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RATIONAL SYMBOLIC CUBATURE RULES OVER THE FIRST QUADRANT IN A CARTESIAN PLANE*

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Abstract. In this paper we introduce a new symbolic Gaussian formula for the evaluation of an integral over the first quadrant in a Cartesian plane, in particular with respect to the weight function $w(x) = \exp(-x^T x - 1/x^T x)$, where $x = (x_1, x_2)^T \in \mathbb{R}^2_+$. It integrates exactly a class of homogeneous Laurent polynomials with coefficients in the commutative field of rational functions in two variables. It is derived using the connection between orthogonal polynomials, two-point Padé approximants, and Gaussian cubatures. We also discuss the connection to two-point Padé-type approximants in order to establish symbolic cubature formulas of interpolatory type. Numerical examples are presented to illustrate the different formulas developed in the paper.

Key words. homogeneous orthogonal polynomials, homogeneous two-point Padé, symbolic Gaussian cubature

AMS subject classifications. 41A21, 41A20, 65D32

1. Introduction.

In different fields of science and engineering including automatic and control theory [2, 3, 19, 20], several methods of computation require the integration of a function of several variables in a given domain. However, there are many integrals for which no analytical solution is provided, and therefore a numerical integration approach is required.

A symbolic (instead of numeric) Gaussian cubature formula over the unit disk, with

$$\left\{\phi_i^{(m)}(\lambda)\right\} \quad \text{and} \quad \left\{A_i^{(m)}(\lambda)\right\}$$

defining the nodes and the weights, respectively, of the form

$$\iint_{r^2+s^2\leq 1} g(\lambda_1 r + \lambda_2 s) w(r,s) dr ds \approx \sum_{i=1}^{m} A_i^{(m)}(\lambda) g(\phi_i^{(m)}(\lambda)), \qquad \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2,$$

is an approach to approximate a finite double integral. The nodes are the zeros of a homogeneous orthogonal polynomial [4]. This formula integrates exactly any homogeneous polynomial of degree $2m - 1$. In the case when the nodes are chosen freely, we then deal with symbolic cubature formulas of interpolatory type, and the degree of exactness becomes $m - 1$.

The so-called two-point Padé approximants have been introduced and studied by several authors, e.g., [1, 10, 11, 13, 14, 17] and others. These approximants are rational functions which provide a good approximation for both small and large values. They are a particular case of multipoint Padé approximants. For general results about multipoint Padé approximants, see, e.g., [15, 16, 18]. In [7], Bultheel et al. have discussed how two-point Padé approximants of a Stieltjes function are related to numerical quadrature formulas. For more details on this issue, the reader is referred to [8, 9].

The main objective of this paper is to construct Gaussian symbolic cubature rules as well as symbolic cubature rules of interpolatory type that are exact not only for bivariate homogeneous polynomials but also for specific bivariate rational functions using the theory of two-point Padé and Padé-type approximants.

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This paper is organized as follows. Section 2 reviews the concept of two-point Padé-type approximants and of two-point Padé approximants, and we derive the error formulas. Section 3 shows the connection between rational quadrature formulas and two-point Padé-type approximants. We present the connections among rational Gaussian quadratures, orthogonal polynomials, and two-point Padé approximants in Section 4. In Section 5, we introduce the properties of the two-dimensional moments. In Section 6, we treat rational symbolic cubature rules of interpolatory type over the first quadrant in a Cartesian plane. In Section 7, we provide rational symbolic Gaussian cubature formulas over the same domain. Numerical examples to illustrate our theoretical results are presented in Section 8.

2. Two-point Padé-type approximants and higher-order approximants. Let $f(z)$ be a function which admits the following expansions

$$f_0(z) = \sum_{i=0}^{+\infty} c_i z^i, \qquad f_\infty(z) = -\sum_{i=1}^{+\infty} c_{-i} z^{-i}.$$

For $l \in \mathbb{Z}$, we define the linear functional $c^{(l)}$ by

$$c^{(l)}(t^i) = c_{l+i}, \qquad i \in \mathbb{Z}.$$

We denote $c^{(0)}$ simply by c. We have

$$f_0(z) = c\left(\frac{1}{1-tz}\right), \qquad z \to 0,$$
$$f_\infty(z) = c\left(\frac{1}{1-tz}\right), \qquad z \to \infty.$$

For $0 \le k \le m$, we consider a polynomial $V_{k,m}$ of the form

$$V_{k,m}(z) = \sum_{i=0}^{m} b_{m-i}^{(k,m)} z^i.$$

Then we define the associated polynomial of degree $m-1$ by

$$W_{k,m}(z) = c^{(k-m)}\Big(\frac{z^{m-k}V_{k,m}(t) - t^{m-k}V_{k,m}(z)}{t-z}\Big).$$

We define

$$\widetilde{V}_{k,m}(z) = z^m V_{k,m}(z^{-1}),$$

and

$$\widetilde{W}_{k,m}(z) = z^{m-1} W_{k,m}(z^{-1}).$$

So, we immediately have the following theorem.

