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A NUMERICAL METHOD FOR SOLVING SYSTEMS OF HYPERSINGULAR INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract. This paper is concerned with a collocation-quadrature method for solving systems of Prandtl's integro-differential equations based on de la Vallée Poussin filtered interpolation at Chebyshev nodes. We prove stability and convergence in Hölder-Zygmund spaces of locally continuous functions. Some numerical tests are presented to examine the method's efficacy.

Key words. Chebyshev nodes, filtered approximation, Hölder-Zygmund spaces, system of Prandtl's integro-differential equations

AMS subject classifications. 41A10, 65D05, 33C45, 45J05

1. Introduction. In this paper, we propose a numerical procedure to solve systems of singular integro-differential equations of the type

$$\begin{cases} \sigma\zeta_2(y) + a\zeta_1'(y) + \dfrac{b}{\pi}\displaystyle\int_{-1}^{1}\frac{\zeta_1'(x)}{x-y}dx - \frac{1}{\pi}\int_{-1}^{1}\kappa_1(x,y)\zeta_1(x)dx = g_1(y), \\ \sigma\zeta_1(y) + a\zeta_2'(y) + \dfrac{b}{\pi}\displaystyle\int_{-1}^{1}\frac{\zeta_2'(x)}{x-y}dx + \frac{1}{\pi}\int_{-1}^{1}\kappa_2(x,y)\zeta_2(x)dx = g_2(y), \end{cases} \quad y \in (-1,1),$$

with $\sigma \in \mathbb{R}\backslash\{0\}$, and where for $i = 1, 2$, $\kappa_i(x, y)$ and $g_i(y)$ are given functions defined in $\Omega := (-1, 1)^2$ and $(-1, 1)$, respectively. The constants $a, b \in \mathbb{R}$ are such that $a^2+b^2 = 1$, and the unknown solution $\mathcal{Z} = (\zeta_1, \zeta_2)$ is a differentiable function satisfying the zero boundary condition

$$\mathcal{Z}(-1) = \mathcal{Z}(1) = 0.$$

In view of the nature of the solution and according to the property

$$\int_{-1}^{1}\frac{G'(x)}{x-y}dx - \frac{G(1)}{1-x} - \frac{G(-1)}{1+x} = \frac{d}{dy}\int_{-1}^{1}\frac{G(x)}{x-y}dx, \qquad y \in (-1,1),$$

holding for any G satisfying $G' \in L_p(-1, 1)$, for some $p > 1$ (see [22, Lemma 6.1, Cap II]), the solution can be rewritten as $\mathcal{Z} = (f_1\varphi, f_2\varphi)$, where $\varphi(x) := \sqrt{1-x^2}$. Taking into account that

$$\int_{-1}^{1}\frac{G(x)}{(x-y)^2}\varphi(x)dx = \frac{d}{dy}\int_{-1}^{1}\frac{G(x)}{x-y}\varphi(x)dx,$$

and that the choice $a = 0, b = 1$ ensures that for $y \in (-1, 1)$ [14] (see also [3, Th.2.1] and [27, Th2.3],[28])

$$\left|\int_{-1}^{1}\frac{G(x)}{x-y}\varphi(x)dx\right| \le \mathcal{C}\|G\|_{Z_r(\varphi)}, \mathcal{C} > 0, \qquad \text{for all } G \in Z_r(\varphi), \;\; r > 0,$$

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with $Z_r(\varphi)$ denoting the Zygmund space defined in (2.1), where the constant $\mathcal{C}$ is independent of G and y. Hence, we are going to consider the following systems of hypersingular integral equations:

(1.1)
$$\begin{cases} \sigma f_2(y) - \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \dfrac{f_1(x)}{(x-y)^2}\varphi(x)dx - \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \log|x-y| f_1(x)\varphi(x)dx \\ \qquad\qquad\qquad + \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} k_1(x,y) f_1(x)\varphi(x)dx = g_1(y), \\ \qquad\qquad\qquad\qquad y \in (-1,1), \\ \sigma f_1(y) - \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \dfrac{f_2(x)}{(x-y)^2}\varphi(x)dx - \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \log|x-y| f_2(x)\varphi(x)dx \\ \qquad\qquad\qquad + \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} k_2(x,y) f_2(x)\varphi(x)dx = g_2(y), \end{cases}$$

for the unknown $\mathbf{f} = (f_1, f_2)$. Here the kernels κ_i, $i = 1, 2$, have been split as

$$\kappa_i(x,y) = \log|x-y| + k_i(x,y).$$

The functions g_1, g_2, k_1, k_2 may have algebraic singularities at the endpoints ± 1 and/or on the boundary $\partial\Omega$, and we show that the solution $\mathbf{f}$ inherits their singular behaviors. For this reason we consider system (1.1) in suitable subspaces of weighted continuous functions. Letting

$$\begin{aligned} D &: f \to Df, & Df(y) &:= -\frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{(x-y)^2}\varphi(x)dx, \\ H &: f \to Hf, & Hf(y) &:= -\frac{1}{\pi}\int_{-1}^{1} \log|x-y| f(x)\varphi(x)dx, \\ K_i &: f \to K_i f, & K_i f(y) &:= \frac{1}{\pi}\int_{-1}^{1} k_i(x,y) f(x)\varphi(x)dx, \qquad i = 1, 2, \end{aligned}$$

the system (1.1) can be rewritten as

$$\begin{cases} \sigma f_2(y) + (D + H + K_1) f_1(y) &= g_1(y), \\ \sigma f_1(y) + (D + H + K_2) f_2(y) &= g_2(y). \end{cases} \tag{1.2}$$

Integro-differential equations are models for many different problems arising for instance in biology, viscoelasticity, fluid mechanics, physics, and engineering (see [4, 7, 8, 9, 11, 12, 13, 18, 19, 20, 22, 25, 26, 35]). In fluid mechanics, singular integro-differential equations of Prandtl's type emerge in problems involving aerofoil and propeller theory, as well as in the contact interaction between a finite-length stringer with a variable along-the-length stiffness in tension-compression. Hence, methods for solving them have got a lot of attention (see, e.g., [1, 2, 5, 6, 10, 15, 23, 24, 32]).

A system of hypersingular integro-differential equations (HIDE) appears, for example, in the model describing the weak interface between two elastic materials containing a periodic array of micro-crazes [35]. Indeed, the boundary conditions for the solution of the problem are given in terms of an HIDE system.

The purpose of this paper is to present a numerical method for solving systems of the type (1.1), seeking the solution in a couple of weighted Zygmund-type spaces equipped with the uniform norm. The approach proposed here is based on the quadrature method described in [6] involving discrete de la Vallée Poussin (VP) polynomials, interpolating a given function

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at the zeros of a Chebyshev polynomial of the second kind. This tool, introduced and studied in [33, 34] in a more general context, appears especially convenient in view of its uniform boundedness in the space of locally continuous functions. Related approximation errors have been recently characterized in [30] in the case of four Chebyshev weights, providing in [31] also error estimates in Zygmund-type subspaces (see also [29]). In particular, in the case $k_1 = k_2$, we combine the aforementioned method with a procedure presented in [21], which converts the system (1.2) into two independent equations.

