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A TIME-STEPPING FINITE ELEMENT METHOD FOR A TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATION OF HIDDEN-MEMORY SPACE-TIME VARIABLE ORDER*

XIANGCHENG ZHENG† AND HONG WANG‡

Abstract. We analyze a time-stepping finite element method for a time-fractional partial differential equation with hidden-memory space-time variable order. Due to the coupling of the space-dependent variable order with the finite element formulation and the hidden memory, the variable fractional order cannot be split from the space and destroys the monotonicity in the time-stepping discretization. Because of these difficulties, the numerical analysis of a fully-discrete finite element method of the proposed model remained untreated in the literature. We develop an alternative analysis to address these issues and to prove an optimal-order error estimate of the fully-discrete finite element scheme without any assumption on the regularity of the true solution and perform numerical experiments to substantiate the theoretical findings.

Key words. fractional differential equation, well-posedness and regularity, hidden memory, space-time variable order, time-stepping finite element discretization, error estimate

AMS subject classifications. 35R11, 65M12

1. Introduction.

Time-fractional diffusion equations (tFDEs)

$$
\begin{aligned}
&\partial_t^\alpha u - \Delta u = 0, \qquad 0 < \alpha < 1,\\
&\partial_t^\alpha g := \int_0^t \frac{g'(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}} ds
\end{aligned}
\tag{1.1}
$$

accurately model physical processes exhibiting power-law memory properties, as they were derived assuming their solutions have power-law decays [23, 24], and they have attracted extensive research [2, 3, 4, 5, 7, 9, 10, 11, 14, 16, 17, 21, 22, 33, 34, 40]. But they yield solutions with initial weak singularities [26, 28] and fail to capture the initial Fickian diffusion behavior of the processes, making the error estimates of their discretizations proved for smooth solutions unrealistic. The reason is that equation (1.1) is derived as the diffusion limit of a continuous time random walk when the number of particle jumps tends to infinity [23, 24], and this limit is only valid for $t \gg 1$ rather than all the way up to the initial time $t = 0$ as is often assumed in the literature. A two-time-scale mobile-immobile tFDE

$$
\partial_t u + \kappa \partial_t^\alpha u - \Delta u = 0, \quad \kappa > 0,
$$

was presented in [27] to improve the modeling of subdiffusive transport of solutes in heterogeneous aquifers, in which a portion of $1/(1+\kappa)$ of the solute mass stays in the mobile phase undergoing Brownian motion while the rest gets trapped in the aquifers forming an immobile phase leading to subdiffusive transport.

In applications such as bioclogging [6] and hydrofracturing in gas and oil recovery [12], the medium structures may evolve with time and change their fractal dimension, which in turn changes the order of the tFDEs [23] and leads to variable-order tFDEs. A widely used

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†School of Mathematical Sciences, Peking University, Beijing 100871, China.

‡Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA (hwang@math.sc.edu).

variable-order tFDE corresponds to $\alpha = \alpha(t)$ in (1.1) [8, 29, 31, 35, 41]. A hidden-memory variable-order tFDE, which corresponds to $\alpha = \alpha(s)$ in (1.1), was investigated in [19, 29]. A variable-order tFDE with general time-dependent hidden-memory variable order $\alpha = \alpha(s,t)$ was proposed in [19], which showed that the variable-order tFDE with $\alpha = \alpha(t)$ has an immediate response to a change of the variable order but has no memory retentiveness of the order history, while the one with $\alpha = \alpha(s)$ has a weak response to a change of the variable order but has an intermediate memory retentiveness of the order history. In addition, the model with $\alpha = \alpha(t-s)$ has a very slow response to a change of the variable order but has strong memory retentiveness of the order history. A general time-dependent variable order $\alpha = \alpha(s,t)$ provides a flexible description between the fading memory of the order history and the response to order variations among other impacts. Another definition, called the variable-order Scarpi derivative, was recently introduced in [13] via the Laplace transform. For highly heterogeneous materials, the variable order α may also depend on the spatial location $\boldsymbol{x}$.

We consider a tFDE with a general space-time variable order

$$
\begin{aligned}
\partial_t u(\boldsymbol{x},t) + \kappa(\boldsymbol{x},t)\partial_t^{\alpha(\boldsymbol{x},:,t)} u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) &= f(\boldsymbol{x},t), && (\boldsymbol{x},t) \in \Omega \times (0,T],\\
u(\boldsymbol{x},t) &= 0, && (\boldsymbol{x},t) \in \partial\Omega \times (0,T],\\
u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}), && \boldsymbol{x} \in \Omega.
\end{aligned}
\tag{1.2}
$$

Here $\Omega \subset \mathbb{R}^d$ is a simply-connected bounded domain with a piecewise smooth boundary $\partial\Omega$ with convex corners, $\boldsymbol{x} := (x_1, \ldots, x_d)$, with $1 \le d \le 3$, denotes the spacial variables, $\kappa \ge 0$ is the partition coefficient, f and u_0 refer to the source term and the initial data, respectively, and $\partial_t^{\alpha(\boldsymbol{x},:,t)}$ is the space-time hidden-memory variable-order fractional differential operator [19]

$$
\begin{aligned}
\partial_t^{\alpha(\boldsymbol{x},:,t)} q &:= I_t^{1-\alpha(\boldsymbol{x},:,t)} \partial_t q,\\
I_t^{1-\alpha(\boldsymbol{x},:,t)} q &:= \int_0^t \frac{q(\boldsymbol{x},s)ds}{\Gamma(1-\alpha(\boldsymbol{x},s,t))(t-s)^{\alpha(\boldsymbol{x},s,t)}},\\
&0 \le \alpha(\boldsymbol{x},s,t) < 1, \quad (\boldsymbol{x},s,t) \in \Omega \times \mathfrak{T},\\
&\mathfrak{T} := \{(s,t) : 0 \le s \le t \le T\}.
\end{aligned}
$$

Note that the symbol ":" in $\alpha(\boldsymbol{x},:,t)$ represents the implicit variable integrated into the fractional integral, e.g., the variable s in the above definitions.

There exists some recent work on the mathematical and numerical analysis for the model (1.2) with a variable order depending on time and/or space [15, 18, 32, 37, 38, 39]. In particular, the semidiscrete-in-time schemes were analyzed for the model (1.2) with space-time dependent variable order in [18, 39]. Without discretization in space, the difficulties in the numerical analysis arising from the coupling of the space-dependent variable order and the spatial inner product of the FEM are avoided (cf. the statements given at (4.6)–(4.7) in Section 4 for details), which significantly facilitates the estimates [39]. For these reasons, error estimates for the fully-discrete finite element scheme of the model (1.2) remained untreated, to the best knowledge of the authors, which motivates the current work.

In this paper we prove an optimal-order error estimate of a time-stepping discretization of problem (1.2). The rest of the paper is organized as follows: In Section 2 we outline the notation and auxiliary results to be used subsequently. In Section 3 we prove the well-posedness and smoothing properties of problem (1.2). In Section 4 we present a time-stepping discretization and prove its optimal-order error estimate. In Section 5 we conduct numerical experiments to substantiate the theoretical findings.

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2. Preliminaries. We present some preliminaries to be used subsequently.

2.1. Notations. For a positive integer m and a real number $1 \le p \le \infty$, let $W^{m,p}(\Omega)$ be the Sobolev space of functions with weak derivatives up to order m in $L^p(\Omega)$, the space of functions with p-th power being Lebesgue integrable on Ω (similarly defined with Ω replaced by an interval $\mathcal{I}$). Let $H^m(\Omega) := W^{m,2}(\Omega)$, and let $H^m_0(\Omega)$ be its subspace with zero boundary conditions up to order $m-1$. For a non-integer $s \ge 0$, $H^s(\Omega)$ is defined via interpolation [1]. Let $0 < \lambda_i \uparrow \infty$ be the eigenvalues and $\{\phi_i\}_{i=1}^\infty$ the corresponding orthonormal eigenfunctions of the problem $-\Delta \phi_i = \lambda_i \phi_i$ with zero boundary conditions. We introduce the Sobolev space $\check{H}^s(\Omega)$, for $s \ge 0$, by

$$\check{H}^s(\Omega) := \Big\{ v \in L^2(\Omega) : \|v\|^2_{\check{H}^s} := \sum_{i=1}^\infty \lambda_i^s (v, \phi_i)^2 < \infty \Big\},$$

which is a subspace of $H^s(\Omega)$ satisfying $\check{H}^0(\Omega) = L^2(\Omega)$ and $\check{H}^2(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$ [30]. For a Banach space $\mathcal{X}$, define $W^{m,p}(\mathcal{I}; \mathcal{X})$ by (see [1])

$$W^{m,p}(0,T;\mathcal{X}) := \Big\{ v : [0,T] \to \mathcal{X} : \big\|\partial_t^k v(\cdot,t)\big\|_{\mathcal{X}} \in L^p(0,T), \ \ 0 \le k \le m, \ 1 \le p \le \infty \Big\},$$

equipped with the norm

$$\|v\|_{W^{m,p}(0,T;\mathcal{X})} := \begin{cases} \left(\sum_{l=0}^m \int_0^T \big\|\partial_t^k v(\cdot,t)\big\|^p_{\mathcal{X}} dt \right)^{1/p}, & 1 \le p < \infty, \\ \max\limits_{0 \le k \le m} \ \operatorname*{ess\,sup}\limits_{t \in (0,T)} \big\|\partial_t^k v(\cdot,t)\big\|_{\mathcal{X}}, & p = \infty. \end{cases}$$

