots i_n}}{m^*}, \frac{m - m_{i_1\dots i_n}}{m - M_*}\Big\},$$

for all $X \in \prod_{k=1}^n I_k$, we can ensure that $m \leq f^\alpha(X) \leq 0$ for all $X \in \prod_{k=1}^n I_k$. The next theorem provides one-sided approximations of a given multivariate real-valued continuous function by fractal functions.

THEOREM 4.20. *Let $f \in \mathcal{C}(\prod_{k=1}^n I_k)$. Assume that the scale function α and the base function b are chosen such that $\alpha(X) \geq 0$ and $b(X) \geq f(X)$ for all $X \in \prod_{k=1}^n I_k$. Then $f^\alpha_{\Delta,b}(X) \leq f(X)$ for all $X \in \prod_{k=1}^n I_k$.*

Proof. Note that

$$\begin{aligned} f^\alpha_{\Delta,b}\big(u_{i_1\dots i_n}(X)\big) - f\big(u_{i_1\dots i_n}(X)\big) &= \alpha(u_{i_1\dots i_n}(X))\big(f^\alpha_{\Delta,b}(X) - b(X)\big) \\ &= \alpha(u_{i_1\dots i_n}(X))\big(f^\alpha_{\Delta,b}(X) - f(X)\big) + \alpha(u_{i_1\dots i_n}(X))\big(f(X) - b(X)\big). \end{aligned}$$

Using the assumptions on the scale function α and the base function b, the proof can be completed on lines similar to the above theorem. See also [40, Theorem 3.18] for the bivariate analogue of this result. □

REMARK 4.21. If $\alpha(X) \geq 0$ and $b(X) \leq f(X)$ for all $X \in \prod_{k=1}^n I_k$ in the previous theorem, then $f^\alpha(X) \geq f(X)$ for all $X \in \prod_{k=1}^n I_k$.

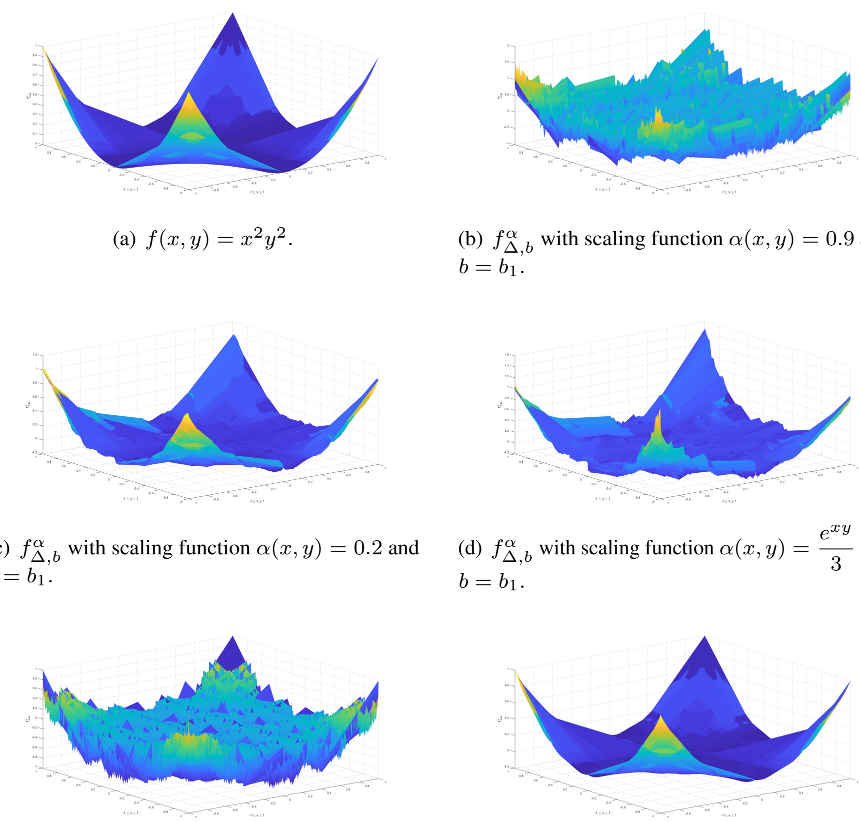
EXAMPLE 4.22. Consider the square, $[-1,1] \times [-1,1] \subset \mathbb{R}^2$. Let

$$\Delta = \{-1, -0.5, 0, 0.5, 1\} \times \{-1, -0.5, 0, 0.5, 1\}$$

be a mesh partition of the square and $f(x,y) = x^2y^2$ be the seed function displayed in Figure 4.1(a). Let us take two base functions satisfying the matching conditions prescribed in Theorem 2.1 but are otherwise arbitrary:

1. $b_1(x, y) = \big(\sin(\pi x) + \cos(\frac{\pi y}{2}) + 1\big) f(x, y)$.
2. $b_2(x, y) = f(\sin(\frac{\pi x}{2}), \sin(\frac{\pi y}{2}))$.