THEOREM 2.1.

$$f_0(z)\widetilde{V}_{k,m}(z) - \widetilde{W}_{k,m}(z) = z^k c^{(k-m)}\left(\frac{\mathcal{V}_{k,m}(t)}{1-tz}\right), \qquad z \to 0,$$
$$f_\infty(z)\widetilde{V}_{k,m}(z) - \widetilde{W}_{k,m}(z) = z^k c^{(k-m)}\left((tz)^{-1}\frac{\mathcal{V}_{k,m}(t)}{(tz)^{-1}-1}\right), \qquad z \to \infty.$$

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The rational approximant $\widetilde{W}_{k,m}/\widetilde{V}_{k,m}$ is called a two-point Padé-type approximant and denoted by $(k/m)_f$. When $k=m$, we deal with the standard Padé-type approximant [5]. If the polynomial $V_{k,m}$ is orthogonal with respect to the functional $c^{(k-2m)},\ 0 \le k \le m$, that is,

$$c^{(k-2m)}\left(t^i V_{k,m}(t)\right) = 0, \qquad 0 \le i \le m-1, \tag{2.1}$$

then it holds

$$f_0(z) - \frac{\widetilde{W}_{k,m}(z)}{\widetilde{V}_{k,m}(z)} = \frac{z^k}{\widetilde{V}_{k,m}(z)} c^{(k-2m)}\left(t^m \frac{V_{k,m}(t)}{1-tz}\right), \qquad z \to 0,$$

and due to the orthogonality conditions, we have

$$c^{(k-2m)}\left(\frac{t^m V_{k,m}(t)}{1-tz}\right) = c^{(k-2m)}\left(\frac{z^{-m} V_{k,m}(t)}{1-tz}\right).$$

Therefore

$$\begin{aligned} f_\infty(z) - \frac{\widetilde{W}_{k,m}(z)}{\widetilde{V}_{k,m}(z)} &= \frac{z^k}{\widetilde{V}_{k,m}(z)} c^{(k-m)}\left(\frac{V_{k,m}(t)}{1-tz}\right) \\ &= \frac{z^{k-m}}{\widetilde{V}_{k,m}(z)} c^{(k-2m)}\left((tz)^{-1}\frac{V_{k,m}(t)}{(tz)^{-1}-1}\right), \qquad z \to \infty. \end{aligned}$$

In the orthogonal case, the rational approximant $\widetilde{W}_{k,m}/\widetilde{V}_{k,m}$ is called a two-point Padé approximant and denoted by $[k/m]_f$. The case $k = 2m$ corresponds to the standard Padé approximant [5].

3. Two-point Padé-type approximants and rational quadrature of interpolatory type. The advantage of the two-point Padé-type approximants compared to the two-point Padé approximants lies in the more flexible choice of the denominator, i.e., in the free choice of the roots of the generating polynomial. Let

$$V_{k,m}(z) = \prod_{i=1}^{m}\left(z - \phi_i^{(k,m)}\right)$$

be an arbitrary polynomial. We define the linear functional c as follows:

$$c(t^i) = c_i = \int_0^{+\infty} t^i w(t)dt, \qquad i = 0, \pm1, \pm2, \dots,$$

where $w(t)$ is a positive weight function such that all the moments $\{c_i\}$ exist. Thus, the relation between the two-point Padé-type approximant and the quadrature formula of interpolatory type is given by

$$f(z) = \int_0^{+\infty} \frac{w(t)}{1-tz}dt = c^{(k-m)}\left(\frac{t^{m-k}}{1-tz}\right) \approx \sum_{i=1}^{m} \frac{A_i^{(k,m)}}{1-\phi_i^{(k,m)}z} = (k/m)_f(z),$$

where $\{A_i^{(k,m)}\}_{1\le i\le m}$ are calculated by the formula

$$A_i^{(k,m)} = c^{(k-2m)}\left(\frac{\left(\phi_i^{(k,m)}\right)^{m-k} V_{k,m}(t)}{\left(t-\phi_i^{(k,m)}\right)V'_{k,m}(\phi_i^{(k,m)})}\right) = \frac{W_{k,m}(\phi_i^{(k,m)})}{V'_{k,m}(\phi_i^{(k,m)})}.$$

Therefore, we define the rational quadrature formula of interpolatory type by

$$\int_0^{+\infty} h(t)w(t)dt \approx \sum_{i=1}^{m} A_i^{(k,m)} h(\phi_i^{(k,m)}), \qquad \text{for all integrable functions } h.$$

Let us complete this section by demonstrating the following theorem.