For instance, if $\mathcal{X} = L^p(\Omega)$, then the above definition becomes

$$\|v\|_{W^{m,p}(0,T;L^p(\Omega))} := \begin{cases} \left(\sum_{l=0}^m \int_0^T \int_\Omega \big|\partial_t^k v(\cdot,t)\big|^p d\boldsymbol{x} dt \right)^{1/p}, & 1 \le p < \infty, \\ \max\limits_{0 \le k \le m} \ \operatorname*{ess\,sup}\limits_{t \in (0,T), \boldsymbol{x} \in \Omega} \big|\partial_t^k v(\boldsymbol{x},t)\big|_{\mathcal{X}}, & p = \infty. \end{cases}$$

In particular, we have $L^p(\mathcal{I}; \mathcal{X}) := W^{0,p}(\mathcal{I}; \mathcal{X})$. Throughout this paper, we assume that there exists a constant $0 < \alpha^* < 1$ such that

$$0 \le \alpha(\boldsymbol{x}, s, t) \le \alpha^*,$$

and we use Q and Q_i to denote generic positive constants, where Q may assume different values at different occurrences. We set $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and $L^p(\mathcal{X})$ for $L^p(0,T;\mathcal{X})$ for brevity and drop the notation Ω in the spaces and norms if no confusion occurs.

2.2. Solution representation and resolvent estimates. For $\theta \in (\pi/2, \pi)$ and $\delta > 0$, let Γ_θ be the contour in the complex plane defined by

$$\Gamma_\theta := \big\{ z \in \mathbb{C} : |\arg(z)| = \theta, |z| \ge \delta \big\} \cup \big\{ z \in \mathbb{C} : |\arg(z)| \le \theta, |z| = \delta \big\}.$$

The following inequalities hold for $1 \le p \le \infty$, $0 < \mu \le 1$, and $Q = Q(\theta, \mu, p)$ [3, 20]:

$$\begin{aligned} \int_{\Gamma_\theta} |z|^{\mu-1} |e^{tz}| \, |dz| &\le Q t^{-\mu}, \\ \Big\| \int_{\Gamma_\theta} z^\mu (z - \Delta)^{-1} e^{tz} \, dz \Big\|_{L^p \to L^p} &\le \frac{Q}{t^\mu}, \end{aligned} \qquad t \in (0,T], \tag{2.1}$$

where the norm $\|\cdot\|_{L^p\to L^p}$ of an operator $\mathcal{S}: L^p(\Omega)\to L^p(\Omega)$ is defined as

$$\|\mathcal{S}\|_{L^p\to L^p} := \sup_{0\neq q\in L^p}\frac{\|\mathcal{S}q\|_{L^p}}{\|q\|_{L^p}}.$$

For any $q\in L^1_{loc}(\mathcal{I})$, the Laplace transform $\mathcal{L}$ of its extension $\tilde{q}(t)$ to zero outside $\mathcal{I}$ and the corresponding inverse transform $\mathcal{L}^{-1}$ are denoted by

$$\mathcal{L}q(z) := \int_0^\infty \tilde{q}(t)e^{-tz}dt, \qquad \mathcal{L}^{-1}(\mathcal{L}q(z)) := \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\theta} e^{tz}\mathcal{L}q(z)dz = q(t).$$

The Riemann-Liouville fractional differential operator ${}^R\partial_t^\gamma q := \partial_t\, I_t^{1-\gamma}q$ satisfies [25]

$$\mathcal{L}\big({}^R\partial_t^\gamma q(t)\big) = z^\gamma\mathcal{L}\big(q(t)\big), \qquad 0\le\gamma<1.$$

The semigroup $E(t) := e^{t\Delta}$ generated by the Dirichlet Laplacian has the spectral decomposition and expression in terms of the inverse Laplace transform

$$E(t)\psi(\boldsymbol{x}) = \sum_{i=1}^\infty e^{-\lambda_i t}(\psi,\phi_i)\phi_i(\boldsymbol{x}) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\theta} e^{zt}(z-\Delta)^{-1}\psi(\boldsymbol{x})\,dz, \quad \psi\in L^2(\Omega). \tag{2.2}$$

The following estimates hold for any $t>0$ and $1\le p\le\infty$ [3, 20, 30]:

$$\|E(t)\|_{L^p\to L^p} + t\|E'(t)\|_{L^p\to L^p} + t\|\Delta E(t)\|_{L^p\to L^p} \le Q,$$

where $E'(t) := e^{t\Delta}\Delta$. The solution u to the heat equation

$$\begin{aligned} 2\partial_t u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) &= f(\boldsymbol{x},t), && (\boldsymbol{x},t)\in\Omega\times(0,T],\\ u(\boldsymbol{x},t) &= 0, && (\boldsymbol{x},t)\in\partial\Omega\times(0,T],\\ u(\boldsymbol{x},0) &= 0, && \boldsymbol{x}\in\Omega, \end{aligned} \tag{2.3}$$

can be expressed in terms of $E(t)$ via Duhamel's principle

$$u(\boldsymbol{x},t) = \int_0^t E(t-\theta)f(\boldsymbol{x},\theta)d\theta \tag{2.4}$$

and allows for the following estimate [3]:

LEMMA 2.1. *For $f\in L^p(L^2)$, for $1<p<\infty$, problem (2.3) has a unique solution $u\in W^{1,p}(L^2)\cap L^p(\check{H}^2)$ given by (2.4) such that*

$$\|u\|_{W^{1,p}(0,t;L^2)} + \|u\|_{L^p(0,t;\check{H}^2)} \le Q\|f\|_{L^p(0,t;L^2)}, \qquad 0<t\le T,$$

where Q is independent of f, t, or T.

3. Analysis of the variable-order tFDE (1.2). We prove the well-posedness and smoothing properties of the space-time hidden-memory variable-order tFDE (1.2).

THEOREM 3.1. *Let $\alpha\in W^{1,\infty}(\mathfrak{T};L^\infty)$, $\kappa\in W^{1,\infty}(L^\infty)$, $\Delta u_0\in L^2$, and $f\in L^p(L^2)$, for $1<p<\infty$. Then problem (1.2) has a unique solution $u\in W^{1,p}(L^2)\cap L^p(\check{H}^2)$ and*

$$\|u\|_{W^{1,p}(L^2)} + \|u\|_{L^p(\check{H}^2)} \le Q\big(\|f\|_{L^p(L^2)} + \|\Delta u_0\|_{L^2}\big), \qquad Q = Q(\alpha_*,T,p).$$

The proof of this theorem could be carried out following that of [39, Theorem 1] and thus is omitted.

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Next we prove more general space-time regularity estimates of the solutions u for the problem (1.2) than those in [39, Theorem 2]. The difference $w(\boldsymbol{x},t) := u(\boldsymbol{x},t) - u_0(\boldsymbol{x})$ satisfies

$$(\partial_t w - \Delta w)(\boldsymbol{x},t) = f(\boldsymbol{x},t) + \Delta u_0(\boldsymbol{x}) - \kappa(\boldsymbol{x},t)\partial_t^{\alpha(\boldsymbol{x},\cdot,t)} w(\boldsymbol{x},t) \tag{3.1}$$

on $(\boldsymbol{x},t) \in \Omega \times (0,T]$ equipped with homogeneous initial and boundary conditions. We use (2.4) to express w as follows:

$$\begin{aligned} w(\boldsymbol{x},t) &= \int_0^t E(t-s)\big(\Delta u_0 + f(\boldsymbol{x},s)\big)ds - \int_0^t E(t-s)(\kappa\,\partial_s^{\alpha(\boldsymbol{x},\cdot,s)} w(\boldsymbol{x},s))ds \\ &:= L_1 - L_2. \end{aligned} \tag{3.2}$$