Hence, the numerical method we get is stable and convergent in suitable Zygmund weighted spaces. Error estimates in a weighted uniform norm are also given, and the well conditioning of the final linear systems is stated. Finally, some numerical experiments are provided to illustrate the agreement between the theoretical estimates and the numerical results.

The paper is organized as follows. Section 2 includes the definition of the spaces in which the current problem is investigated as well as the relevant properties of the VP operator. In Section 3 mapping features of the operators involved in (1.1) are given, and sufficient conditions assuring existence and uniqueness of the solution are proved. In Section 4, we describe the procedure we propose in both the cases, i.e., for the complete system and the system separated into two independent equations. In Section 5 some numerical tests are presented. Section 6 includes proofs of the main results, and in Appendix A it is shown how to compute the matrices of the final linear systems.

2. Preliminaries. Throughout the paper $\mathcal{C}$ stands for any positive constant having different values at different occurrences, and $\mathcal{C} \neq \mathcal{C}(n, f, \ldots)$ means that $\mathcal{C} > 0$ is independent of $n, f, \ldots$ Moreover, $\mathbb{P}_m$ denotes the space of all algebraic polynomials of degree at most m. For any bivariate function $g(x, y)$, we denote by g_y the function of the variable x only, with y fixed, and similarly by g_x the function of the variable y only.

2.1. Function spaces. With $\varphi(x) = \sqrt{1 - x^2}$, let

$$C_\varphi = \left\{ f \in C^0((-1,1)) \ : \ \lim_{x \to \pm 1} f(x)\varphi(x) = 0 \right\},$$

endowed with the norm

$$\|f\|_\varphi = \max_{x \in [-1,1]} |f(x)\varphi(x)|.$$

Denote by

$$E_m(f)_\varphi = \inf_{P \in \mathbb{P}_m} \|f - P\|_\varphi$$

the error of the best approximation of $f \in C_\varphi$ by polynomials. The limit conditions assure the validity of the Weierstrass theorem in C_φ, i.e., [16]

$$\lim_m E_m(f)_\varphi = 0, \quad \Leftrightarrow \quad f \in C_\varphi.$$

By means of $E_m(f)_\varphi$ it is possible to define in C_φ the Zygmund-type subspaces of order $s \in \mathbb{R}^+$,

$$Z_s(\varphi) = \left\{ f \in C_\varphi \ : \ \sup_{m>0} (m+1)^s E_m(f)_\varphi < +\infty \right\}, \tag{2.1}$$

equipped with the norm

$$\|f\|_{Z_s(\varphi)} = \|f\|_\varphi + \sup_{m>0} (m+1)^s E_m(f)_\varphi.$$

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Finally, setting $\mathbf{f} = (f_1, f_2)$, we consider the product spaces

$$C_\varphi \times C_\varphi = \{(f_1, f_2) : f_1, f_2 \in C_\varphi\}, \quad Z_s(\varphi) \times Z_s(\varphi) = \{(f_1, f_2) : f_1, f_2 \in Z_s(\varphi)\},$$

equipped with the norms

$$\|\mathbf{f}\|_{C_\varphi \times C_\varphi} = \max\{\|f_1\|_{C_\varphi}, \|f_2\|_{C_\varphi}\}, \quad \|\mathbf{f}\|_{Z_s(\varphi)\times Z_s(\varphi)} = \max\{\|f_1\|_{Z_s(\varphi)}, \|f_2\|_{Z_s(\varphi)}\},$$

respectively.

2.2. Discrete de la Vallée Poussin interpolating polynomial. Let $\{p_j\}_j$ be the orthonormal polynomial sequence with respect to the Chebyshev weight φ, i.e.,

$$p_j(x) = \sqrt{\frac{2}{\pi}} \frac{\sin{(j+1)t}}{\sin t}, \qquad t = \arccos x, \ |x| \le 1,$$

and for a given even integer $N \in \mathbb{N}$, denote by

$$x_k = \cos\left(\frac{2k\pi}{3N+2}\right), \qquad k = 1, \dots, \frac{3}{2}N,$$

the zeros of $p_{\frac{3}{2}N}$. Moreover, let $\{\lambda_k\}_{k=1}^{\frac{3}{2}N}$ be the Christoffel numbers related to φ. Then, the N-th discrete de la Vallée Poussin polynomial with respect to the weight φ, introduced in [33, 34] in a more general context, is defined as

$$V_N f(x) = \sum_{j=0}^{\frac{3}{2}N-1} c_j(f) q_j(x),$$

where

$$q_j(x) := \begin{cases} p_j(x) & \text{if } \ j = 0, \dots, N, \\ \dfrac{2N-j}{N} p_j(x) - \dfrac{j-N}{N} p_{3N-j}(x) & \text{if } \ N+1 \le j \le \frac{3}{2}N - 1, \end{cases}$$

and

$$c_j(f) = \sum_{k=1}^{\frac{3}{2}N} \lambda_k p_j(x_k) f(x_k)$$

are discretizations of the Chebyshev-Fourier coefficients by the $\frac{3}{2}N$-th Gauss-Chebyshev quadrature rule with respect to φ. The polynomial $V_N f$ interpolates the function f at the knots x_k, $k = 1, \dots, \frac{3}{2}N$, and reproduces polynomials of degree at most N. On the other hand, as proved in [33], V_N is a projection onto the so-called VP *space* defined as

$$S_N = \text{span}\left\{q_j \ : \ j = 0, \dots, \frac{3}{2}N - 1\right\},$$

for which the polynomials $\{q_j\}_{j=0}^{\frac{3}{2}N-1}$, which are orthogonal with respect to the inner product

$$\langle f, g\rangle_\varphi = \int_{-1}^{1} f(x) g(x) \varphi(x) dx,$$

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represent an orthogonal basis. The space S_N is nested between two classical polynomial spaces, i.e.,

$$\mathbb{P}_N \subset S_N \subset \mathbb{P}_{2N-1}.$$

By the specific feature of preserving polynomials in S_N, $V_N : f \to V_N(f)$ belongs to the so-called *polynomial quasi projectors*.