THEOREM 3.1. *Let $\phi_1^{(k,m)}, \ldots, \phi_m^{(k,m)}$ be the zeros of $V_{k,m}$, and let $R_{k,m}$ be a function of the form*

$$R_{k,m}(t) = \sum_{i=k-m}^{k-1} a_i t^i.$$

Then we have

$$\int_0^{+\infty} R_{k,m}(t)w(t)dt = \sum_{i=1}^{m} A_i^{(k,m)} R_{k,m}(\phi_i^{(k,m)}).$$

Proof. The function $R_{k,m}(t)$ can be written as

$$R_{k,m}(t) = t^{k-m}P(t),$$

where $P(t)$ is a polynomial of degree $m-1$. We have

$$c(R_{k,m}(t)) = c^{(k-m)}(P(t)).$$

The polynomial $P(t)$ can be written as a Lagrange interpolation polynomial of degree $m-1$ at the points $\phi_1^{(k,m)}, \ldots, \phi_m^{(k,m)}$,

$$P(t) = \sum_{i=1}^{m} \frac{V_{k,m}(t)}{(t-\phi_i^{(k,m)})V'_{k,m}(\phi_i^{(k,m)})} P(\phi_i^{(k,m)}).$$

Applying the functional $c^{(k-m)}$ to both sides of this equation gives the result

$$\int_0^{+\infty} R_{k,m}(t)w(t)dt = c^{(k-m)}(P(t)) = \sum_{i=1}^{m} A_i^{(k,m)} R_{k,m}(\phi_i^{(k,m)}). \qquad \square$$

4. Two-point Padé approximants and rational Gaussian quadrature. For $n \geq 0$ and $l \in \mathbb{Z}$, the Hankel determinant $H_n^{(l)}$ is defined by

$$H_n^{(l)} = \begin{vmatrix} c_l & c_{l+1} & \cdots & c_{l+n-1} \\ c_{l+1} & c_{l+2} & \cdots & c_{l+n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{l+n-1} & c_{l+n} & \cdots & c_{l+2n-2} \end{vmatrix}, \qquad H_0^{(l)} = 1.$$

For a fixed l, let $\{V_{l+2n,n}\}$ be the family of orthogonal polynomials with respect to the linear functional $c^{(l)}$. To guarantee the existence of this family we assume $H_n^{(l)} \neq 0$ for any $n > 0$. The linear functional $c^{(l)}$ is said to be positive definite if

$$H_n^{(l)} > 0, \qquad \forall n > 0.$$

THEOREM 4.1. *If $c^{(l)}$ is positive definite, then the zeros of the orthogonal polynomial $V_{l+2n,n}$ are real and distinct.*

Proof. It is similar to the proof of Theorem 2.15 a) in [5]. □

Let k and m be fixed with $0 \leq k \leq 2m$. We now assume that $c^{(k-2m)}$ is positive definite, so the m zeros of the orthogonal polynomial $V_{k,m}(z)$, $\phi_1^{(k,m)}, \dots, \phi_m^{(k,m)}$, are distinct.

THEOREM 4.2.

$$c^{(k-2m)}\left(p_{k,m}(t,z)\right) = \sum_{i=1}^{m} \frac{A_i^{(k,m)}}{1-\phi_i^{(k,m)} z} = [k/m]_f(z),$$

where $p_{k,m}(t,z)$ is the Lagrange interpolation polynomial of degree $m-1$ in t of the function $\frac{t^{2m-k}}{1-tz}$ at the points $\phi_1^{(k,m)}, \dots, \phi_m^{(k,m)}$, where z is considered a parameter.

Proof. This Lagrange interpolation polynomial is given by

$$p_{k,m}(t,z) = \sum_{i=1}^{m} \frac{V_{k,m}(t)}{(t-\phi_i^{(k,m)})V'_{k,m}(\phi_i^{(k,m)})} \frac{\left(\phi_i^{(k,m)}\right)^{2m-k}}{1-\phi_i^{(k,m)} z}, \tag{4.1}$$

and using the orthogonality conditions (2.1), the polynomial $W_{k,m}$ can be written as

$$W_{k,m}(z) = c^{(k-2m)}\left(\frac{z^{2m-k}V_{k,m}(t) - t^{2m-k}V_{k,m}(z)}{t-z}\right).$$

Applying the functional $c^{(k-2m)}$ to (4.1), we get

$$c^{(k-2m)}\left(p_{k,m}(t,z)\right) = \sum_{i=1}^{m} \frac{A_i^{(k,m)}}{1-\phi_i^{(k,m)} z},$$

where

$$A_i^{(k,m)} = c^{(k-2m)}\left(\frac{\left(\phi_i^{(k,m)}\right)^{2m-k} V_{k,m}(t)}{\left(t-\phi_i^{(k,m)}\right) V'_{k,m}(\phi_i^{(k,m)})}\right) = \frac{W_{k,m}(\phi_i^{(k,m)})}{V'_{k,m}(\phi_i^{(k,m)})}.$$

It is obvious that

$$c^{(k-2m)}\left(p_{k,m}(t,z)\right) = \frac{1}{z}\frac{W_{k,m}(z^{-1})}{V_{k,m}(z^{-1})} = [k/m]_f(z). \tag{4.2}$$ □