As in the previous example, it is easy to see that the above choices of b satisfy the conditions sought in (3.1).



(a) $f(x, y) = x^2y^2$.

(b) $f^{\alpha}_{\Delta,b}$ with scaling function $\alpha(x, y) = 0.9$ and $b = b_1$.

(c) $f^{\alpha}_{\Delta,b}$ with scaling function $\alpha(x, y) = 0.2$ and $b = b_1$.

(d) $f^{\alpha}_{\Delta,b}$ with scaling function $\alpha(x, y) = \dfrac{e^{xy}}{3}$ and $b = b_1$.

(e) $f^{\alpha}_{\Delta,b}$ with scaling function $\alpha(x, y) = 0.9$ and $b = b_2$.

(f) $f^{\alpha}_{\Delta,b}$ with scaling function $\alpha(x, y) = \dfrac{1}{16}$ and $b = b_2$.

FIG. 4.1. *Fractal functions corresponding to the seed function $f(x, y) = x^2y^2$ associated with different choices of scaling functions and base functions.*

Figures 4.1(b)–4.1(d) display the surfaces corresponding to the fractal perturbations of f with the base function b_1 and the scaling function $\alpha : [-1, -1] \times [-1, 1] \to (-1, 1)$ given by $\alpha(x, y) = 0.9$, $\alpha(x, y) = 0.2$, and $\alpha(x, y) = \dfrac{e^{xy}}{3}$, respectively. Figures 4.1(e)–4.1(f) provide the graphs of the fractal functions of f associated with the base function b_2 and the scaling function $\alpha(x, y) = 0.9$ and $\alpha(x, y) = \dfrac{1}{16}$, respectively. It can be seen easily that $0 \le f(x, y) \le 1$ for all $(x, y) \in [-1, 1] \times [-1, 1]$. In particular, $f(x, y)$ is nonnegative on $[-1, 1] \times [-1, 1]$. However, the fractal counterparts $f^{\alpha}_{\Delta,b}$, in general, may not preserve the nonnegativity of f; see, for instance, Figures 4.1(b)–4.1(e). In view of Theorem 4.18,

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we see that the fractal perturbation of f associated with $b = b_2$ and $\alpha(x, y) = \dfrac{1}{16}$ satisfies $0 \le f^{\alpha}_{\Delta,b}(x, y) \le 1$ for all $(x, y) \in [-1, 1] \times [-1, 1]$. As indicated by Figures 4.1(a)–4.1(f), the properties such as fractal dimensions and smoothness of the fractal perturbation of a given function depend on the selection of the parameters-base function and scaling function. These additional parameters may find applications to tackle problems combined with optimization and approximation. The optimal selection of α and b depends on the modeling problem at hand, and we leave it open.

Appendix A. Proof of Theorem 2.1. This appendix is devoted to the detailed proof of Theorem 2.1.

Proof. For convenience of the reader, we shall divide the proof into several steps.

Step I: Considering an appropriate function space and the Read-Bajraktarević operator. Let $\mathcal{C}\big(\prod_{k=1}^{n} I_k\big)$ denote the Banach space of all real-valued continuous functions defined on the n-dimensional hyperrectangle $\prod_{k=1}^{n} I_k$ endowed with the uniform norm. The subset

$$\mathcal{C}^*\Big(\prod_{k=1}^{n} I_k\Big) = \Big\{ g \in \mathcal{C}\Big(\prod_{k=1}^{n} I_k\Big) : g(x_{1,j_1}, x_{2,j_2}, \ldots, x_{n,j_n}) = y_{j_1 j_2 \ldots j_n} : \\ (j_1, j_2, \ldots, j_n) \in \prod_{k=1}^{n} \partial\Sigma_{N_k,0} \Big\}$$

is a closed (and hence a complete) metric subspace of $\mathcal{C}\big(\prod_{k=1}^{n} I_k\big)$ with the uniform metric. Consider the so-called Read-Bajraktarević (RB) operator (see also [1])