As proved in [33, 34] the map $V_N : C_\varphi \to C_\varphi$ is uniformly bounded with respect to N. Hence it is an optimal tool for approximating functions and allows for the following estimate [33]:

$$\|f - V_N f\|_\varphi \le \mathcal{C} E_N(f)_\varphi, \qquad \text{for all } f \in C_\varphi, \quad \mathcal{C} \ne \mathcal{C}(N, f).$$

Besides the VP space S_N, we consider the *modified* VP *space*

$$\widetilde{S}_N = \text{span}\left\{\tilde{q}_j \; : \; j = 0, \dots, \frac{3}{2}N - 1\right\}$$

generated by the polynomials

$$\tilde{q}_j(x) := \begin{cases} \frac{p_j(x)}{j+1} & \text{if} \quad j = 0, \dots, N, \\ \dfrac{2N-j}{N} \dfrac{p_j(x)}{j+1} - \dfrac{j-N}{N} \dfrac{p_{3N-j}(x)}{3N-j+1} & \text{if} \quad N+1 \le j \le \frac{3}{2}N - 1. \end{cases}$$

The introduction of the space $\widetilde{S}_N$ is crucial. Indeed, as shown in [6, Proposition 3.1], the operator D is a bijective map from $\widetilde{S}_N$ into S_N, and the important relation holds

$$V_N D f = D f, \qquad \text{for all } f \in \widetilde{S}_N. \tag{2.2}$$

Finally, we recall the following result [6, Lemma 5.1]:

LEMMA 2.1. *For any polynomial* $\widetilde{P}_N$

$$\widetilde{P}_N(y) = \sum_{j=0}^{\frac{3}{2}N-1} a_j \tilde{q}_j(x) \qquad \Rightarrow \qquad V_N \widetilde{P}_N(y) = \sum_{j=0}^{\frac{3}{2}N-1} a_j w_j q_j(x),$$

with

$$w_j = \begin{cases} \frac{1}{j+1}, & 0 \le j \le N \\ \frac{1}{N}\left\{\frac{2N-j}{j+1} + \frac{j-N}{3N-j+1}\right\}, & N+1 \le j \le \frac{3}{2}N - 1. \end{cases} \tag{2.3}$$

3. Main results. Introduce the matrices

$$\mathbf{J} := \begin{bmatrix} O & I \\ I & O \end{bmatrix}, \quad \mathbf{D} := \begin{bmatrix} D & O \\ O & D \end{bmatrix}, \quad \mathbf{H} := \begin{bmatrix} H & O \\ O & H \end{bmatrix}, \quad \mathbf{K} := \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix},$$

where I and O are the identity and null operators, respectively, and the arrays

$$\mathbf{f} := \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad \mathbf{g} := \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

Equation (1.2) can be written as

$$(\sigma \mathbf{J} + \mathbf{D} + \mathbf{H} + \mathbf{K})\mathbf{f} = \mathbf{g}. \tag{3.1}$$

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The following assertions are crucial in order to study the solvability of the system (3.1).

LEMMA 3.1. *Under the assumptions* $k_i(x,y)\varphi(y) \in C^0([-1,1]^2)$, $i = 1,2$, *and*

$$k_i(x,\cdot) \in Z_s(\varphi) \quad \textit{uniformly with respect to } x \in [-1,1] \textit{ for some } s > 0, \quad i = 1,2,$$

$\mathbf{K} : Z_{s+1}(\varphi) \times Z_{s+1}(\varphi) \to Z_s(\varphi) \times Z_s(\varphi)$ *is a compact operator.*

LEMMA 3.2. *For any* $s > 0$, $\mathbf{H} : Z_s(\varphi) \times Z_s(\varphi) \to Z_s(\varphi) \times Z_s(\varphi)$ *is a compact operator.*

LEMMA 3.3. *For any* $s > 0$, $\mathbf{D} : Z_{s+1}(\varphi) \times Z_{s+1}(\varphi) \to Z_s(\varphi) \times Z_s(\varphi)$ *is a bounded map, having a bounded inverse.*

As a consequence of the above lemmas and of the classical Fredholm's alternative theorem, the next result provides sufficient conditions so that the system (3.1) is uniquely solvable.

THEOREM 3.4. *Under the assumptions of Lemmas 3.1, if* $Ker(\sigma\mathbf{J}+\mathbf{D}+\mathbf{H}+\mathbf{K}) = \{\mathbf{0}\}$ *in* $Z_{s+1}(\varphi) \times Z_{s+1}(\varphi)$, *then for any* $\mathbf{g} \in Z_s(\varphi) \times Z_s(\varphi)$ *the equation*

$$(\sigma\mathbf{J} + \mathbf{D} + \mathbf{H} + \mathbf{K})\mathbf{f} = \mathbf{g}$$

admits a unique and stable solution $\mathbf{f} \in Z_{s+1}(\varphi) \times Z_{s+1}(\varphi)$.

Now, we describe the discretization method proposed to approximate the solution of the system (1.2), which is an extended application of the method proposed in [6]. Using the N-th discrete de la Vallée Poussin polynomial in (2.2), let us define the following discrete operators

$$H_N := V_N H,$$

and

$$K_{N,i} := V_N \widetilde{K}_{N,i}, \quad \text{with} \quad \widetilde{K}_{N,i} f(y) := -\frac{1}{\pi}\int_{-1}^{1} V_N k_{i,y}(x) f(x)\varphi(x)dx, \quad i = 1,2.$$

Moreover, letting

$$\mathbf{V}_N := \begin{bmatrix} V_N & O \\ O & V_N \end{bmatrix}, \qquad \widetilde{\mathbf{K}}_N = \begin{bmatrix} \widetilde{K}_{N,1} & O \\ O & \widetilde{K}_{N,2} \end{bmatrix},$$

we define the following matrices of approximating operators

$$\bar{\mathbf{V}}_N := \sigma\mathbf{V}_N\mathbf{J}, \qquad \mathbf{H}_N := \mathbf{V}_N\mathbf{H}, \qquad \mathbf{K}_N := \mathbf{V}_N\widetilde{\mathbf{K}}_N,$$

and the following array

$$\mathbf{g}_N := \begin{bmatrix} g_{1,N} \\ g_{2,N} \end{bmatrix} = \mathbf{V}_N\mathbf{g}.$$

Then, the proposed numerical method consists of solving in place of the system (3.1) the following finite-dimensional system

$$(\bar{\mathbf{V}}_N + \mathbf{D} + \mathbf{H}_N + \mathbf{K}_N)\mathbf{f}_N = \mathbf{g}_N \tag{3.2}$$

for the unknown solution

$$\mathbf{f}_N = \begin{bmatrix} f_{1,N} \\ f_{2,N} \end{bmatrix}.$$

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The following proposition holds true:

PROPOSITION 3.5. *If a solution* $\mathbf{f}_N$ *of* (3.2) *exists, it belongs to* $\widetilde{S}_N \times \widetilde{S}_N$*, where* $\widetilde{S}_N \times \widetilde{S}_N := \{(f_1, f_2) : f_i \in \widetilde{S}_N\}$.

Letting

$$\mathbf{T}_N := \bar{\mathbf{V}}_N + \mathbf{D} + \mathbf{H}_N + \mathbf{K}_N \qquad \mathbf{T} := \sigma \mathbf{J} + \mathbf{D} + \mathbf{H} + \mathbf{K},$$

the following theorem gives the assumptions under which $\mathbf{f}_N$ is unique when it exists.