LEMMA 3.2. *Assume that $\alpha \in W^{3,\infty}(\mathfrak{T};L^\infty)$ and $\kappa \in W^{2,\infty}(L^\infty)$. Then there is a positive $Q = Q(\varepsilon, \alpha_*, \|\alpha\|_{W^{3,\infty}(\mathfrak{T};L^\infty)}, \|\kappa\|_{W^{2,\infty}(L^\infty)}, T)$ such that for $0 < t \le T$ and $0 \le \varepsilon \ll 1$,*

$$\big|{}^R\partial_t^\varepsilon \partial_t\big(\kappa\partial_t^{\alpha(\boldsymbol{x},\cdot,t)} w\big)\big| \le Q\int_0^t \frac{\big|\partial_\theta^2 w(\boldsymbol{x},\theta)\big|d\theta}{(t-\theta)^{\alpha_*+\varepsilon}} + \frac{Q(|f(\boldsymbol{x},0)| + |\Delta u_0|)}{t^{\alpha(\boldsymbol{x},0,0)+\varepsilon}}. \tag{3.3}$$

Proof. The proof of this lemma is given in Appendix A. ∎

THEOREM 3.3. *Suppose that $\alpha \in W^{3,\infty}(\mathfrak{T};L^\infty)$, $\kappa \in W^{2,\infty}(L^\infty)$, $\Delta u_0, \Delta^2 u_0 \in L^2$, and $f \in H^1(L^2) \cap L^2(\check{H}^2)$. Then, for $1 \le p \le 2$ and $p < 1/\alpha_0$ with $\alpha_0 := \|\alpha(\cdot,0,0)\|_{L^\infty}$,*

$$\begin{aligned} &\|u\|_{W^{2,p}(L^2)} + \|u\|_{W^{1,\infty}(L^2)} + \|u\|_{W^{1,p}(\check{H}^2)} \\ &\qquad \le Q\left(\|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta u_0\| + \|\Delta^2 u_0\|\right), \end{aligned}$$

where $Q = Q(p, \alpha_*, \|\alpha\|_{W^{3,\infty}(\mathfrak{T};L^\infty)}, \|\kappa\|_{W^{2,\infty}(L^\infty)}, T)$.

Proof. We apply (2.2) to directly evaluate $\partial_t^2 L_1$ by

$$\partial_t^2 L_1 \quad = \partial_t f + \Delta(\Delta u_0 + f) + \textstyle\int_0^t \sum_{i=1}^\infty \lambda_i^2 e^{-\lambda_i(t-s)}(\Delta u_0 + f, \phi_i)\phi_i ds.$$

An application of Young's inequality yields

$$\begin{aligned} \|\partial_t^2 L_1\|_{L^2(L^2)} &\le \left[\sum_{i=1}^\infty \Big\| \int_0^t \lambda_i^2 e^{-\lambda_i(t-s)}(\Delta u_0 + f(\cdot,s), \phi_i)ds\Big\|_{L^2(0,T)}^2\right]^{1/2} \\ &\qquad + \|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta^2 u_0\| \\ &\le Q\big(\|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta^2 u_0\|\big). \end{aligned} \tag{3.4}$$

We differentiate L_2 in (3.2) with respect to t to get

$$\partial_t L_2 = \int_0^t e^{(t-s)\Delta}\,\Delta\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot,s)} w(\boldsymbol{x},s)\big)ds + \kappa\partial_t^{\alpha(\boldsymbol{x},\cdot,t)} w(\boldsymbol{x},t). \tag{3.5}$$

We utilize the commutativity of the convolution operator to obtain

$$\begin{aligned} \partial_t \int_0^t e^{(t-s)\Delta}\Delta\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot,s)} w(\boldsymbol{x},s)\big)ds &= \partial_t \int_0^t e^{s\Delta}\Delta\big(\kappa\partial_y^{\alpha(\boldsymbol{x},\cdot,y)} w(\boldsymbol{x},y)\big)\big|_{y=t-s} ds \\ &= \int_0^t e^{s\Delta}\Delta\partial_t\big(\kappa\partial_y^{\alpha(\boldsymbol{x},\cdot,y)} w(\boldsymbol{x},y)\big)\big|_{y=t-s}\big)ds \\ &= -\int_0^t e^{(t-s)\Delta}\Delta\partial_s\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot,s)} w(\boldsymbol{x},s)\big)ds, \end{aligned}$$

and we use (3.5) to evaluate $\partial_t^2 L_2$ as follows:

$$\partial_t^2 L_2 = -\int_0^t e^{(t-s)\Delta}\Delta\partial_s\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot;\cdot,s)}w(\boldsymbol{x},s)\big)ds + \partial_t\big(\kappa\partial_t^{\alpha(\boldsymbol{x},\cdot;\cdot,t)}w(\boldsymbol{x},t)\big). \tag{3.6}$$

We use Lemma 3.2 with $\varepsilon = 0$ to bound the second term directly and use (2.2) and apply the Laplace transform to the first term to conclude that for $0 < \varepsilon \ll 1$,

$$\begin{aligned}
&\mathcal{L}\Big[\int_0^t e^{(t-s)\Delta}\Delta\partial_s\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot;\cdot,s)}w(\boldsymbol{x},s)\big)ds\Big]\\
&\quad= \mathcal{L}\Big[\int_0^t \partial_t\Big(\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\theta} e^{z(t-s)}(z-\Delta)^{-1}\mathrm{d}z\Big)\partial_s\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot;\cdot,s)}w(\boldsymbol{x},s)\big)ds\Big]\\
&\quad= \big(z^{1-\varepsilon}(z-\Delta)^{-1}\big)\big(z^\varepsilon\mathcal{L}\big[\partial_t\big(\kappa\partial_t^{\alpha(\boldsymbol{x},\cdot;\cdot,t)}w\big)\big]\big),
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_0^t e^{(t-s)\Delta}\Delta\partial_s\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot;\cdot,s)}w(\boldsymbol{x},s)\big)ds\\
&\quad= \int_0^t \Big[\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\theta} z^{1-\varepsilon}(z-\Delta)^{-1}e^{z(t-s)}dz\Big]\big({}^R\partial_s^\varepsilon\partial_s(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot;\cdot,s)}w(\boldsymbol{x},s))\big)ds.
\end{aligned}$$

Using (2.1) to bound the integral in the square brackets and Lemma 3.2 to bound the first term on the right-hand side of $\partial_t^2 L_2$ in (3.6) yields

$$\begin{aligned}
&\Big\|\int_0^t e^{(t-s)\Delta}\Delta\partial_s\big(\kappa\partial_s^{\alpha(\boldsymbol{x},\cdot;\cdot,s)}w(\boldsymbol{x},s)\big)ds\Big\| \le Q\int_0^t \frac{\big\|{}^R\partial_s^\varepsilon\partial_s(\kappa\partial_s^{\alpha(\cdot,\cdot;\cdot,s)}w(\boldsymbol{x},s))\big\|ds}{(t-s)^{1-\varepsilon}}\\
&\quad\le Q\int_0^t \frac{1}{(t-s)^{1-\varepsilon}}\Big(\int_0^s \frac{\|\partial_\theta^2 w(\cdot,\theta)\|}{(s-\theta)^{\alpha_*+\varepsilon}}d\theta + \big(\|f(\cdot,0)\|+\|\Delta u_0\|\big)s^{-\varepsilon-\alpha_0}\Big)ds\\
&\quad\le Q\int_0^t \frac{\|\partial_\theta^2 w(\cdot,\theta)\|}{(t-\theta)^{\alpha_*}}d\theta + Q\big(\|f(\cdot,0)\|+\|\Delta u_0\|\big)t^{-\alpha_0}.
\end{aligned} \tag{3.7}$$

We multiply (3.6) by $e^{-\sigma t}$ and take the $\|\cdot\|_{L^p(0,T)}$-norm on both sides of the resulting equation and then invoke (3.7) and Young's convolution inequality to obtain

$$\begin{aligned}
&\big\|e^{-\sigma t}\partial_t^2 L_2\big\|_{L^p(L^2)}\\
&\quad\le Q\big\|\big(e^{-\sigma t}t^{-\alpha_*}\big)*\big(e^{-\sigma t}\|\partial_t^2 w(\cdot,t)\|\big)\big\|_{L^p(0,T)} + Q\big(\|f(\cdot,0)\|+\|\Delta u_0\|\big)\\
&\quad\le Q\sigma^{\alpha_*-1}\|e^{-\sigma t}\partial_t^2 w\|_{L^p(L^2)} + Q\big(\|f(\cdot,0)\|+\|\Delta u_0\|\big).
\end{aligned} \tag{3.8}$$

Differentiating (3.2) twice in time, applying the $\|\cdot\|_{L^p(0,T)}$-norm on both sides of the resulting equation multiplied by $e^{-\lambda t}$ and invoking (3.4) and (3.8) yields

$$\begin{aligned}
&\|e^{-\sigma t}\partial_t^2 w\|_{L^p(L^2)} \le \big\|e^{-\sigma t}\partial_t^2 L_1\big\|_{L^p(L^2)} + \big\|e^{-\sigma t}\partial_t^2 L_2\big\|_{L^p(L^2)}\\
&\quad\le Q\big(\|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta u_0\| + \|\Delta^2 u_0\|\big) + Q\sigma^{\alpha_*-1}\big\|e^{-\sigma t}\partial_t^2 w\big\|_{L^p(L^2)}.
\end{aligned}$$