Thus, from (4.2), a quadrature formula is given by

$$f(z) = \int_0^{+\infty} \frac{w(t)}{1-tz}dt = c^{(k-2m)}\left(\frac{t^{2m-k}}{1-tz}\right) \approx \sum_{i=1}^{m} \frac{A_i^{(k,m)}}{1-\phi_i^{(k,m)} z} = [k/m]_f(z).$$

This establishes a connection between two-point Padé approximants and numerical quadratures in the similar way as for one-point Padé approximants [5, 6]. We can therefore propose the following rational Gaussian quadrature

$$\int_0^{+\infty} h(t)w(t)dt \approx \sum_{i=1}^{m} A_i^{(k,m)} h(\phi_i^{(k,m)}).$$

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THEOREM 4.3. *If $c^{(k-2m)}$ is positive definite, then the zeros of the orthogonal polynomial $V_{k,m}$ belong to $[0,+\infty]$.*

Proof. See the proof of Theorem 2.16 in [5]. ∎

THEOREM 4.4. *Let $\phi_1^{(k,m)},\ldots,\phi_m^{(k,m)}$ be the distinct zeros of $V_{k,m}$, and let $R_{k,m}$ be a rational function of the form*

$$R_{k,m}(t)=\sum_{i=k-2m}^{k-1} a_i t^i.$$

Then we have

$$\int_0^{+\infty} R_{k,m}(t)w(t)dt=\sum_{i=1}^{m} A_i^{(k,m)} R_{k,m}(\phi_i^{(k,m)}).$$

Proof. The function $R_{k,m}(t)$ can be written as

$$R_{k,m}(t)=t^{k-2m}P(t),$$

where $P(t)$ is a polynomial of degree $2m-1$ that can be written as

$$P(t)=Q(t)V_{k,m}(t)+S(t).$$

Here $V_{k,m}$ is an orthogonal polynomial with respect to the linear functional $c^{(k-2m)}$, and the quotient Q and the remainder S are polynomials of degree less than $m-1$. Thus,

$$c(R_{k,m}(t))=c^{(k-2m)}(P(t))=c^{(k-2m)}(Q(t)V_{k,m}(t)+S(t))=c^{(k-2m)}(S(t)).$$

The polynomial $S(t)$ is equal to its Lagrange interpolant of degree $m-1$ for the interpolation points $\phi_1^{(k,m)},\ldots,\phi_m^{(k,m)}$ and is given by

$$S(t)=\sum_{i=1}^{m} \frac{V_{k,m}(t)}{(t-\phi_i^{(k,m)})V'_{k,m}(\phi_i^{(k,m)})} S(\phi_i^{(k,m)}).$$

Applying the functional $c^{(k-2m)}$ on both sides of this equation, we obtain

$$\begin{aligned}
c^{(k-2m)}(S(t)) &= c^{(k-2m)}\left(\sum_{i=1}^{m} \frac{V_{k,m}(t)}{(t-\phi_i^{(k,m)})V'_{k,m}(\phi_i^{(k,m)})} S(\phi_i^{(k,m)})\right)\\
&= c^{(k-2m)}\left(\sum_{i=1}^{m} \frac{V_{k,m}(t)}{(t-\phi_i^{(k,m)})V'_{k,m}(\phi_i^{(k,m)})} \left(\phi_i^{(k,m)}\right)^{2m-k} R_{k,m}(\phi_i^{(k,m)})\right)\\
&= \sum_{i=1}^{m} A_i^{(k,m)} R_{k,m}(\phi_i^{(k,m)}).
\end{aligned}$$

Therefore,

$$\int_0^{+\infty} R_{k,m}(t)w(t)dt=c(R_{k,m}(t))=c^{(k-2m)}(S(t))=\sum_{i=1}^{m} A_i^{(k,m)} R_{k,m}(\phi_i^{(k,m)}). \qquad ∎$$

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5. Properties of two-dimensional moments. In this section we establish a recurrence relation that allows computing recursively the two-dimensional moments over the first quadrant. This enables us to approximate a finite integral of the form

$$\iint_{\Omega} g(r\lambda_1 + s\lambda_2) w(\|(r,s)\|_2) drds, \qquad \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2,$$

where

- $\Omega = \{(r,s) \in \mathbb{R}^2 \mid r, s > 0\}$;
- w is a positive weight function on Ω;
- $\|(r,s)\|_2 = \sqrt{r^2 + s^2}$;
- g is an integrable function on Ω.

The above integral can be written as

$$\begin{aligned}\iint_{\Omega} g(r\lambda_1 + s\lambda_2) w(\|(r,s)\|_2) drds &= \iint_{\Omega} g\left(\mu\left(r\cos(\alpha) + s\sin(\alpha)\right)\right) w(\|(r,s)\|_2) drds \\ &= \iint_{\Omega} h\left(r\cos(\alpha) + s\sin(\alpha)\right) w(\|(r,s)\|_2) drds,\end{aligned}$$

where $\lambda_1 = \mu\cos(\alpha)$, $\lambda_2 = \mu\sin(\alpha)$, $\alpha \in \left[0, \frac{\pi}{2}\right]$, $\mu > 0$, and $h(z) = g(\mu z)$.