THEOREM 3.6. *Let us assume that the kernels* k_i*,* $i = 1, 2$*, satisfy* $k_i(x, y)\varphi(y) \in C^0([-1, 1]^2)$ *and*

$$k_i(x, \cdot) \in Z_s(\varphi) \quad \textit{uniformly with respect to } x \in [-1, 1] \textit{ for some } s > 0, \quad i = 1, 2,$$

and that $Ker(\sigma\mathbf{J} + \mathbf{D} + \mathbf{H} + \mathbf{K}) = \{\mathbf{0}\}$ *in* $Z_{s+1}(\varphi) \times Z_{s+1}(\varphi)$*. Then for* $N > N_0$ *with* N_0 *being a fixed positive and sufficiently large integer, the matrices of operators*

$$\mathbf{T}_N : Z_{s+1}(\varphi) \times Z_{s+1}(\varphi) \to Z_s(\varphi) \times Z_s(\varphi)$$

have bounded inverses, and

$$\sup_N \|\mathbf{T}_N^{-1}\|_{Z_s(\varphi)\times Z_s(\varphi)\to Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)} < +\infty. \tag{3.3}$$

Moreover, the condition numbers of $\mathbf{T}_N$ *tend to the condition number of* $\mathbf{T}$*, i.e.,*

$$\lim_{N\to\infty} \frac{\|\mathbf{T}_N\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)} \|\mathbf{T}_N^{-1}\|_{Z_s(\varphi)\times Z_s(\varphi)\to Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)}}{\|\mathbf{T}\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)} \|\mathbf{T}^{-1}\|_{Z_s(\varphi)\times Z_s(\varphi)\to Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)}} = 1. \tag{3.4}$$

In view of the previous result, for any $\mathbf{g} \in Z_s(\varphi) \times Z_s(\varphi)$, the approximating system (3.2) admits a unique solution $\mathbf{f}_N \in Z_{s+1}(\varphi) \times Z_{s+1}(\varphi)$.

The next theorem provides conditions under which the sequence $\{\mathbf{f}_N\}_N$ converges to the unique solution $\mathbf{f}$ of the system (3.1) in $C_\varphi \times C_\varphi$.

THEOREM 3.7. *Let the assumptions of Theorem 3.6 be satisfied. For every* $\mathbf{g} \in Z_s(\varphi) \times Z_s(\varphi)$ *and for* $N > N_0$*, with* N_0 *being a fixed positive and sufficiently large integer, we have*

$$\|\mathbf{f} - \mathbf{f}_N\|_{C_\varphi \times C_\varphi} \le \frac{\mathcal{C}}{N^s} \|\mathbf{g}\|_{Z_s(\varphi)\times Z_s(\varphi)}, \qquad \mathcal{C} \neq \mathcal{C}(N, \mathbf{f}). \tag{3.5}$$

4. Computation of the approximate solution. We consider the general case with $K_1 \neq K_2$ and the special case when $K_1 = K_2$ separately.

4.1. The general case $K_1 \neq K_2$. Since $f_{1,N}, f_{2,N} \in \widetilde{S}_N$, we express these functions in terms of the orthogonal basis $\{\tilde{q}_n^m\}_n$ of $\widetilde{S}_N$, i.e.,

$$\begin{aligned} \mathbf{f}_N(y) = (f_{1,N}, f_{2,N})^T &= \left(\sum_{j=0}^{\frac{3}{2}N-1} f_j^{(1)}\, \tilde{q}_j(y), \ \sum_{j=0}^{\frac{3}{2}N-1} f_j^{(2)}\, \tilde{q}_j(y) \right)^T \\ &= \left(\widetilde{\mathrm{Q}} \cdot F^{(1)}, \widetilde{\mathrm{Q}} \cdot F^{(2)} \right)^T, \end{aligned} \tag{4.1}$$

where

$$\widetilde{\mathrm{Q}} = (\tilde{q}_0, \ldots, \tilde{q}_{\frac{3}{2}N-1}), \quad F^{(1)} = (f_0^{(1)}, \ldots, f_{\frac{3}{2}N-1}^{(1)})^T, \quad \text{and} \quad F^{(2)} = (f_0^{(2)}, \ldots, f_{\frac{3}{2}N-1}^{(2)})^T.$$

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Taking into account Lemma 2.1, we have

$$V_N f_{1,N}(y) = \mathrm{Q}(y)\mathcal{V}_N F^{(1)} \qquad \text{and} \qquad V_N f_{2,N}(y) = \mathrm{Q}(y)\mathcal{V}_N F^{(2)},$$

where $\mathrm{Q} = (q_0, \dots, q_{\frac{3}{2}N-1})$ and $\mathcal{V}_N := \mathrm{diag}(w_j)_{j=0,..,\frac{3}{2}N-1}$, with w_j defined in (2.3). Denoting by $\mathcal{I}_N$ the identity matrix of order $\frac{3}{2}N$ and recalling the definitions of the matrices $\mathcal{A}_N, \mathcal{B}_N$ introduced in [6, pp. 693–695] (reported for the reader's convenience in Appendix A), by (3.2), we have

$$\begin{aligned}
(D + H_N)f_{1,N}(y) &= \mathrm{Q}(y) \cdot (\mathcal{I}_N + \mathcal{A}_N)F^{(1)}, \\
(D + H_N)f_{2,N}(y) &= \mathrm{Q}(y) \cdot (\mathcal{I}_N + \mathcal{A}_N)F^{(2)}, \\
K_{N,1}f_{1,N}(y) &= \mathrm{Q}(y) \cdot \mathcal{B}_N^{(1)} F^{(1)}, \qquad & K_{N,2}f_{2,N}(y) &= \mathrm{Q}(y) \cdot \mathcal{B}_N^{(2)} F^{(2)}, \\
g_{1,N}(y) &= \mathrm{Q}(y) \cdot G^{(1)}, & g_{2,N}(y) &= \mathrm{Q}(y) \cdot G^{(2)},
\end{aligned}$$

where

$$\begin{aligned}
G^{(1)} &:= \left(g_0^{(1)}, \dots, g_{\frac{3}{2}N-1}^{(1)}\right)^T, \qquad & g_j^{(1)} &:= c_j(g_1) = \sum_{k=1}^{\frac{3}{2}N} \lambda_k p_j(x_k) g_1(x_k), \\
G^{(2)} &:= \left(g_0^{(2)}, \dots, g_{\frac{3}{2}N-1}^{(2)}\right)^T, & g_j^{(2)} &:= c_j(g_2) = \sum_{k=1}^{\frac{3}{2}N} \lambda_k p_j(x_k) g_2(x_k).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathrm{Q}(y)\left(\sigma\mathcal{V}_N F^{(2)} + (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(1)})F^{(1)}\right) &= \mathrm{Q}(y) \cdot G^{(1)}, \\
\mathrm{Q}(y)\left(\sigma\mathcal{V}_N F^{(1)} + (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(2)})F^{(2)}\right) &= \mathrm{Q}(y) \cdot G^{(2)},
\end{aligned}$$

and the unknown vector $(F^{(1)},\ F^{(2)})^T$ will be the solution of the following linear system

$$\begin{aligned}
\sigma\mathcal{V}_N F^{(2)} + (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(1)})F^{(1)} &= G^{(1)}, \\
\sigma\mathcal{V}_N F^{(1)} + (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(2)})F^{(2)} &= G^{(2)}
\end{aligned}$$

having the block matrix form

$$\begin{bmatrix} (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(1)}) & \sigma\mathcal{V}_N \\ \sigma\mathcal{V}_N & (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(2)}) \end{bmatrix} \begin{bmatrix} F^{(1)} \\ F^{(2)} \end{bmatrix} = \begin{bmatrix} G^{(1)} \\ G^{(2)} \end{bmatrix}. \tag{4.2}$$

By solving the above linear system we compute the approximating array $\mathbf{f}_N$ by (4.1).