We set σ large enough to hide the last term on the right-hand side to get

$$\big\|\partial_t^2 w\big\|_{L^p(L^2)} \le Q\big(\|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta u_0\| + \|\Delta^2 u_0\|\big). \tag{3.9}$$

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Applying Lemma 3.2 with $\varepsilon = 0$ yields

$$\big\|\partial_t\big(\kappa\partial_t^{\alpha(\boldsymbol{x},:,t)}w\big)\big\|_{L^p(L^2)} \le Q\big(\|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta u_0\| + \|\Delta^2 u_0\|\big).$$

We use these estimates to arrive at

$$\begin{aligned}\big\|\partial_t w\big\|_{L^p(\check{H}^2)} &= \big\|\partial_t \Delta w\big\|_{L^p(L^2)} = \big\|\partial_t^2 w + \partial_t(\kappa\partial_t^{\alpha(\boldsymbol{x},:,t)}w) - \partial_t f\big\|_{L^p(L^2)} \\ &\le Q\big(\|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta u_0\| + \|\Delta^2 u_0\|\big).\end{aligned} \tag{3.10}$$

Finally, we combine the two expression

$$\partial_t w = \int_0^t \partial_s^2 w(\boldsymbol{x},s)ds + \partial_t w(\boldsymbol{x},0) \quad \text{and} \quad \partial_t w(\boldsymbol{x},0) = f(\boldsymbol{x},0) + \Delta u_0,$$

and estimate (3.9) to bound $\|\partial_t w\|_{L^\infty(L^2)}$ in terms of the right-hand side of (3.10). □

REMARK 3.4. The proved regularity results in Theorem 3.3 are required in the estimates of the truncation errors proved in Step 2 of the proof of Theorem 4.1. Furthermore, if the solutions are smooth enough, then some compatibility conditions such as $f(\boldsymbol{x},0) + \Delta u_0 = 0$ for $\boldsymbol{x} \in \partial\Omega$ may be required.

4. A numerical discretization and its error estimate. Define a uniform temporal partition on $[0,T]$ by $t_n := n\tau$, for $\tau := T/N$ and $0 \le n \le N$. Define a quasi-uniform partition of Ω with mesh diameter h, and let S_h be the space of continuous and piecewise linear functions on Ω with respect to the partition. Let I be the identity operator. The Ritz projection $\Pi_h : H_0^1(\Omega) \to S_h(\Omega)$ defined by $\big(\nabla(g - \Pi_h g), \nabla\chi\big) = 0$, for any $\chi \in S_h$, has the approximation property

$$\|(I - \Pi_h)g\| \le Qh^2\|g\|_{H^2}, \qquad \forall g \in H^2 \cap H_0^1. \tag{4.1}$$

For brevity of expression, we skip $\boldsymbol{x}$ whenever there is no confusion and denote $u_n := u(\boldsymbol{x},t_n)$, $\kappa_n := \kappa(\boldsymbol{x},t_n)$, and $f_n := f(\boldsymbol{x},t_n)$. We discretize $\partial_t u$ and $\partial_t^{\alpha(\boldsymbol{x},:,t)}u$ at $t = t_n$, for $1 \le n \le N$, by

$$\begin{aligned}\partial_t u|_{t=t_n} &= \delta_\tau u_n + E_n := \frac{u_n - u_{n-1}}{\tau} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_t^2 u(\boldsymbol{x},t)(t - t_{n-1})dt, \\ \partial_t^{\alpha(\boldsymbol{x},:,t)}u|_{t=t_n} &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{(t_n - s)^{-\alpha(\boldsymbol{x},s,t_n)}\partial_s u(\boldsymbol{x},s)ds}{\Gamma(1 - \alpha(\boldsymbol{x},s,t_n))} \\ &= \delta_\tau^{\alpha(\boldsymbol{x},:,t_n)}u_n + R_n + \hat{R}_n,\end{aligned} \tag{4.2}$$

where $\delta_\tau^{\alpha(\boldsymbol{x},:,t_n)}u_n$, $\hat{R}_n$, and R_n are defined by

$$\delta_\tau^{\alpha(\boldsymbol{x},:,t_n)}u_n := \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{(t_n - s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1 - \alpha(\boldsymbol{x},t_k,t_n))}\delta_\tau u_k ds = \sum_{k=1}^n b_{n,k}(u_k - u_{k-1}),$$

$$\begin{aligned}b_{n,k} &:= \frac{(t_n - t_{k-1})^{1-\alpha(\boldsymbol{x},t_k,t_n)} - (t_n - t_k)^{1-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(2 - \alpha(\boldsymbol{x},t_k,t_n))\tau}, \\ \hat{R}_n &:= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left(\frac{(t_n - s)^{-\alpha(\boldsymbol{x},s,t_n)}}{\Gamma(1 - \alpha(\boldsymbol{x},s,t_n))} - \frac{(t_n - s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1 - \alpha(\boldsymbol{x},t_k,t_n))}\right)\partial_s u(\boldsymbol{x},s)ds,\end{aligned}$$

$$
\begin{aligned}
R_n &:= \sum_{k=1}^{n} R_{n,k} := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{(\partial_s u(\boldsymbol{x},s) - \delta_\tau u_k(\boldsymbol{x}))ds}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))(t_n-s)^{\alpha(\boldsymbol{x},t_k,t_n)}} \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\tau\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))} \bigg(\int_{t_{k-1}}^{t_k} \int_{z}^{s} \partial_\theta^2 u(\boldsymbol{x},\theta) d\theta dz \bigg) ds.
\end{aligned}
$$

The idea of the discretization of $\partial_t^{\alpha(\boldsymbol{x},:,t)} u$ arises from the approximation of each sub-integral

$$
\int_{t_{k-1}}^{t_k} \frac{(t_n-s)^{-\alpha(\boldsymbol{x},s,t_n)} \partial_s u(\boldsymbol{x},s) ds}{\Gamma(1-\alpha(\boldsymbol{x},s,t_n))}
$$

by replacing $\partial_s u(\boldsymbol{x},s)$ and $\alpha(\boldsymbol{x},s,t_n)$ by $\delta_\tau u_k$ and $\alpha(\boldsymbol{x},t_k,t_n)$, respectively, in order to obtain an explicit formula $\delta_\tau^{\alpha(\boldsymbol{x},:,t_n)} u_n$ for the numerical implementation; the resulting truncation error is split into two parts $\hat{R}_n$ and R_n for the convenience of error estimates.

We plug these discretizations into (1.2) and integrate the resulting equation multiplied by $\chi \in H_0^1(\Omega)$ on Ω to obtain its weak formulation for $n = 1, \ldots, N$:

$$
\begin{aligned}
(\delta_\tau u_n, \chi) + (\kappa_n\, \delta_\tau^{\alpha(\boldsymbol{x},:,t_n)} u_n, \chi) + \big(\nabla u_n, \nabla\chi\big) \\
= (f_n,\chi) - \big(\kappa_n(\hat{R}_n + R_n) + E_n, \chi\big), \qquad \forall \chi \in H_0^1(\Omega).
\end{aligned} \tag{4.3}
$$

We drop the local truncation error term to obtain a finite element scheme for (1.2): find $U_n \in S_h$ with $U_0 := \Pi_h u_0$ such that for $n = 1, \ldots, N$,

$$
(\delta_\tau U_n, \chi) + (\kappa_n\, \delta_\tau^{\alpha(\boldsymbol{x},:,t_n)} U_n, \chi) + \big(\nabla U_n, \nabla\chi\big) = (f_n,\chi), \qquad \forall \chi \in S_h. \tag{4.4}
$$

We subtract equation (4.4) from equation (4.3) to obtain the error equation

$$
\begin{aligned}
(\delta_\tau (U-u)_n, \chi) + (\kappa_n\, \delta_\tau^{\alpha(\boldsymbol{x},:,t_n)} (U-u)_n, \chi) + \big(\nabla (U-u)_n, \nabla\chi\big) \\
= \big(\kappa_n(\hat{R}_n + R_n) + E_n, \chi\big), \qquad \forall \chi \in S_h.
\end{aligned} \tag{4.5}
$$

Let $\Pi_h u$ be the Ritz projection of u and $\eta := \Pi_h u - u$ be bounded in (4.1). We split the error into $U_n - u_n = \xi_n + \eta_n$, and it remains to bound $\xi_n := U_n - \Pi_h u_n \in S_h$. We set $\chi = \xi_n$ in (4.5) and rewrite the error equation as

$$
\begin{aligned}
(\delta_\tau \xi_n, \xi_n) + \big(\kappa_n \delta_\tau^{\alpha(\boldsymbol{x},:,t_n)} \xi_n, \xi_n\big) + \big(\nabla \xi_n, \nabla\xi_n\big) \\
= \big(\kappa_n[\hat{R}_n + R_n - \delta_\tau^{\bar{\alpha}(\boldsymbol{x},t_n)} \eta_n] + E_n - \delta_\tau \eta_n, \xi_n\big).
\end{aligned} \tag{4.6}
$$