We define the moments $\{d_i(\alpha)\}_{i\in\mathbb{Z}}$ by

$$d_i(\alpha) = \iint_{\Omega} \left(r\cos(\alpha) + s\sin(\alpha)\right)^i w(\|(r,s)\|_2) drds, \qquad i = 0, \pm 1, \pm 2, \ldots,$$

and establish the following propositions.

PROPOSITION 5.1.

$$d_i(\alpha) = \left(\int_0^{\frac{\pi}{2}} \left(\cos(\theta - \alpha)\right)^i d\theta\right)\left(\int_0^{+\infty} z^{i+1} w(z) dz\right).$$

Proof. The change of variables $(r,s) = (z\cos(\theta), z\sin(\theta))$, with $(z,\theta) \in \Delta$ and $\Delta = [0, +\infty] \times \left[0, \frac{\pi}{2}\right]$, gives

$$\begin{aligned} d_i(\alpha) &= \iint_{\Omega} \left(r\cos(\alpha) + s\sin(\alpha)\right)^i w(\|(r,s)\|_2) drds \\ &= \iint_{\Delta} \Big(\left(\cos(\alpha)\cos(\theta)z + \sin(\alpha)\sin(\theta)z\right)^i w(z)\Big) z d\theta dz \\ &= \iint_{\Delta} \cos(\theta - \alpha)^i z^{i+1} w(z) d\theta dz. \end{aligned}$$

The result is obtained due to the fact that the variables can be separated. ∎

The following proposition will be useful.

PROPOSITION 5.2. *Let us define $J_i(\alpha)$ as*

$$J_i(\alpha) = \int_0^{\frac{\pi}{2}} \left(\cos(\theta - \alpha)\right)^i d\theta.$$

The integrals $\{J_i(\alpha)\}_{i\in\mathbb{Z}}$ satisfy the recurrence relation

$$iJ_i(\alpha) = \left(\cos(\alpha)^{i-1}\sin(\alpha) + \cos(\alpha)\sin(\alpha)^{i-1} + (i-1)J_{i-2}(\alpha)\right), \quad i \geq 2.$$

Proof. The result is immediately obtained by using integration by parts. ∎

Thus, according to the latter proposition, the integrals $\{J_i(\alpha)\}_{i\geq 0}$ are obtained inductively by

$$J_i(\alpha) = \frac{1}{i}\left(\cos(\alpha)^{i-1}\sin(\alpha) + \cos(\alpha)\sin(\alpha)^{i-1} + (i-1)J_{i-2}(\alpha)\right),$$

with

$$J_0(\alpha) = \frac{\pi}{2}, \qquad J_1(\alpha) = \cos(\alpha) + \sin(\alpha),$$

and the integrals $\{J_i(\alpha)\}_{i<0}$ are determined using the relation

$$J_{-2-i}(\alpha) = \frac{1}{1+i}\left(\frac{\sin(\alpha)}{\cos(\alpha)^{1+i}} + \frac{\cos(\alpha)}{\sin(\alpha)^{1+i}} + iJ_{-i}(\alpha)\right),$$

with

$$J_{-1}(\alpha) = \ln\left(\frac{1+\cos(\alpha)}{\sin(\alpha)}\right) + \ln\left(\frac{\cos(\alpha)}{1-\sin(\alpha)}\right).$$

PROPOSITION 5.3. *The moments* $d_i\left(\frac{\pi}{2}-\alpha\right)$ *and* $d_i(\alpha)$ *are the same.*

Proof. We have

$$\begin{aligned} d_i\left(\frac{\pi}{2}-\alpha\right) &= \iint_\Omega \left(r\cos\left(\frac{\pi}{2}-\alpha\right) + s\sin\left(\frac{\pi}{2}-\alpha\right)\right)^i w(\|(r,s)\|_2)drds \\ &= \iint_\Omega (r\sin(\alpha) + s\cos(\alpha))^i w(\|(r,s)\|_2)drds \\ &= d_i(\alpha). \end{aligned}$$ ∎

6. Rational cubature rules of interpolatory type. The aim of this section is to establish symbolic cubature formulas of interpolatory type using two-point Padé-type approximants. Let us now associate to the sequence $\{d_i(\alpha)\}_{i\in\mathbb{Z}}$ a linear functional d defined on the space of polynomials in the variable t with coefficients from $\mathbb{R}(\cos(\alpha),\sin(\alpha))$, the commutative field of rational functions in $\cos(\alpha)$ and $\sin(\alpha)$ with real coefficients, by

$$d(t^i) = d_i(\alpha), \qquad i\in\mathbb{Z}.$$

We define the bivariate Stieltjes function as follows [12]:

$$\mathfrak{I}(x,y) = \iint_\Omega \frac{w(\|(r,s)\|_2)}{1-(rx+sy)}drds.$$

By defining

$$\mathcal{I}_\alpha(z) = \mathfrak{I}(z\cos(\alpha), z\sin(\alpha)) = \iint_\Omega \frac{w(\|(r,s)\|_2)}{1-(r\cos(\alpha)+s\sin(\alpha))z}drds,$$

we have

$$\begin{aligned} \mathcal{I}_\alpha(z) &= \sum_{i=0}^{+\infty} d_i(\alpha)z^i = d\left(\frac{1}{1-tz}\right), && z\to 0, \\ \mathcal{I}_\alpha(z) &= -\sum_{i=-\infty}^{-1} d_i(\alpha)z^i = -d\left(\frac{(tz)^{-1}}{1-(tz)^{-1}}\right), && z\to\infty. \end{aligned}$$

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Fix an arbitrary $0 \leq k \leq m$. We define a polynomial $\mathcal{V}_{k,m}$ of the form

$$\mathcal{V}_{k,m}(\alpha, z) = \prod_{i=1}^{m} \left(z - \phi_i^{(k,m)}(\alpha)\right),$$

where the zeros $\{\phi_i^{(k,m)}(\alpha)\}_{1 \leq i \leq m}$ are chosen arbitrarily. It will be assumed that

$$\alpha \notin \bigcup_{1 \leq i < j \leq m} E_{i,j}, \quad \text{where} \quad E_{i,j} = \left\{\alpha \in \left[0, \frac{\pi}{2}\right] : \ \phi_i^{(k,m)}(\alpha) = \phi_j^{(k,m)}(\alpha)\right\}.$$

Using the two-point Padé-type approximant, we propose the following approximation

$$\begin{aligned}\mathcal{I}_\alpha(z) &= \iint_\Omega \frac{1}{1 - (r\cos(\alpha) + s\sin(\alpha))z} w(\|(r,s)\|_2) dr ds \\ &\approx \sum_{i=1}^{m} \frac{A_i^{(k,m)}(\alpha)}{1 - \phi_i^{(k,m)}(\alpha) z} = (k/m)_{\mathcal{I}_\alpha}(z),\end{aligned}$$

where $\{A_i^{(k,m)}(\alpha)\}$ are computed by

$$A_i^{(k,m)}(\alpha) = \frac{\mathcal{W}_{k,m}(\alpha, \phi_i^{(k,m)}(\alpha))}{\mathcal{V}'_{k,m}(\alpha, \phi_i^{(k,m)}(\alpha))}$$

and $\mathcal{W}_{k,m}(\alpha, z)$ is defined by

$$\mathcal{W}_{k,m}(\alpha, z) = d^{(k-m)}\Big(\frac{z^{m-k}\mathcal{V}_{k,m}(\alpha, t) - t^{m-k}\mathcal{V}_{k,m}(\alpha, z)}{t - z}\Big).$$

We can then define rational symbolic cubature rules of interpolatory type as

$$\iint_\Omega h(r\cos(\alpha) + s\sin(\alpha)) w(\|(r,s)\|_2) dr ds \approx \sum_{i=1}^{m} A_i^{(k,m)}(\alpha) h(\phi_i^{(k,m)}(\alpha)),$$

which integrate exactly any function of the form

$$\sum_{i=k-m}^{k-1} a_i(\alpha)(r\cos(\alpha) + s\sin(\alpha))^i, \qquad a_i(\alpha) \in \mathbb{R}(\cos(\alpha), \sin(\alpha)).$$

7. Rational symbolic Gaussian cubature rules. The purpose of this section is to construct symbolic cubature rules using two-point Padé approximants. In other words, we use the zeros of orthogonal polynomials related to two-point Padé approximants depending on the parameter α to construct symbolic cubature rules.

Let k and m be fixed with $0 \leq k \leq 2m$. We consider a polynomial in z of degree m and whose coefficients belong to $\mathbb{R}(\cos(\alpha), \sin(\alpha))$, denoted by $\mathcal{V}_{k,m}(\alpha, z)$, of the form

$$\mathcal{V}_{k,m}(\alpha, z) = \sum_{i=0}^{m} B_{m-i}^{(k,m)}(\alpha) z^i$$

that satisfies the orthogonality conditions

$$d^{(k-2m)}\left(t^i \mathcal{V}_{k,m}(\alpha, t)\right) \equiv 0, \qquad 0 \leq i \leq m-1, \tag{7.1}$$

where the linear functional $d^{(l)}$ is defined by $d^{(l)}(t^i) = d_{l+i}(\alpha)$. We remind that its associated polynomial is defined by

$$\mathcal{W}_{k,m}(\alpha, z) = d^{(k-2m)}\Big(\frac{z^{2m-k}\mathcal{V}_{k,m}(\alpha,t) - t^{2m-k}\mathcal{V}_{k,m}(\alpha,z)}{t-z}\Big).$$