4.2. The special case $K_1 = K_2 = K$. Following a technique introduced in [21], we transform (3.2) into a separable system of two independent finite-dimensional equations. Setting

$$\begin{aligned}
\hat{f}_N &:= f_{1,N} + f_{2,N}, \qquad & \tilde{f}_N &:= f_{1,N} - f_{2,N}, \\
\hat{g}_N &:= g_{1,N} + g_{2,N}, & \tilde{g}_N &:= g_{1,N} - g_{2,N},
\end{aligned} \tag{4.3}$$

the following proposition holds:

PROPOSITION 4.1. *The finite-dimensional system (3.2) can be reformulated as*

$$(\sigma I + D + K_N + H_N)\hat{f}_N = \hat{g}_N, \tag{4.4}$$

$$(-\sigma I + D + K_N + H_N)\tilde{f}_N = \tilde{g}_N. \tag{4.5}$$

Hence, for determining $\mathbf{f}_N$, we have to compute $\hat{f}_N, \tilde{f}_N$, i.e., we have to apply twice the method in [6]. For the convenience of the reader we report in the following the main steps to perform this computation. Taking into account that $\hat{f}_N, \tilde{f}_N \in \widetilde{S}_N$ and recalling the definition of $\widetilde{\mathrm{Q}}$ given in the previous section, we write

$$\hat{f}_N(y) = \sum_{j=0}^{\frac{3}{2}N-1} \hat{f}_j\, \tilde{q}_j(y) = \widetilde{\mathrm{Q}} \cdot \widehat{\mathrm{F}}, \tag{4.6}$$

$$\tilde{f}_N(y) = \sum_{j=0}^{\frac{3}{2}N-1} \tilde{f}_j\, \tilde{q}_j(y) = \widetilde{\mathrm{Q}} \cdot \widetilde{\mathrm{F}}, \tag{4.7}$$

where

$$\widehat{\mathrm{F}} = (\hat{f}_0, \dots, \hat{f}_{\frac{3}{2}N-1})^T, \qquad \widetilde{\mathrm{F}} = (\tilde{f}_0, \dots, \tilde{f}_{\frac{3}{2}N-1})^T.$$

Since

$$\forall f \in \widetilde{S}_N \quad \Rightarrow \quad V_N f,\ Df,\ H_N f,\ K_N f \in S_N,$$

with the notation used in the previous section, we get

$$\begin{aligned}
V_N \hat{f}_N(y) &= \mathrm{Q}(y) \cdot \mathcal{V}_N \widehat{\mathrm{F}}, & V_N \tilde{f}_N(y) &= \mathrm{Q}(y) \cdot \mathcal{V}_N \widetilde{\mathrm{F}},\\
D\hat{f}_N(y) &= \mathrm{Q}(y) \cdot \widehat{\mathrm{F}}, & D\tilde{f}_N(y) &= \mathrm{Q}(y) \cdot \widetilde{\mathrm{F}},\\
H_N \hat{f}_N(y) &= \mathrm{Q}(y) \cdot \mathcal{A}_N\ \widehat{\mathrm{F}}, & H_N \tilde{f}_N(y) &= \mathrm{Q}(y) \cdot \mathcal{A}_N \widetilde{\mathrm{F}},\\
K_N \hat{f}_N(y) &= \mathrm{Q}(y) \cdot \mathcal{B}_N\ \widehat{\mathrm{F}}, & K_N \tilde{f}_N(y) &= \mathrm{Q}(y) \cdot \mathcal{B}_N \widetilde{\mathrm{F}},\\
\hat{g}_N(y) &= \mathrm{Q}(y) \cdot \widehat{\mathrm{G}}, & \tilde{g}_N(y) &= \mathrm{Q}(y) \cdot \widetilde{\mathrm{G}},
\end{aligned}$$

where

$$\widehat{\mathrm{G}} := \left(\hat{g}_0, \dots, \hat{g}_{\frac{3}{2}N-1}\right)^T, \qquad \hat{g}_j := c_j(\hat{g}) = \sum_{k=1}^{\frac{3}{2}N} \lambda_k p_j(x_k) \hat{g}(x_k),$$

$$\widetilde{\mathrm{G}} := \left(\tilde{g}_0, \dots, \tilde{g}_{\frac{3}{2}N-1}\right)^T, \qquad \tilde{g}_j := c_j(\tilde{g}) = \sum_{k=1}^{\frac{3}{2}N} \lambda_k p_j(x_k) \tilde{g}(x_k).$$

Summing up, we can rewrite the approximate equations (4.4)–(4.5) as

$$\begin{cases} \mathrm{Q}(y)(\sigma \mathcal{V}_N + \mathcal{R}_N)\widehat{\mathrm{F}} = \mathrm{Q}(y) \cdot \widehat{\mathrm{G}}, \\ \mathrm{Q}(y)(-\sigma \mathcal{V}_N + \mathcal{R}_N)\widetilde{\mathrm{F}} = \mathrm{Q}(y) \cdot \widetilde{\mathrm{G}}, \end{cases}$$

where

$$\mathcal{R}_N = \mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N.$$

Consequently, the unknowns arrays $\widehat{\mathrm{F}}$ and $\widetilde{\mathrm{F}}$ are the unique solutions of the two linear systems of equations,

$$\begin{cases} (\sigma\mathcal{V}_N + \mathcal{R}_N)\widehat{\mathrm{F}} = \widehat{\mathrm{G}}, & \text{(4.8a)} \\ (-\sigma\mathcal{V}_N + \mathcal{R}_N)\widetilde{\mathrm{F}} = \widetilde{\mathrm{G}}. & \text{(4.8b)} \end{cases}$$

After solving these systems, we compute the approximate solutions of (4.4)–(4.5) using (4.6) and (4.7) and then the approximation

$$\mathbf{f}_N = \begin{bmatrix} f_{1,N} \\ f_{2,N} \end{bmatrix}, \qquad \text{with} \quad f_{1,N} = \frac{\hat{f}_N + \tilde{f}_N}{2}, \quad f_{2,N} = \frac{\hat{f}_N - \tilde{f}_N}{2},$$

of the unique solution $\mathbf{f}$ of system (3.1).

REMARK 4.2. It is hardly necessary to note that in the case $K_1 = K_2 = 0$, the system reduces to two separable linear systems as in (4.4)–(4.5), both of which involving matrices of coefficients of bandwidth 2 and having dominant diagonal. Of course, this special structure enables the realization of a strong computational reduction in solving the linear systems.