For $\alpha = \alpha(t)$ and $\kappa = \kappa(t)$, κ_n and $\{b_{n,k}\}$ are independent of $\boldsymbol{x}$ and can be moved outside of the inner product in the second term on the left-hand side. Equation (4.6) reduces to

$$
\begin{aligned}
\big[1 + \tau k(t_n) b_{n,n}\big] \|\xi_n\|^2 + \big\|\nabla \xi_n\big\|^2 \\
&= (\xi_{n-1}, \xi_n) + \tau k(t_n) \sum_{k=1}^{n-1} \big(b_{n,k+1} - b_{n,k}\big)(\xi_k, \xi_n) \\
&\quad + \tau\big(k(t_n)\big[\hat{R}_n + R_n - \delta_\tau^{\alpha(t_n)} \eta_n\big] + E_n - \delta_\tau \eta_n, \xi_n\big) \\
&\le \|\xi_n\| \bigg(\|\xi_{n-1}\| + \tau k(t_n) \sum_{k=1}^{n-1} \big(b_{n,k+1} - b_{n,k}\big) \|\xi_k\| \\
&\quad + \tau\big[k(t_n) \|\hat{R}_n + R_n - \delta_\tau^{\alpha(t_n)} \eta_n\| + \|E_n - \delta_\tau \eta_n\|\bigg).
\end{aligned} \tag{4.7}
$$

Dropping the $\|\nabla\xi_n\|^2$-term, canceling $\|\xi_n\|$ on both sides, and using the monotonicity of $\{b_{n,k}\}$ and an induction argument yields a stability estimate [36]. For problem (1.2) with $\alpha = \alpha(\boldsymbol{x}, s, t)$, the expressions $\{b_{n,k}\}$ and κ_n depend on $\boldsymbol{x}$ and cannot be moved out of the inner product in the second term on the left-hand side of equation (4.6), to which the analysis in (4.7) does not apply. We adopt a novel approach to prove the following theorem.

THEOREM 4.1. *Under the assumptions of Theorem 3.3, scheme (4.4) has an optimal-order error estimate for τ small enough:*

$$\|U - u\|_{\hat{L}^\infty(L^2)} := \max_{1\le n\le N} \|U_n - u_n\| \le Q\hat{Q}\big(\tau + h^2\big). \tag{4.8}$$

Here $\hat{Q} := \|f\|_{H^1(L^2)} + \|f\|_{L^2(\check{H}^2)} + \|\Delta u_0\| + \|\Delta^2 u_0\|$, and Q is independent of h, τ, N.

Proof. We multiply equation (4.6) by 2τ, cancel $\|\xi_n\|^2$ from the (ξ_{n-1}, x_n)-term, and use $\xi_0 := U_0 - \Pi_h u_0 = 0$ to deduce the following inequality from (4.6):

$$\begin{aligned} &\|\xi_n\|^2 + 2\tau(\kappa_n b_{n,n}\xi_n, \xi_n) + 2\tau\|\nabla\xi_n\|^2 \\ &\qquad \le \|\xi_{n-1}\|^2 + 2\tau\sum_{k=1}^{n-1}\big(\kappa_n(b_{n,k+1} - b_{n,k})\xi_k, \xi_n\big) + \tau\big(G_n, \xi_n\big), \end{aligned} \tag{4.9}$$

where $G_n := 2\big(\kappa_n\big[\hat{R}_n + R_n - \delta_\tau^{\alpha(\boldsymbol{x},:,t_n)}\eta_n\big] + E_n - \delta_\tau\eta_n\big)$.

We prove the theorem in two steps.

Step 1. Stability estimate of $\|\xi_n\|$. There is a constant $Q_* = Q_*(T)$ such that

$$\|\xi\|_{\hat{L}^\infty(L^2)} \le Q_*\|G\|_{\hat{L}^1(L^2)}, \qquad \|G\|_{\hat{L}^1(L^2)} := \tau\sum_{n=1}^{N}\|G_n\|. \tag{4.10}$$

Since $b_{n,k} = O(\tau^{-\alpha})$, a naive estimate of the second term on the right-hand side of (4.9) will blow up. We present a novel splitting to bound the sum of the terms $\big|b_{n,k+1} - b_{n,k}\big|$.

$$\begin{aligned} &\sum_{k=1}^{n-1}\big|b_{n,k+1} - b_{n,k}\big| \\ &\quad = \sum_{k=1}^{n-1}\left|\frac{1}{\tau}\int_{t_k}^{t_{k+1}}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_{k+1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{k+1},t_n))}ds - \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}ds\right| \\ &\quad \le \frac{1}{\tau}\sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}}\left|\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_{k+1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{k+1},t_n))} - \frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}\right|ds \\ &\qquad + \frac{1}{\tau}\sum_{k=1}^{n-1}\left|\int_{t_k}^{t_{k+1}}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}ds - \int_{t_{k-1}}^{t_k}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}ds\right| \\ &\quad =: J_1 + J_2. \end{aligned} \tag{4.11}$$

We bound the first term on the right-hand side by

$$\begin{aligned} J_1 &= \frac{1}{\tau}\sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}}\left|\int_{t_k}^{t_{k+1}}\partial_z\frac{(t_n-s)^{-\alpha(\boldsymbol{x},z,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},z,t_n))}dz\right|ds \\ &\le Q\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\frac{\big|\ln(t_n-s)\big|}{(t_n-s)^{\alpha_*}}ds = Q\int_0^{t_n}\frac{\big|\ln(t_n-s)\big|}{(t_n-s)^{\alpha_*}}ds \le Q. \end{aligned} \tag{4.12}$$

The summands in J_2 are nonnegative. We further split J_2 as

$$\begin{aligned}J_2 &= \frac{1}{\tau}\sum_{k=1}^{n-1}\left[\int_{t_k}^{t_{k+1}}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}ds-\int_{t_{k-1}}^{t_k}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_{k-1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{k-1},t_n))}ds\right]\\ &\quad+\frac{1}{\tau}\sum_{k=1}^{n-1}\left[\int_{t_{k-1}}^{t_k}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_{k-1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{k-1},t_n))}ds-\int_{t_{k-1}}^{t_k}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}ds\right]\\ &=: J_{2,1}+J_{2,2}.\end{aligned}\tag{4.13}$$

We note that the first term is a telescoping sum with respect to k to get

$$\begin{aligned}0\le J_{2,1} &= \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_{n-1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{n-1},t_n))}ds-\int_0^{t_1}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_0,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_0,t_n))}ds\\ &\le \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_{n-1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{n-1},t_n))}ds\\ &= b_{n,n}+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\left[\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_{n-1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{n-1},t_n))}-\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_n,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_n,t_n))}\right]ds\\ &= b_{n,n}+\left|\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n}\partial_z\frac{(t_n-s)^{-\alpha(\boldsymbol{x},z,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},z,t_n))}dzds\right|\\ &\le b_{n,n}+Q\int_{t_{n-1}}^{t_n}(t_n-s)^{-\alpha_*}|\ln(t_n-s)|ds\le b_{n,n}+Q.\end{aligned}\tag{4.14}$$

We bound the second term on the right-hand side of (4.13) similarly to (4.12) by

$$|J_{2,2}|\le\frac{1}{\tau}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k}\left|\partial_z\frac{(t_n-s)^{-\alpha(\boldsymbol{x},z,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},z,t_n))}\right|dzds\le Q.\tag{4.15}$$

We apply the estimates (4.14) and (4.15) to (4.13) and combine the resulting estimate with (4.11) and (4.12) to conclude that $\sum_{k=1}^{n-1}\left|b_{n,k+1}-b_{n,k}\right|\le b_{n,n}+Q_0$ for a Q_0 that is independent of τ, n, and N. We use this estimate to bound the second term on the right-hand side of (4.9) by

$$\begin{aligned}&2\tau\sum_{k=1}^{n-1}\big(\kappa_n(b_{n,k+1}-b_{n,k})\xi_k,\xi_n\big)\\ &\quad\le\tau\left(\sum_{k=1}^{n-1}|b_{n,k+1}-b_{n,k}|\kappa_n\xi_n,\xi_n\right)+\tau\sum_{k=1}^{n-1}\big(\kappa_n|b_{n,k+1}-b_{n,k}|\xi_k,\xi_k\big)\\ &\quad\le\tau\big(\kappa_nb_{n,n}\xi_n,\xi_n\big)+Q_0\tau\big(\kappa_n\xi_n,\xi_n\big)+\tau\sum_{k=1}^{n-1}\big(\kappa_n|b_{n,k+1}-b_{n,k}|\xi_k,\xi_k\big).\end{aligned}\tag{4.16}$$