If the linear functional $d^{(k-2m)}$ is defined, which means that the Hankel determinant

$$\mathcal{H}_m^{(k-2m)}(\alpha) = \begin{vmatrix} d_{k-2m}(\alpha) & d_{k-2m+1}(\alpha) & \dots & d_{k-m-1}(\alpha) \\ d_{k-2m+1}(\alpha) & d_{k-2m+2}(\alpha) & \dots & d_{k-m}(\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ d_{k-m-1}(\alpha) & d_{k-m}(\alpha) & \dots & d_{k-2}(\alpha) \end{vmatrix}, \quad \mathcal{H}_0^{(k-2m)}(\alpha) = 1,$$

is not identically null for any $\alpha \in \Big[0, \dfrac{\pi}{2}\Big]$, then a solution of (7.1) is given by

$$\mathcal{V}_{k,m}(\alpha, z) = \begin{vmatrix} d_{k-2m}(\alpha) & d_{k-2m+1}(\alpha) & \cdots & d_{k-m}(\alpha) \\ d_{k-2m+1}(\alpha) & d_{k-2m+2}(\alpha) & \cdots & d_{k-m+1}(\alpha) \\ \vdots & & & \vdots \\ d_{k-m-1}(\alpha) & d_{k-m}(\alpha) & \cdots & d_{k-1}(\alpha) \\ 1 & z & \cdots & z^m \end{vmatrix}, \quad \mathcal{V}_{k,0}(\alpha, z) = 1.$$

In the following, we assume that $d^{(k-2m)}$ is positive definite, which means $\mathcal{H}_n^{(k-2m)}(\alpha) > 0$, for all $n = 1, 2, \dots$ and all $\alpha \in \Big[0, \dfrac{\pi}{2}\Big]$. In this case, for each α, the m zeros $\{\phi_i^{(k,m)}(\alpha)\}_{1 \le i \le m}$ of $\mathcal{V}_{k,m}(\alpha, z)$ are real and distinct. Thus, the connection between the two-point Padé approximant and the symbolic Gaussian cubature formula is given by

$$\begin{aligned}\mathcal{I}_\alpha(z) &= \iint_\Omega \frac{w(\|(r,s)\|_2)}{1-(r\cos(\alpha)+s\sin(\alpha))z} drds = d^{(k-2m)}\left(\frac{t^{2m-k}}{1-tz}\right) \\ &\approx \sum_{i=1}^m \frac{A_i^{(k,m)}(\alpha)}{1-\phi_i^{(k,m)}(\alpha)z} = [k/m]_{\mathcal{I}_\alpha}(z),\end{aligned}$$

where $A_i^{(k,m)}(\alpha)$ are defined by

$$A_i^{(k,m)}(\alpha) = \frac{\mathcal{W}_{k,m}(\alpha, \phi_i^{(k,m)}(\alpha))}{\mathcal{V}'_{k,m}(\alpha, \phi_i^{(k,m)}(\alpha))}.$$

This can also be obtained by solving the linear system of equations

$$\sum_{i=1}^m A_i^{(k,m)}(\alpha)(\phi_i^{(k,m)}(\alpha))^{k-2m+j} = d_{k-2m+j}(\alpha), \qquad \text{for } j = 0, 1, \dots, 2m-1. \tag{7.2}$$

The matrix of the linear system (7.2) has full rank since we assume that $V_{k,m}(\alpha, z)$ has m real and distinct non-zero roots. Moreover, with the orthogonality conditions (7.1), we have

$$\begin{vmatrix} (\phi_1^{(k,m)}(\alpha))^{k-2m} & (\phi_2^{(k,m)}(\alpha))^{k-2m} & \cdots & (\phi_m^{(k,m)}(\alpha))^{k-2m} & d_{k-2m}(\alpha) \\ \vdots & \vdots & & \vdots & \vdots \\ (\phi_1^{(k,m)}(\alpha))^{k-m-1} & (\phi_2^{(k,m)}(\alpha))^{k-m-1} & \cdots & (\phi_m^{(k,m)}(\alpha))^{k-m-1} & d_{k-m-1}(\alpha) \\ (\phi_1^{(k,m)}(\alpha))^{k-m+l} & (\phi_2^{(k,m)}(\alpha))^{k-m+l} & \cdots & (\phi_m^{(k,m)}(\alpha))^{k-m+l} & d_{k-m+l}(\alpha) \end{vmatrix} = 0,$$

for $l = 0, 1, \ldots, m-1$.