5. Numerical tests. In this section, we offer some numerical examples to demonstrate the theoretical results obtained in the previous sections. Denoting by X a sufficiently large mesh of equally spaced points in $[-1, 1]$, in each test we report the absolute weighted errors

$$\mathcal{E}_N := \max_{x\in X} \Big(|\mathbf{f}(x) - \mathbf{f}_N(x)|\varphi(x) \Big),$$

where $n := \frac{3}{2}N$ denotes the number of collocation nodes and $\mathbf{f}_N$ the numerical solution computed by the proposed method. Moreover, we also compute the condition numbers (defined for any $A \in \mathbb{R}^{n\times n}$ as $\mathrm{cond(A)} = \|\mathrm{A}\|_\infty \|\mathrm{A}^{-1}\|_\infty$) of the involved linear systems, providing in the general case (systems (4.8a)–(4.8b))

$$\mathrm{cond_N} = \mathrm{cond} \begin{bmatrix} (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(1)}) & \sigma\mathcal{V}_N \\ \sigma\mathcal{V}_N & (\mathcal{I}_N + \mathcal{A}_N + \mathcal{B}_N^{(2)}) \end{bmatrix},$$

and in the special case $K_1 = K_2 = K$ (system (4.2))

$$\widetilde{\mathrm{cond}}_N := \max\{\mathrm{cond}(\sigma\mathcal{V}_\mathrm{N} + \mathcal{R}_\mathrm{N}), \mathrm{cond}(-\sigma\mathcal{V}_\mathrm{N} + \mathcal{R}_\mathrm{N})\}.$$

We point out that all the computations were performed with 16 decimal digits, and the solutions of the linear systems have been computed by the Gaussian elimination method. Moreover, in the cases where the exact solution is unknown, the errors shown in the tables have been computed assuming as the exact solution the values obtained for $n = 1024$ or equivalently $N = 1536$.

Now we present the following examples:

EXAMPLE 5.1. $\sigma = 1, \quad k(x, y) = |y| + |x|,$

$$\mathbf{g}(y) = \begin{bmatrix} 7/(15\pi) + \frac{5|y|}{16} + \frac{1}{64}(-9 + 120y^2 - 8y^4 + 10\log 4) \\ \frac{1}{320\pi}(64 + 60\pi|y| + 5\pi(25 - 152y^2 + 8y^4 + 6\log 4)) \end{bmatrix},$$

having as exact solution

$$\mathbf{f}(y) = \left(\frac{y^2+1}{2}, \frac{y^2-1}{2} \right)^T.$$

EXAMPLE 5.2. $\sigma = 1, \quad k(x,y) = \left|\cos\left(y - \frac{\pi}{4}\right)\right|^{\frac{9}{2}} + |\sin(x)|^{\frac{7}{2}},$

$$\mathbf{g}(y) = \left(\frac{1}{2}\left(|y|^{\frac{11}{2}} + y\cos(y)\right),\ \frac{1}{2}\left(|y|^{\frac{11}{2}} - y\cos(y)\right)\right)^T.$$

EXAMPLE 5.3. $H \equiv 0, \quad \sigma = 1, \quad k(x,y) = (x^2+y^2)\cos(xy),$

$$\mathbf{g}(y) = \left(\frac{y|y| + |y+0.2|^{\frac{5}{2}}}{2},\ \frac{y|y| - |y+0.2|^{\frac{5}{2}}}{2}\right).$$

EXAMPLE 5.4. $K \equiv 0, \quad \sigma = 1,$

$$\mathbf{g}(y) = \left(1,\ \frac{1}{2}\left(1+\cos(2y)\right)\right)^T.$$

EXAMPLE 5.5. $\sigma = 1,$

$$k_1(x,y) = \left|\cos\left(y - \frac{\pi}{4}\right)\right|^{4.5} + |\sin(x)|^{3.5}, \quad k_2(x,y) = (x+y)^2,$$
$$\mathbf{g}(y) = \left(|y|^{5.5},\ y\cos(y)\right)^T.$$

EXAMPLE 5.6. $\sigma = \frac{1}{2}, \quad k_1(x,y) = \exp((x+y)^3), \quad k_2(x,y) = 1 + |x-y|^{3.5},$

$$\mathbf{g}(y) = \left((1-y^2)\arccos(y),\ (1-y^2)\arcsin(y)\right)^T.$$

The numerical results are given in Table 5.1.

5.1. Comments to the numerical tests. Examples 5.1–5.4 deal with the case $k_1(x,y) = k_2(x,y) = k(x,y)$, while Examples 5.5–5.6 deal with the general case $k_1(x,y) \neq k_2(x,y)$.

Referring to Example 5.1, this is the only case where the solution is known. By solving a well-conditioned linear system of order $n = 3072$, the solution is approximated with at least 7 exact decimal digits. This means that the numerical error is much smaller than the theoretical estimate. Since $\sup_{x\in[-1,1]} k_x \in Z_1(\varphi)$, $\mathbf{g} \in Z_1(\varphi) \times Z_1(\varphi)$ and according to Theorem 3.7 the theoretical errors goes like $\mathcal{O}\left(\frac{1}{N}\right)$.

In Example 5.2, $\sup_{x\in[-1,1]} k_x \in Z_{3.5}(\varphi)$, $\mathbf{g} \in Z_{3.5}(\varphi) \times Z_{3.5}(\varphi)$, and the expected rate of convergence is $\mathcal{O}\left(\frac{1}{N^{3.5}}\right)$. Also in this case the numerical results exceed the expectations from the theoretical estimates since with $n = 384$ we get an error of almost machine precision. The condition number of the corresponding linear systems is ≤ 3.

In Example 5.3 the rate of convergence is $\mathcal{O}\left(\frac{1}{N^2}\right)$ since $\mathbf{g} \in Z_2(\varphi) \times Z_{3.5}(\varphi)$, and $\sup_{x\in[-1,1]} k_x \in Z_s(\varphi)$ for any s. The errors confirm this behavior, and the condition numbers of the linear system are less than 5.

In Example 5.4, a solution with 13 exact decimal digits is achieved by solving a well conditioned linear system of order only 36 (corresponding to $n = 24$). This fast convergence agrees with the theoretical expectation since g and k_x are very smooth functions. The theoretical estimate assures that the error behaves like $\mathcal{O}\left(\frac{1}{N^{3.5}}\right)$ in both Examples 5.5–5.6. Hence, we conclude that in all the tests our method's high performance has been established.

TABLE 5.1
Numerical results for Examples 5.1–5.6.

	Example 5.1			Example 5.2	
n	$\widetilde{\text{cond}}_N$	$\mathcal{E}_N^{VP}$	n	$\widetilde{\text{cond}}_N$	$\mathcal{E}_N^{VP}$
12	2.64	1.76e-3	12	2.48	6.92e-6
24	2.77	4.74e-4	24	2.61	8.50e-8
48	2.84	1.23e-4	48	2.67	1.08e-9
96	2.87	3.14e-5	96	2.70	1.40e-11
192	2.89	7.95e-6	192	2.71	2.03e-13
384	2.89	1.99e-6	384	2.72	4.54e-15
768	2.90	5.01e-7			
1536	2.90	1.25e-7			
3072	2.91	3.13e-8			

	Example 5.3			Example 5.4	
n	$\widetilde{\text{cond}}_N$	$\mathcal{E}_N^{VP}$	n	$\widetilde{\text{cond}}_N$	$\mathcal{E}_N^{VP}$
12	4.16	2.81e-04	12	2.29	4.20e-9
24	4.40	3.83e-05	24	2.40	1.07e-14
48	4.50	3.67e-06	48	2.46	1.19e-14
96	4.55	4.24e-07	96	2.49	1.08e-14
192	4.58	5.26e-08	192	2.50	1.27e-14
384	4.59	5.62e-09			
768	4.60	3.99e-10			
1536	4.60	1.07e-10			