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Johann Radon Institute (RICAM)

We use the estimate (4.16) in (4.9), sum the resulting equations from $n = 1$ to $m\,(\le N)$, and cancel the like terms to get

$$\begin{aligned}
&\|\xi_m\|^2 + \tau\sum_{n=1}^{m}\big(\kappa_n b_{n,n}\xi_n,\xi_n\big) + 2\tau\sum_{n=1}^{m}\|\nabla\xi_n\|^2\\
&\le \tau\sum_{n=2}^{m}\sum_{k=1}^{n-1}\big(\kappa_n|b_{n,k+1}-b_{n,k}|\xi_k,\xi_k\big) + Q_1\tau\sum_{n=1}^{m}\|\xi_n\|^2 + \tau\sum_{n=1}^{m}\|G_n\|\|\xi_n\|\\
&= \tau\sum_{k=1}^{m-1}\Big(\xi_k,\xi_k\sum_{n=k+1}^{m}\kappa_n|b_{n,k+1}-b_{n,k}|\Big) + Q_1\tau\sum_{n=1}^{m}\|\xi_n\|^2 + \tau\sum_{n=1}^{m}\|G_n\|\|\xi_n\|.
\end{aligned}\tag{4.17}$$

We bound the sum in the inner product in the first term on the right-hand side by

$$\begin{aligned}
&\sum_{n=k+1}^{m}\kappa_n|b_{n,k+1}-b_{n,k}|\\
&\quad=\sum_{n=k+1}^{m}\frac{\kappa_n}{\tau}\bigg|\int_{t_{k-1}}^{t_k}\bigg(\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_{k+1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{k+1},t_n))}-\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}\bigg)ds\bigg|\\
&\quad\le\sum_{n=k+1}^{m}\frac{\kappa_n}{\tau}\int_{t_{k-1}}^{t_k}\bigg|\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_{k+1},t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_{k+1},t_n))}-\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}\bigg|ds\\
&\qquad+\sum_{n=k+1}^{m}\frac{\kappa_n}{\tau}\int_{t_{k-1}}^{t_k}\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}-(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}ds\\
&\quad=: L_1+L_2.
\end{aligned}\tag{4.18}$$

We use the substitution of variable $\theta := t_k + (s - t_{n-1})$ to bound L_1 by

$$\begin{aligned}
L_1 &\le \frac{\|\kappa\|_{L^\infty(L^\infty)}}{\tau}\sum_{n=k+1}^{m}\int_{t_{k-1}}^{t_k}\int_{t_k}^{t_{k+1}}\bigg|\partial_z\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},z,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},z,t_n))}\bigg|dzds\\
&\le Q\sum_{n=k+1}^{m}\int_{t_{k-1}}^{t_k}\frac{\big|\ln(t_{n-1}-s)\big|}{(t_{n-1}-s)^{\alpha_*}}ds \le Q\sum_{n=k+1}^{m}\int_{t_{k-1}}^{t_k}\frac{ds}{(t_{n-1}-s)^{(1+\alpha_*)/2}}\\
&= Q\sum_{n=k+1}^{m}\int_{t_{n-1}}^{t_n}\frac{d\theta}{(\theta-t_k)^{(1+\alpha_*)/2}} = Q\int_{t_k}^{t_m}\frac{d\theta}{(\theta-t_k)^{(1+\alpha_*)/2}} \le Q.
\end{aligned}$$

As the integrand of L_2 in (4.18) is nonnegative, we split and bound L_2 by

$$\begin{aligned}
0\le L_2 &= \sum_{n=k+1}^{m}\frac{\kappa_n-\kappa_{n-1}}{\tau}\int_{t_{k-1}}^{t_k}\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}ds\\
&\quad+\sum_{n=k+1}^{m}\frac{\kappa_{n-1}}{\tau}\int_{t_{k-1}}^{t_k}\bigg[\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}-\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_{n-1})}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_{n-1}))}\bigg]ds\\
&\quad+\sum_{n=k+1}^{m}\int_{t_{k-1}}^{t_k}\bigg[\frac{\kappa_{n-1}}{\tau}\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_{n-1})}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_{n-1}))}-\frac{\kappa_n}{\tau}\frac{(t_n-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))}\bigg]ds
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{n=k+1}^{m} \frac{\kappa_n - \kappa_{n-1}}{\tau} \int_{t_{k-1}}^{t_k} \frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))} ds \\
&\quad + \sum_{n=k+1}^{m} \frac{\kappa_{n-1}}{\tau} \int_{t_{k-1}}^{t_k} \left[\frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_n)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_n))} - \frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,t_{n-1})}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_{n-1}))} \right] ds \\
&\quad + \frac{\kappa_k}{\tau} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{-\alpha(\boldsymbol{x},t_k,t_k)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,t_k))} ds =: L_{21} + L_{22} + \kappa_k b_{k,k}.
\end{aligned}$$

Here we have used the fact that the last term on the right-hand side of the equal sign is a telescoping sum. We bound L_{21} and L_{22} in a similar manner to (4.12) by

$$\begin{aligned}
|L_{21}| + |L_{22}| \le Q \sum_{n=k+1}^{m} \Bigg[&\int_{t_{k-1}}^{t_k} \frac{ds}{(t_{n-1}-s)^{\alpha_*}} \\
&+ \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \int_{t_{n-1}}^{t_n} \left| \partial_z \frac{(t_{n-1}-s)^{-\alpha(\boldsymbol{x},t_k,z)}}{\Gamma(1-\alpha(\boldsymbol{x},t_k,z))} \right| dz ds \Bigg] \le Q.
\end{aligned}$$

We invoke these estimates in (4.18) to find that $\sum_{n=k+1}^{m} \kappa_n |b_{n,k+1} - b_{n,k}| \le Q_2 + \kappa_k b_{k,k}$ for some $Q_2 > 0$ (independent of τ, k, m, and N). Consequently, we bound the first term on the right-hand side of (4.17) by

$$\tau \sum_{k=1}^{m-1} \left(\xi_k, \xi_k \sum_{n=k+1}^{m} \kappa_n |b_{n,k+1} - b_{n,k}| \right) \le Q_2 \tau \sum_{k=1}^{m-1} \|\xi_k\|^2 + \tau \sum_{k=1}^{m-1} \big(\kappa_k b_{k,k} \xi_k, \xi_k\big).$$

We invoke this in (4.17) and cancel the like terms to conclude that for $1 \le m \le N$,

$$\|\xi_m\|^2 + 2\tau \sum_{n=1}^{m} \|\nabla \xi_n\|^2 \le (Q_1 + Q_2)\tau \sum_{n=1}^{m} \|\xi_n\|^2 + \tau \sum_{n=1}^{m} \|G_n\| \|\xi_n\|.$$

We drop the second term on the left-hand side, choose $\tau > 0$ such that $(Q_1 + Q_2)\tau < 1/2$, and apply the discrete Gronwall inequality to obtain

$$\|\xi_m\|^2 \le 2e^{2(Q_1+Q_2)T} \tau \sum_{n=1}^{m} \|G_n\| \|\xi_n\|, \qquad 1 \le m \le N. \tag{4.19}$$

Let $\|\xi_{m_*}\| := \max_{1 \le m \le N} \|\xi_m\|$ (assumed positive without loss of generality). Set $m = m_*$ in (4.19) and divide the resulting inequality by $\|\xi_{m_*}\|$ to arrive at (4.10).

Step 2. We use Theorem 3.3 to bound E_n, R_n, and $\hat{R}_n$ introduced in (4.2) by

$$\begin{aligned}
\sum_{n=1}^{N} \|E_n\| &\le \frac{Q}{\tau} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\partial_t^2 u\| (t - t_{n-1}) dt \le Q \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\partial_t^2 u\| dt \le Q\hat{Q}, \\
\sum_{n=1}^{N} \|R_n\| &\le Q \sum_{n=1}^{N} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha_*} \int_{t_{k-1}}^{t_k} \|\partial_\theta^2 u(\cdot,\theta)\| d\theta ds \\
&= Q \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \|\partial_\theta^2 u(\cdot,\theta)\| d\theta \sum_{n=k}^{N} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha_*} ds \\
&\le Q \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \|\partial_\theta^2 u(\cdot,\theta)\| d\theta \le Q \|u\|_{W^{2,1}(L^2)} \le Q\hat{Q},
\end{aligned}$$

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$$
\begin{aligned}
\sum_{n=1}^{N} \|\hat{R}_n\| &\le Q \sum_{n=1}^{N} \|u\|_{W^{1,\infty}(L^2)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| \frac{(t_n - s)^{-\alpha_n(s)}}{\Gamma(1-\alpha_n(s))} - \frac{(t_n - s)^{-\alpha_n(t_k)}}{\Gamma(1-\alpha_n(t_k))} \right| ds \\
&\le Q\hat{Q} \sum_{n=1}^{N} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| \partial_z \left(\frac{(t_n - s)^{-\alpha_n(z)}}{\Gamma(1-\alpha_n(z))} \right) \right|_{z=\zeta} \left| (t_k - s) ds \\
&\le Q\hat{Q} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{-(1+\alpha_*)/2} ds \le Q\hat{Q}.
\end{aligned}
$$