$$T : \mathcal{C}^*\Big(\prod_{k=1}^{n} I_k\Big) \to \mathcal{C}^*\Big(\prod_{k=1}^{n} I_k\Big)$$

defined by

$$\begin{aligned}(Tg)&(x_1, x_2, \ldots, x_n) \\ &= F_{i_1 i_2 \ldots i_n}\Big(u^{-1}_{1,i_1}(x_1), \quad \ldots, u^{-1}_{n,i_n}(x_n), g\big(u^{-1}_{1,i_1}(x_1), \ldots, u^{-1}_{n,i_n}(x_n)\big)\Big),\end{aligned} \tag{A.1}$$

for all $(x_1, x_2, \ldots, x_n) \in \prod_{k=1}^{n} I_{k,i_k}$ and $(i_1, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$.

Step II: Proving that T is well-defined.
Let $g \in \mathcal{C}^*(\prod_{k=1}^{n} I_k)$. For simplicity, let us consider $X = (x_1, \ldots, x_r, \ldots, x_s, \ldots, x_n) \in \prod_{k=1}^{n} I_{k,i_k}$ such that $x_r \in I_{r,i_r} \cap I_{r,i_r+1}$ and $x_s \in I_{s,i_s} \cap I_{s,i_s+1}$ for some $r, s \in \Sigma_n$ and $(i_r, i_s) \in \operatorname{int}\Sigma_{N_r,0} \times \operatorname{int}\Sigma_{N_s,0}$. This is possible if and only if $x_r = x_{r,i_r}$ and $x_s = x_{s,i_s}$. Without loss of generality we assume that $r < s$ and deal with the following four possible cases.

i) Case 1. Treating $x_r = x_{r,i_r}$ as a point in I_{r,i_r} and $x_s = x_{s,i_s}$ as a point in I_{s,i_s}, we have

$$T(g)(X) = F_{i_1 \ldots i_r \ldots i_s \ldots i_n}\Big(u^{-1}_{1,i_1}(x_1), \ldots, u^{-1}_{r,i_r}(x_{r,i_r}), \ldots, u^{-1}_{s,i_s}(x_{s,i_s}), \ldots, u^{-1}_{n,i_n}(x_n), \\ g\big(u^{-1}_{1,i_1}(x_1), \ldots, u^{-1}_{n,i_n}(x_n)\big)\Big).$$

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ii) Case 2. Consider $x_r = x_{r,i_r}$ as a point in I_{r,i_r+1} and $x_s = x_{s,i_s}$ as a point in I_{s,i_s}. Bearing (2.2) and (2.6) in mind, one gets

$$\begin{aligned}T(g)(X) &= F_{i_1\ldots i_r+1\ldots i_s\ldots i_n}\Big(u_{1,i_1}^{-1}(x_1),\ldots u_{r,i_r+1}^{-1}(x_{r,i_r}),\ldots,u_{s,i_s}^{-1}(x_{s,i_s}),\ldots,u_{n,i_n}^{-1}(x_n),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_1),\ldots,u_{n,i_n}^{-1}(x_n)\big)\Big)\\ &= F_{i_1\ldots i_r\ldots i_s\ldots i_n}\Big(u_{1,i_1}^{-1}(x_1),\ldots,u_{r,i_r}^{-1}(x_{r,i_r}),\ldots,u_{s,i_s}^{-1}(x_{s,i_s}),\ldots,u_{n,i_n}^{-1}(x_n),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_1),\ldots,u_{n,i_n}^{-1}(x_n)\big)\Big).\end{aligned}$$

iii) Case 3. Considering $x_r = x_{r,i_r}$ as an element in I_{r,i_r} and $x_s = x_{s,i_s}$ as an element in I_{s,i_s+1}, similar to the previous case we have

$$\begin{aligned}T(g)(X) &= F_{i_1\ldots i_r\ldots i_s+1\ldots i_n}\Big(u_{1,i_1}^{-1}(x_1),\ldots,u_{r,i_r}^{-1}(x_{r,i_r}),\ldots,u_{s,i_s+1}^{-1}(x_{s,i_s}),\ldots,u_{n,i_n}^{-1}(x_n),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_1),\ldots,u_{n,i_n}^{-1}(x_n)\big)\Big)\\ &= F_{i_1\ldots i_r\ldots i_s\ldots i_n}\Big(u_{1,i_1}^{-1}(x_1),\ldots,u_{r,i_r}^{-1}(x_{r,i_r}),\ldots,u_{s,i_s}^{-1}(x_{s,i_s}),\ldots,u_{n,i_n}^{-1}(x_n),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_1),\ldots,u_{n,i_n}^{-1}(x_n)\big)\Big).\end{aligned}$$