	Example 5.5			Example 5.6	
n	cond_N	$\mathcal{E}_N^{VP}$	n	cond_N	$\mathcal{E}_N^{VP}$
12	4.69	7.77e-06	12	12.22	1.88e-4
24	4.69	9.83e-08	24	12.22	9.06e-7
48	4.69	1.48e-09	48	12.22	1.56e-8
96	4.69	3.09e-11	96	12.22	2.03e-10
192	4.69	9.52e-13	192	12.22	4.06e-12
384	4.69	3.67e-14	384	12.22	9.18e-14
768	4.69	5.61e-15	768	12.22	8.07e-15

6. The proofs.

Proof of Lemma 3.1. The operator $\mathbf{K} : Z_{s+1}(\varphi) \times Z_{s+1}(\varphi) \to Z_s(\varphi) \times Z_s(\varphi)$ is compact if and only the operators $O : Z_{s+1}(\varphi) \to Z_s(\varphi)$ and $K_i : Z_{s+1}(\varphi) \to Z_s(\varphi), i = 1, 2$, are compact (see [32, p. 153]). The null operator is trivially compact, and the operator $K_i, i = 1, 2$, are compact as a consequence of [6, Proposition 2.3] with $\nu = \xi = 0$ and [6, (28)]. ∎

Proof of Lemma 3.2. Analogously to the operator $\mathbf{K}$ (see the proof of Lemma 3.1), the matrix operator $\mathbf{H}$ is compact if and only if the operator H is compact. This is true as a consequence of [6, Theorem 2.2]. ∎

Proof of Lemma 3.3. It is easy to verify that

$$\begin{aligned}\|\mathbf{Df}\|_{Z_s(\varphi)\times Z_s(\varphi)} &= \max\{\|Df_1\|_{Z_s(\varphi)}, \|Df_2\|_{Z_s(\varphi)}\} \\ &\leq \|D\|_{Z_{s+1}(\varphi)\to Z_s(\varphi)}\|\mathbf{f}\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)},\end{aligned}$$

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and then

$$\|\mathbf{D}\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)} \le \|D\|_{Z_{s+1}(\varphi)\to Z_s(\varphi)}.$$

Thus, the boundedness of the matrix operator $\mathbf{D}$ as a map from $Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)$ into $Z_s(\varphi)\times Z_s(\varphi)$ follows from the boundedness of the operator $D : Z_{s+1}(\varphi)\to Z_s(\varphi)$ [6, Theorem 2.1]. Moreover, it is easy to see that its inverse is the matrix operator

$$\mathbf{D}^{-1} = \begin{bmatrix} D^{-1} & O \\ O & D^{-1}\end{bmatrix},$$

where $D^{-1} : Z_s(\varphi)\to Z_{s+1}(\varphi)$ is the bounded inverse of the operator D [6, (15)]. Thus, since

$$\|\mathbf{D}^{-1}\|_{Z_s(\varphi)\times Z_s(\varphi)\to Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)} \le \|D^{-1}\|_{Z_s(\varphi)\to Z_{s+1}(\varphi)},$$

the matrix operator $\mathbf{D}^{-1}$ is the bounded inverse of $\mathbf{D}$. $\square$

Proof of Proposition 3.5. Recalling that V_N is a projection in the space S_N defined in (2.2), it is easy to see that $\mathbf{V}_N$ is a projection in the space $S_N\times S_N$, and

$$\begin{aligned}\mathbf{D}\mathbf{f}_N &= \mathbf{g}_N - \bar{\mathbf{V}}_N\mathbf{f}_N - \mathbf{H}_N\mathbf{f}_N - \mathbf{K}_N\mathbf{f}_N \\ &= \mathbf{V}_N\mathbf{g} - \sigma\mathbf{V}_N\mathbf{J}\mathbf{f}_N - \mathbf{V}_N\mathbf{H}\mathbf{f}_N - \mathbf{V}_N\widetilde{\mathbf{K}}_N\mathbf{f}_N \quad \in S_N\times S_N,\end{aligned}$$

where $S_N\times S_N = \{(f_1,f_2) : f_i\in S_N\}$. Moreover, since D is a bijective map from $\widetilde{S}_N$ into S_N, it is easy to deduce that $\mathbf{D} : \widetilde{S}_N\times\widetilde{S}_N\to S_N\times S_N$ is bijective, too. Then, if a solution $\mathbf{f}_N$ of (3.2) exists, it belongs to $\widetilde{S}_N\times\widetilde{S}_N$. $\square$

Proof of Theorem 3.6. Taking into account that from (2.2) we get $\mathbf{V}_N\mathbf{D} = \mathbf{D}$, the system (3.2) can be expressed in the following form:

$$\mathbf{V}_N(\sigma\mathbf{J}+\mathbf{D}+\mathbf{H}+\widetilde{\mathbf{K}}_N)\mathbf{f}_N = \mathbf{V}_N\mathbf{g},$$

i.e., as a projection of the equation

$$(\sigma\mathbf{J}+\mathbf{D}+\mathbf{H}+\widetilde{\mathbf{K}}_N)\mathbf{f}_N = \mathbf{g}$$

by the projector $\mathbf{V}_N$. Consequently we can deduce the solvability of the approximating system (3.2) by standard arguments of projection methods (see, for example, [17, Theorem 4.2]). In particular, if

$$\lim_N \|(\sigma\mathbf{J}+\mathbf{H}+\mathbf{K}) - (\bar{\mathbf{V}}_N+\mathbf{H}_N+\mathbf{K}_N)\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)} = 0, \tag{6.1}$$

then we deduce the uniqueness of the solutions of the approximating systems (3.2) by the uniqueness of the solution of system (3.1), i.e., (3.3), and

$$\lim_N \|\mathbf{T}_N\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)} = \|\mathbf{T}\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)}$$

and

$$\lim_N \|\mathbf{T}_N^{-1}\|_{Z_s(\varphi)\times Z_s(\varphi)\to Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)} = \|\mathbf{T}^{-1}\|_{Z_s(\varphi)\times Z_s(\varphi)\to Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)},$$

i.e., (3.4). Taking into account that

$$\|\sigma\mathbf{J}\mathbf{f} - \bar{\mathbf{V}}_N\mathbf{f}\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)} = \sigma\max\{\|f_2 - V_Nf_2\|_{Z_{s+1}(\varphi)}, \|f_1 - V_Nf_1\|_{Z_{s+1}(\varphi)}\}, \tag{6.2}$$