We use (4.1) to bound the remaining two terms in G_n in (4.9) by

$$
\begin{aligned}
\tau \sum_{n=1}^{N} \|\delta_\tau \eta_n\| &= \sum_{n=1}^{N} \left\| \int_{t_{n-1}}^{t_n} (I - \Pi_h) \partial_t u dt \right\| \le Q h^2 \|u\|_{W^{1,1}(H^2)} \le Q\hat{Q} h^2, \\
\tau \sum_{n=1}^{N} \|\delta_\tau^{\bar{\alpha}(\boldsymbol{x},t_n)} \eta_n\| &= \tau \sum_{n=1}^{N} \left\| \sum_{k=1}^{n} b_{n,k} (I - \Pi_h) \int_{t_{k-1}}^{t_k} \partial_t u dt \right\| \\
&\le Q \sum_{n=1}^{N} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha_*} ds \int_{t_{k-1}}^{t_k} \|(I - \Pi_h) \partial_t u\| dt \\
&\le Q h^2 \sum_{n=1}^{N} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha_*} ds \|\partial_t u\|_{L^1(t_{k-1}, t_k; H^2)} \\
&= Q h^2 \sum_{k=1}^{N} \|u\|_{W^{1,1}(t_{k-1}, t_k; H^2)} \sum_{n=k}^{N} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha_*} ds \\
&\le Q h^2 \|u\|_{W^{1,1}(H^2)} \le Q\hat{Q} h^2.
\end{aligned}
$$

We incorporate these local error estimates into (4.10) to obtain

$$
\begin{aligned}
\|\xi\|_{\hat{L}^\infty(L^2)} &\le Q_* \tau \sum_{n=1}^{N} \left(\|E_n\| + \|R_n\| + \|\hat{R}_n\| + \|\delta_\tau \eta_n\| + \|\delta_\tau^{\bar{\alpha}(\boldsymbol{x},t_n)} \eta_n\| \right) \\
&\le Q\hat{Q}(\tau + h^2).
\end{aligned}
\tag{4.20}
$$

We combine (4.20) with (4.1) to prove the estimate (4.8). □

5. Numerical experiments. We numerically test the convergence rate of the finite element approximation (4.4) by measuring the error $\|u - U\|_{\hat{L}^\infty(L^2)}$. In all numerical experiments, we apply a uniform spatial partition with rectangular elements of mesh size h in each direction.

TABLE 5.1
Convergence of the scheme (4.4) in Example 5.1 with $a = 3$ and $f_0 = 0$.

τ	$h = 1/36$	conv. rate	h	$\tau = 1/360$	conv. rate
1/20	4.70e-02		1/8	3.12e-03	
1/30	3.22e-02	0.93	1/16	7.78e-04	2.00
1/40	2.41e-02	1.00	1/24	3.44e-04	2.01
1/60	1.56e-02	1.07	1/30	2.19e-04	2.03

EXAMPLE 5.1 (Simulation of problem (1.2) in one space dimension). The data are $\Omega = (0,1)$, $[0,T] = [0,1]$, $\kappa(x,t) = 1$, $u_0(x) = \sin(\pi x)$, $f(x,t) = f_0$, and

$$\alpha(x,s,t) = \frac{a+s+2t}{10}\Big(1 + \frac{\sin(0.5\pi x)}{10}\Big), \qquad 0 \le a \in \mathbb{R}.$$

As closed-form solutions are not available, we use the scheme (4.4) with fine mesh sizes $h = 1/240$ and $\tau = 1/360$ to compute a reference solution to test the spatial convergence rate of (4.4), and we use $h = 1/36$ and $\tau = 1/360$ to compute a reference solution to test the temporal convergence rate. We present the numerical results in Tables 5.1–5.3 for $a = 0$ or 3, respectively, which illustrates the second-order accuracy in space and the first-order accuracy in time of the scheme (4.4) as proved in Theorem 4.1.

TABLE 5.2
Convergence of the scheme (4.4) in Example 5.1 with $a = 0$ and $f_0 = 0$.

τ	$h = 1/36$	conv. rate	h	$\tau = 1/360$	conv. rate
11/20	4.97e-02		1/8	3.20e-03	
1/30	3.41e-02	0.93	1/16	7.97e-04	2.00
1/40	2.56e-02	1.00	1/24	3.53e-04	2.01
1/60	1.65e-02	1.08	1/30	2.24e-04	2.03

TABLE 5.3
Convergence of the scheme (4.4) in Example 5.1 with $a = 0$ and $f_0 = 1$.

τ	$h = 1/36$	conv. rate	h	$\tau = 1/360$	conv. rate
11/20	4.33e-02		1/8	4.35e-03	
1/30	2.97e-02	0.93	1/16	1.10e-03	1.99
1/40	2.23e-02	1.00	1/24	4.87e-04	2.00
1/60	1.44e-02	1.08	1/30	3.10e-04	2.02

EXAMPLE 5.2 (Initial singularity of the solutions). We use the same model data as in Example 5.1 with $f_0 = 1$ and plot the curves of the approximations of $\partial_t u(0.5,t)$ near $t = 0$ under $a = 6$ and $a = 0$. Recall that $\alpha_0 := \|\alpha(\cdot,0,0)\|_{L^\infty}$ in Theorem 3.3, which implies $\alpha_0 = 0.66$ and $\alpha_0 = 0$, respectively. We observe from Figure 5.1 that when $\alpha_0 = 0$, the function $\partial_t u(0.5,t)$ appears to be smooth near the initial time, while when $\alpha_0 = 0.66$, it becomes steeper and thus exhibits an initial singularity. These observations demonstrate the analysis in Theorem 3.3 in that a larger α_0 leads to a smaller p, which may imply a stronger singularity of the solutions.

EXAMPLE 5.3 (Simulation for problem (1.2) in two space dimensions). The data are $\Omega = (0,1)^2$, $[0,T] = [0,1]$,

$$u_0(x,y) = \sin(\pi x)\sin(\pi y), \qquad f(x,y,t) = 0, \qquad \alpha(x,y,s,t) = s^2 t e^{0.1(x+y)}/2,$$

and

$$\text{(i)}\ \ \kappa(x,y,t) = txy^2 \qquad \text{or} \qquad \text{(ii)}\ \ \kappa(x,y,t) = e^t\big(1 + \sin x \cos y\big).$$

We focus on the temporal convergence rates of the scheme for problem (1.2) for the cases (i)–(ii) and compute the reference solution with $h = 1/24$ and $\tau = 1/240$. We present the numerical results in Table 5.4, which illustrates the first-order accuracy in time of the scheme (4.4) as proved in Theorem 4.1.

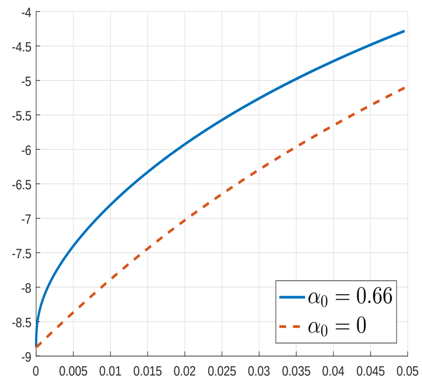

FIG. 5.1. *Curves of approximations of $\partial_t u(0.5, t)$ under $\alpha_0 = 0.66$ and $\alpha_0 = 0$ in Example 5.2.*

TABLE 5.4
Temporal convergence of the scheme (4.4) in Example 5.3 under $h = 1/24$.

τ	(i)	conv. rate	(ii)	conv. rate
1/30	2.00e-01		1.95e-01	
1/40	1.50e-01	1.00	1.49e-01	0.94
1/48	1.23e-01	1.07	1.21e-01	1.14
1/60	9.49e-02	1.17	9.45e-02	1.10

Conflict of interest. The authors declare that they have no conflict of interest.