iv) Case 4. Finally, let us view $x_r = x_{r,i_r}$ as a point in I_{r,i_r+1} and $x_s = x_{s,i_s}$ as a point in I_{s,i_s+1}. Using (2.2) and (2.6) we obtain

$$\begin{aligned}&T(g)(X)\\ &= F_{i_1\ldots i_r+1\ldots i_s+1\ldots i_n}\Big(u_{1,i_1}^{-1}(x_1),\ldots,u_{r,i_r+1}^{-1}(x_{r,i_r}),\ldots,u_{s,i_s+1}^{-1}(x_{s,i_s}),\ldots,u_{n,i_n}^{-1}(x_n),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_1),\ldots,u_{n,i_n}^{-1}(x_n)\big)\Big)\\ &= F_{i_1\ldots i_r\ldots i_s+1\ldots i_n}\Big(u_{1,i_1}^{-1}(x_1),\ldots,u_{r,i_r}^{-1}(x_{r,i_r}),\ldots,u_{s,i_s+1}^{-1}(x_{s,i_s+1}),\ldots,u_{n,i_n}^{-1}(x_n),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_1),\ldots,u_{n,i_n}^{-1}(x_n)\big)\Big)\\ &= F_{i_1\ldots i_r\ldots i_s\ldots i_n}\Big(u_{1,i_1}^{-1}(x_1),\ldots,u_{r,i_r}^{-1}(x_{r,i_r}),\ldots,u_{s,i_s}^{-1}(x_{s,i_s})m\ldots,u_{n,i_n}^{-1}(x_n),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_1),\ldots,u_{n,i_n}^{-1}(x_n)\big)\Big).\end{aligned}$$

We see that the value of $T(g)(X)$ is determined univocally in all the four cases. Similarly, all other possibilities can be worked out to conclude that $T(g)$ is well defined on the boundary of the hyperrectangle $\prod_{k=1}^{n} I_{k,i_k}$. Furthermore, $T(g)$ is continuous on $\prod_{k=1}^{n} I_k$.

Let $(i_1, i_2, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k,0}$. Choose $(j_1, j_2, \ldots, j_n) \in \prod_{k=1}^{n} \partial\Sigma_{N_k,0}$ such that $(i_1, i_2, \ldots, i_n) = (\tau(i_1, j_1), \tau(i_2, j_2), \ldots, \tau(i_n, j_n))$. By definition of τ, we have

$$(x_{1,j_1}, x_{2,j_2}, \ldots, x_{n,j_n}) = \big(u_{1,i_1}^{-1}(x_{1,i_1}), u_{2,i_2}^{-1}(x_{2,i_2}), \ldots, u_{n,i_n}^{-1}(x_{n,i_n})\big).$$

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Thus,

$$\begin{aligned} T(g)(x_{1,i_1},\dots,x_{n,i_n}) &= F_{i_1 i_2 \dots i_n}\Big(u_{1,i_1}^{-1}(x_{1,i_1}), u_{2,i_2}^{-1}(x_{2,i_2}),\dots,u_{n,i_n}^{-1}(x_{n,i_n}),\\ &\qquad g\big(u_{1,i_1}^{-1}(x_{1,i_1}), u_{2,i_2}^{-1}(x_{2,i_2}),\dots,u_{n,i_n}^{-1}(x_{n,i_n})\big)\Big)\\ &= F_{i_1 i_2 \dots i_n}\Big(x_{1,j_1}, x_{2,j_2},\dots,x_{n,j_n}, g\big(x_{1,j_1}, x_{2,j_2},\dots,x_{n,j_n}\big)\Big)\\ &= F_{i_1 i_2 \dots i_n}\Big(x_{1,j_1}, x_{2,j_2},\dots,x_{n,j_n}, y_{j_1 j_2 \dots j_n}\Big)\\ &= y_{\tau(i_1,j_1)\tau(i_2,j_2)\dots\tau(i_n,j_n)}\\ &= y_{i_1 i_2 \dots i_n}, \end{aligned}$$

showing that $T(g)$ interpolates the data in Δ for all $g \in \mathcal{C}^*(\prod_{k=1}^n I_k)$. In particular, T maps $\mathcal{C}^*(\prod_{k=1}^n I_k)$ into $\mathcal{C}^*(\prod_{k=1}^n I_k)$.