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$$\|\mathbf{H}-\mathbf{H}_N\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)} \le \|H-H_N\|_{Z_{s+1}(\varphi)\to Z_s(\varphi)}, \tag{6.3}$$

and

$$\begin{aligned}&\|\mathbf{K}-\mathbf{K}_N\|_{Z_{s+1}(\varphi)\times Z_{s+1}(\varphi)\to Z_s(\varphi)\times Z_s(\varphi)}\\ &\qquad\le \max\{\|K_1-K_{N,1}\|_{Z_{s+1}(\varphi)\to Z_s(\varphi)}, \|K_2-K_{N,2}\|_{Z_{s+1}(\varphi)\to Z_s(\varphi)}\},\end{aligned} \tag{6.4}$$

using [6, Theorems 3.2, 4.1 and 4.2], the identity (6.1) follows. □

Proof of Theorem 3.7. We note that

$$\mathbf{f}-\mathbf{f}_N = \mathbf{T}_N^{-1}\left[(\mathbf{g}-\mathbf{V}_N\mathbf{g}) + (\sigma\mathbf{J}\mathbf{f}-\bar{\mathbf{V}}_N\mathbf{f}) + (\mathbf{K}-\mathbf{K}_N)\mathbf{f} + (\mathbf{H}-\mathbf{H}_N)\mathbf{f}\right].$$

Taking into account (3.3), (6.2), (6.3), (6.4), and

$$\|\mathbf{g}-\mathbf{V}_N\mathbf{g}\|_{Z_s(\varphi)\times Z_s(\varphi)} = \max\{\|g_1-V_Ng_1\|_{Z_s(\varphi)}, \|g_2-V_Ng_2\|_{Z_s(\varphi)}\},$$

from [6, Theorems 3.2, 4.1, and 4.2] under the assumptions $g_1, g_2 \in Z_s(\varphi)$ and $f_1, f_2 \in Z_{s+1}(\varphi)$, we deduce

$$\|\mathbf{f}-\mathbf{f}_N\|_{Z_r(\varphi)\times Z_r(\varphi)} \le \frac{\mathcal{C}}{N^{s-r}}\|\mathbf{f}\|_{Z_{s+1}(\varphi)\times Z_{s+}(\varphi)}.$$

The bound (3.5) follows from the above estimate when $r \to 0^+$. □

Proof of Proposition 4.1. By (4.3)

$$f_{1,N} = \frac{\hat{f}_N+\tilde{f}_N}{2}, \qquad f_{2,N} = \frac{\hat{f}_N-\tilde{f}_N}{2},$$
$$g_{1,N} = \frac{\hat{g}_N+\tilde{g}_N}{2}, \qquad g_{2,N} = \frac{\hat{g}_N-\tilde{g}_N}{2},$$

and by replacing them in (1.2), we get

$$\sigma\left(\hat{f}_N-\tilde{f}_N\right) + (D+\widetilde{K}_N+H_N)\left(\hat{f}_N+\tilde{f}_N\right) = (\hat{g}_N+\tilde{g}_N), \tag{6.5}$$

$$\sigma\left(\hat{f}_N+\tilde{f}_N\right) + (D+\widetilde{K}_N+H_N)\left(\hat{f}_N-\tilde{f}_N\right) = (\hat{g}_N-\tilde{g}_N). \tag{6.6}$$

Then, (4.4) and (4.5) follow by adding and subtracting (6.5) and (6.6), respectively. □

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Appendix A. Matrices of the linear system. For the convenience of the reader, we report here the matrices of the final linear system, already given in [6] in a more general form.

$$\mathcal{V}_N := \mathrm{diag}(w_j)_{j=0,..,\frac{3}{2}N-1},$$

with w_j defined in (2.3). The matrix $\mathcal{A}_N$ is a bandwidth-2 matrix defined as

$$\begin{aligned}\mathcal{A}_N(i,i) &= \beta_i, && 0 \le i \le \frac{3}{2}N-1,\\ \mathcal{A}_N(i,i+2) &= \alpha_{i+2}, && 0 \le i \le \frac{3}{2}N-3,\\ \mathcal{A}_N(i,i-2) &= \gamma_{i-2}, && 2 \le i \le \frac{3}{2}N-1,\end{aligned}$$

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with

$$\alpha_\ell := \begin{cases} -\frac{1}{4\ell(\ell+1)} & 2 \le \ell \le N, \\ -\frac{1}{4N}\left[\frac{2N-\ell}{\ell(\ell+1)} + \frac{\ell-N}{(3N-\ell+1)(3N-\ell+2)}\right] & N+1 \le \ell \le \frac{3}{2}N-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta_\ell := \begin{cases} \frac{1}{4}\left(\frac{1}{2}+2\log 2\right) & \ell = 0, \\ \frac{1}{2\ell(\ell+2)} & \ell = 1,2,\dots,N, \\ \frac{1}{2N}\left[\frac{2N-\ell}{\ell(\ell+2)} + \frac{\ell-N}{(3N-\ell)(3N-\ell+2)}\right] & N+1 \le \ell \le \frac{3}{2}N-2, \\ \frac{1}{4N}\left[\frac{2(N+2)(9N-2)}{3N(9N^2-4)} + \frac{2(N-2)(9N+14)}{(3N+2)(3N+4)(3N+6)}\right] & \ell = \frac{3}{2}N-1, \end{cases}$$

$$\gamma_\ell := \begin{cases} -\frac{1}{8} & \ell = 0, \\ -\frac{1}{4(\ell+1)(\ell+2)} & \ell = 1,2,\dots,N, \\ -\frac{1}{4N}\left[\frac{2N-\ell}{(\ell+1)(\ell+2)} + \frac{\ell-N}{(3N-\ell+1)(3N-\ell)}\right] & \ell = N+1,\dots,\frac{3}{2}N-3, \\ 0 & \text{otherwise}; \end{cases}$$

$$\begin{aligned} \mathcal{B}_N &:= \frac{1}{\pi}\left(\mathcal{P}_N\Lambda_N\right)\mathcal{K}_N\left(\mathcal{P}_N\Lambda_N\right)^T\mathcal{Q}_N, \\ \mathcal{B}_N^{(1)} &:= \frac{1}{\pi}\left(\mathcal{P}_N\Lambda_N\right)\mathcal{K}_N^{(1)}\left(\mathcal{P}_N\Lambda_N\right)^T\mathcal{Q}_N, \\ \mathcal{B}_N^{(2)} &:= \frac{1}{\pi}\left(\mathcal{P}_N\Lambda_N\right)\mathcal{K}_N^{(2)}\left(\mathcal{P}_N\Lambda_N\right)^T\mathcal{Q}_N, \end{aligned}$$

with

$$\begin{aligned} \Lambda_N &:= \operatorname{diag}(\lambda_j)_{j=0,..,\frac{3}{2}N-1}, & \mathcal{Q}_N &:= \operatorname{diag}(\langle q_j,\tilde{q}_j\rangle)_{j=0,..,\frac{3}{2}N-1}, \\ \mathcal{P}_N &= \{p_{i-1}(x_j)\}_{i,j=1,\dots,\frac{3}{2}N}, & \mathcal{K}_N &= \{k(x_j,x_i)\}_{i,j=1,\dots,\frac{3}{2}N}, \\ \mathcal{K}_N^{(1)} &= \{k_1(x_j,x_i)\}_{i,j=1,\dots,\frac{3}{2}N}, & \mathcal{K}_N^{(2)} &= \{k_2(x_j,x_i)\}_{i,j=1,\dots,\frac{3}{2}N}. \end{aligned}$$

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