Appendix A. The proof of Lemma 3.2. We begin with $0 < \varepsilon \ll 1$. A straightforward calculation yields

$$ {}^R\partial_t^{\varepsilon} \partial_t \big(\kappa \partial_t^{\alpha(\boldsymbol{x},:,t)} w\big) = {}^R\partial_t^{\varepsilon} \big((\partial_t \kappa) \partial_t^{\alpha(\boldsymbol{x},:,t)} w + \kappa \partial_t \partial_t^{\alpha(\boldsymbol{x},:,t)} w\big). \tag{A.1} $$

The second term in (A.1) is leading, which can be decomposed as

$$ \begin{aligned} {}^R\partial_t^{\varepsilon} \big(\kappa \partial_t \partial_t^{\alpha(\boldsymbol{x},:,t)} w\big) &= \partial_t I_t^{1-\varepsilon} \big(\kappa \partial_t \partial_t^{\alpha(\boldsymbol{x},:,t)} w\big) \\ &= \kappa(\boldsymbol{x}, t) \partial_t I_t^{1-\varepsilon} \partial_t \partial_t^{\alpha(\boldsymbol{x},:,t)} w + (\partial_t \kappa) I_t^{1-\varepsilon} \partial_t \partial_t^{\alpha(\boldsymbol{x},:,t)} w \\ &\quad + \partial_t \int_0^t \frac{\kappa(\boldsymbol{x}, s) - \kappa(\boldsymbol{x}, t)}{\Gamma(1-\varepsilon)(t-s)^{\varepsilon}} \partial_s \partial_s^{\alpha(\boldsymbol{x},:,s)} w ds. \end{aligned} \tag{A.2} $$

We estimate the first term on the right-hand side since it dominates. As ${}_0I_t^{1-\varepsilon}$ and ∂_t are commutative for an integrand vanishing at $t = 0$ [25], we perform integration by parts on the right-hand side to obtain

$$
\begin{aligned}
\partial_t\, I_t^{1-\varepsilon}\partial_t\partial_t^{\alpha(\boldsymbol{x},:,t)}w(\boldsymbol{x},t) &= \partial_t^2\, {}_0I_t^{1-\varepsilon}\partial_t^{\alpha(\boldsymbol{x},:,t)}w(\boldsymbol{x},t)\\
&= \partial_t^2\int_0^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\int_0^s \frac{(s-y)^{-\alpha(\boldsymbol{x},y,s)}}{\Gamma(1-\alpha(\boldsymbol{x},y,s))}\partial_y w(\boldsymbol{x},y)dyds\\
&= \partial_t^2\int_0^t \partial_y w(\boldsymbol{x},y)\int_y^t \frac{(t-s)^{-\varepsilon}(s-y)^{-\alpha(\boldsymbol{x},y,s)}ds}{\Gamma(1-\varepsilon)\Gamma(1-\alpha(\boldsymbol{x},y,s))}dy\\
&= \partial_t^2\Big[-\partial_y w(\boldsymbol{x},y)\int_y^t\int_\theta^t \frac{(t-s)^{-\varepsilon}(s-\theta)^{-\alpha(\boldsymbol{x},\theta,s)}ds}{\Gamma(1-\varepsilon)\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}d\theta\Big|_{y=0}^{y=t}\\
&\qquad + \int_0^t \partial_y^2 w(\boldsymbol{x},y)\int_y^t\int_\theta^t \frac{(t-s)^{-\varepsilon}(s-\theta)^{-\alpha(\boldsymbol{x},\theta,s)}dsd\theta}{\Gamma(1-\varepsilon)\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}dy\Big]\\
&= \int_0^t \partial_y^2 w(\boldsymbol{x},y)\,\partial_t^2\int_y^t\int_\theta^t \frac{(t-s)^{-\varepsilon}(s-\theta)^{-\alpha(\boldsymbol{x},\theta,s)}dsd\theta}{\Gamma(1-\varepsilon)\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}dy\\
&\qquad + \partial_t w(\boldsymbol{x},0)\,\partial_t^2\int_0^t\int_\theta^t \frac{(t-s)^{-\varepsilon}(s-\theta)^{-\alpha(\boldsymbol{x},\theta,s)}dsd\theta}{\Gamma(1-\varepsilon)\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}.
\end{aligned}
\tag{A.3}
$$

To bound (A.3), we focus on evaluating the following term for $0 \le y < t$:

$$
\begin{aligned}
&\partial_t^2\int_y^t\int_\theta^t \frac{(t-s)^{-\varepsilon}(s-\theta)^{-\alpha(\boldsymbol{x},\theta,s)}dsd\theta}{\Gamma(1-\varepsilon)\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}\\
&= \partial_t^2\int_y^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\int_y^s \frac{(s-\theta)^{-\alpha(\boldsymbol{x},\theta,s)}d\theta}{\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}ds\\
&= -\partial_t^2\int_y^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\Big[\int_y^s \frac{(s-\theta)^{\alpha(\boldsymbol{x},s,s)-\alpha(\boldsymbol{x},\theta,s)}}{\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}d\frac{(s-\theta)^{1-\alpha(\boldsymbol{x},s,s)}}{1-\alpha(\boldsymbol{x},s,s)}\Big]ds\\
&= \partial_t^2\int_y^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\Big[\frac{(s-y)^{1-\alpha(\boldsymbol{x},y,s)}}{(1-\alpha(\boldsymbol{x},s,s))\Gamma(1-\alpha(\boldsymbol{x},y,s))}\\
&\qquad + \int_y^s \frac{(s-\theta)^{1-\alpha(\boldsymbol{x},s,s)}}{1-\alpha(\boldsymbol{x},s,s)}\partial_\theta\frac{(s-\theta)^{\alpha(\boldsymbol{x},s,s)-\alpha(\boldsymbol{x},\theta,s)}}{\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}d\theta\Big]ds\\
&=: \partial_t^2 I_1 + \partial_t^2 I_2.
\end{aligned}
\tag{A.4}
$$

Direct calculations show that I_2 can be expressed as

$$
\begin{aligned}
I_2 &= \int_y^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\int_y^s (s-\theta)^{1-\alpha(\boldsymbol{x},\theta,s)}K_2(\boldsymbol{x},\theta,s)d\theta ds,\\
K_2(\boldsymbol{x},\theta,s) &:= \frac{(1-\alpha(\boldsymbol{x},s,s))^{-1}}{\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}\Big[-\partial_\theta\alpha(\boldsymbol{x},\theta,s)\ln(s-\theta)\\
&\qquad - \frac{\alpha(\boldsymbol{x},s,s)-\alpha(\boldsymbol{x},\theta,s)}{s-\theta}\\
&\qquad + \frac{\Gamma'(1-\alpha(\boldsymbol{x},\theta,s))\partial_\theta\alpha(\boldsymbol{x},\theta,s)}{\Gamma(1-\alpha(\boldsymbol{x},\theta,s))}\Big].
\end{aligned}
\tag{A.5}
$$

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We apply integration by parts to get

$$
\begin{aligned}
|\partial_t^2 I_2| &= \left|\partial_t^2 \int_y^t \frac{(t-s)^{2-\varepsilon}}{\Gamma(3-\varepsilon)}\partial_s \int_y^s \partial_s\big[(s-\theta)^{1-\alpha(\boldsymbol{x},\theta,s)}K_2(\boldsymbol{x},\theta,s)\big]d\theta ds\right| \\
&= \left|\int_y^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\partial_s \int_y^s \partial_s\big[(s-\theta)^{1-\alpha(\boldsymbol{x},\theta,s)}K_2(\boldsymbol{x},\theta,s)\big]d\theta ds\right| \\
&\leq Q\int_y^t (t-s)^{-\varepsilon}(s-y)^{-\alpha_*-\varepsilon}ds \leq Q(t-y)^{1-\alpha_*-2\varepsilon}.
\end{aligned}
\tag{A.6}
$$

We then split I_1 as

$$
\begin{aligned}
I_1 = &\int_y^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\frac{(s-y)^{1-\alpha(\boldsymbol{x},y,y)}}{\Gamma(2-\alpha(\boldsymbol{x},y,y))}ds \\
&+\int_y^t \frac{(t-s)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\int_y^s \partial_z \frac{(s-y)^{1-\alpha(\boldsymbol{x},y,z)}}{(1-\alpha(\boldsymbol{x},z,z))\Gamma(1-\alpha(\boldsymbol{x},y,z))}dzds =: I_{1a}+I_{1b}.
\end{aligned}
$$

$\partial_t^2 I_{1a}$ can be bounded by

$$
|\partial_t^2 I_{1a}| = \left|\partial_t^2 \frac{(t-y)^{2-\alpha(\boldsymbol{x},y,y)-\varepsilon}}{\Gamma(3-\alpha(\boldsymbol{x},y,y)-\varepsilon)}\right| \leq Q(t-y)^{-\alpha(\boldsymbol{x},y,y)-\varepsilon}.
$$

$\partial_t^2 I_{1b}$ can be bounded in a similar manner to that of $\partial_t^2 I_2$. Furthermore, passing to the limit $t \to 0$ in problem (3.1) yields

$$
\partial_t w(\boldsymbol{x},0) = f(\boldsymbol{x},0) + \Delta u_0. \tag{A.7}
$$

We incorporate the estimates (A.4), (A.5), (A.6), (A.7), and the estimates for $\partial_t^2 I_1$ into (A.3) and combine the resulting estimate with (A.1) and (A.2) to prove (3.3) for $0 < \varepsilon \ll 1$. The estimate (3.3) with $\varepsilon = 0$ can be obtained by letting $\varepsilon \downarrow 0$ in (3.3).

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