Step III: Proving that T is a contraction.
Let $g, h \in \mathcal{C}^*(\prod_{k=1}^n I_k)$ and $X = (x_{1,i_1},\dots,x_{n,i_n}) \in \prod_{k=1}^n I_{k,i_k}$. Using (2.4) and (A.1) we have

$$\begin{aligned} &\big|T(g)(X) - T(h)(X)\big|\\ &= \Big|F_{i_1 i_2 \dots i_n}\Big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n), g\big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n)\big)\Big)\\ &\qquad - F_{i_1 i_2 \dots i_n}\Big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n), h\big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n)\big)\Big)\Big|\\ &\le \alpha_{i_1 i_2 \dots i_n}\Big|g\big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n)\big) - h\big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n)\big)\Big|\\ &\le \|\alpha\|_\infty \|g-h\|_\infty, \end{aligned}$$

where $\|\alpha\|_\infty = \max\{\alpha_{i_1 i_2 \dots i_n} : (i_1, i_2, \dots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}\}$. Since the above inequality holds for all $X \in \prod_{k=1}^n I_k$, it follows that

$$\|T(g) - T(h)\|_\infty \le \|\alpha\|_\infty \|g-h\|_\infty.$$

This yields that T is a contraction on $\mathcal{C}^*(\prod_{k=1}^n I_k)$.

Step IV: Proving existence by an application of the Banach fixed point theorem.
By the Banach fixed point theorem T has a unique fixed point. That is, there exists a unique function $\tilde{f} \in \mathcal{C}^*(\prod_{k=1}^n I_k)$ such that

$$\begin{aligned} \tilde{f}(x_1,\dots,x_n) =& F_{i_1 i_2 \dots i_n}\Big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n), \tilde{f}\big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n)\big)\Big),\\ &\forall (x_1,\dots,x_n) \in \prod_{k=1}^n I_{k,i_k} \text{ and } (i_1,\dots,i_n) \in \prod_{k=1}^n \Sigma_{N_k}. \end{aligned}$$

Writing $X = (x_1, x_2, \dots, x_n)$,

$$u_{i_1,\dots i_n}^{-1}(X) = \big(u_{1,i_1}^{-1}(x_1),\dots,u_{n,i_n}^{-1}(x_n)\big), \text{ and } u_{i_1,\dots i_n}(X) = \big(u_{1,i_1}(x_1),\dots,u_{n,i_n}(x_n)\big),$$

we have the self-referential equation

$$\begin{aligned} \tilde{f}(X) &= F_{i_1 i_2 \dots i_n}\big(u_{i_1,\dots i_n}^{-1}(X), \tilde{f}(u_{i_1,\dots i_n}^{-1}(X))\big),\\ &\qquad \forall X \in \prod_{k=1}^n I_{k,i_k} \text{ and } (i_1,\dots,i_n) \in \prod_{k=1}^n \Sigma_{N_k}. \end{aligned}$$

Equivalently,

$$\tilde{f}\big(u_{i_1,\dots i_n}(X)\big) = F_{i_1 i_2 \dots i_n}\big(X, \tilde{f}(X)\big), \qquad \forall X \in \prod_{k=1}^{n} I_k \text{ and } (i_1,\dots,i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}. \tag{A.2}$$

Let $G = \big\{(X, \tilde{f}(X)) : X \in \prod_{k=1}^{n} I_k\big\}$ be the graph of $\tilde{f}$. In view of (2.5) and (A.2),

$$\begin{aligned}
&\bigcup \Big\{ W_{i_1 i_2 \dots i_n}(G) : (i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k} \Big\} \\
&\quad = \bigcup \Big\{ W_{i_1 i_2 \dots i_n}(X, \tilde{f}(X)) : X \in \prod_{k=1}^{n} I_k, (i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k} \Big\} \\
&\quad = \bigcup \Big\{ \big(u_{i_1 \dots i_n}(X), F_{i_1 i_2 \dots i_n}(X, \tilde{f}(X))\big) : X \in \prod_{k=1}^{n} I_k, (i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k} \Big\} \\
&\quad = \bigcup \Big\{ \big(u_{i_1 \dots i_n}(X), \tilde{f}(u_{i_1,\dots,i_n}(X))\big) : X \in \textstyle\prod_{k=1}^{n} I_k, (i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k} \Big\} \\
&\quad = \bigcup \Big\{ (X, f(X)) : X \in \prod_{k=1}^{n} I_k \Big\} \\
&\quad = G,
\end{aligned}$$

completing the proof. □

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