It follows that the system (7.2) has a unique solution. We can therefore propose the following cubature formula

$$\iint_\Omega h(r\cos(\alpha) + s\sin(\alpha)) w(\|(r,s)\|_2) drds \approx \sum_{i=1}^{m} A_i^{(k,m)}(\alpha) h(\phi_i^{(k,m)}(\alpha)),$$

called the rational symbolic Gaussian cubature, which integrates exactly any function of the form

$$\sum_{i=k-2m}^{k-1} a_i(\alpha)(r\cos(\alpha) + s\sin(\alpha))^i, \qquad a_i(\alpha) \in \mathbb{R}(\cos(\alpha), \sin(\alpha)).$$

PROPOSITION 7.1. *The nodes and weights for* $\frac{\pi}{2} - \alpha$ *and* α *are the same.*

Proof. Since the moments $d_i\left(\frac{\pi}{2} - \alpha\right)$ and $d_i(\alpha)$ are equal, we have

$$\mathcal{V}_{k,m}\left(\frac{\pi}{2} - \alpha, z\right) = \mathcal{V}_{k,m}(\alpha, z).$$

Thus,

$$\phi_i^{(k,m)}\left(\frac{\pi}{2} - \alpha\right) = \phi_i^{(k,m)}(\alpha),$$

and consequently

$$A_i^{(k,m)}\left(\frac{\pi}{2} - \alpha\right) = \frac{\mathcal{W}_{k,m}\left(\frac{\pi}{2} - \alpha, \phi_i^{(k,m)}\left(\frac{\pi}{2} - \alpha\right)\right)}{\mathcal{V}'_{k,m}\left(\frac{\pi}{2} - \alpha, \phi_i^{(k,m)}\left(\frac{\pi}{2} - \alpha\right)\right)} = \frac{\mathcal{W}_{k,m}(\alpha, \phi_i^{(k,m)}(\alpha))}{\mathcal{V}'_{k,m}(\alpha, \phi_i^{(k,m)}(\alpha))} = A_i^{(k,m)}(\alpha). \qquad \square$$

8. Numerical examples. This section is devoted to establish numerical examples to illustrate the theoretical results obtained in the above sections. All computations were carried out using Maple on a computer with an Intel Core i7 processor and with about 16 significant digits. We define the moments $\{d_i(\alpha)\}_{i\in\mathbb{Z}}$ by

$$d_i(\alpha) = \iint_\Omega (r\cos(\alpha) + s\sin(\alpha))^i w(\|(r,s)\|_2) drds,$$

with the weight function given by

$$w(\|(r,s)\|_2) = \exp\left(-(r^2+s^2) - \frac{1}{(r^2+s^2)}\right).$$

The coefficients $d_i(\alpha)$, $i \in \mathbb{Z}$, are given by

$$\begin{aligned}
&\vdots \\
d_{-4}(\alpha) &= \frac{1}{3}\left(\frac{\cos(\alpha)}{\sin^3(\alpha)} + 2\left(\frac{\cos(\alpha)}{\sin(\alpha)} + \frac{\sin(\alpha)}{\cos(\alpha)}\right)\right)\mathrm{BesselK}(1,2),\\
d_{-3}(\alpha) &= \frac{1}{4}\left(\frac{\cos(\alpha)}{\sin^2(\alpha)} + \frac{\sin(\alpha)}{\cos^2(\alpha)} + \ln\left(\frac{1+\cos(\alpha)}{\sin(\alpha)}\right) - \ln\left(\frac{1-\sin(\alpha)}{\cos(\alpha)}\right)\right)\sqrt{\pi}e^{-2},\\
d_{-2}(\alpha) &= \frac{1}{4}\left(\frac{\cos(\alpha)}{\sin(\alpha)} + \frac{\sin(\alpha)}{\cos(\alpha)}\right)\mathrm{BesselK}(0,2),\\
d_{-1}(\alpha) &= \frac{1}{2}\left(\ln\left(\frac{1+\cos(\alpha)}{\sin(\alpha)}\right) - \ln\left(\frac{1-\sin(\alpha)}{\cos(\alpha)}\right)\right)\sqrt{\pi}e^{-2},\\
d_0(\alpha) &= \frac{\pi}{2}\,\mathrm{BesselK}(1,2),\\
d_1(\alpha) &= \frac{3}{4}\left(\cos(\alpha)+\sin(\alpha)\right)\sqrt{\pi}e^{-2},\\
d_2(\alpha) &= \left(\frac{\pi}{4} + \cos(\alpha) + \sin(\alpha)\right)\mathrm{BesselK}(2,2),\\
d_3(\alpha) &= \frac{13}{24}\left(\cos^2(\alpha)\sin(\alpha) + \cos(\alpha)\sin^2(\alpha) + 2(\cos(\alpha)+\sin(\alpha)\right)\sqrt{\pi}e^{-2},\\
d_4(\alpha) &= \frac{1}{4}\left(\cos^3(\alpha)\sin(\alpha) + \cos(\alpha)\sin^3(\alpha) + 3\cos(\alpha)\sin(\alpha) + \frac{3}{4}\pi\right)\mathrm{BesselK}(3,2),\\
&\vdots
\end{aligned}$$

EXAMPLE 8.1. Let us take $k = 2$ and $m = 4$, and let the nodes $\{\phi_i^{(2,4)}(\alpha)\}$ be taken as

$$\phi_1^{(2,4)}(\alpha) = \frac{5}{2}\cos(\alpha), \qquad \phi_2^{(2,4)}(\alpha) = \frac{5}{3}\sin(\alpha),$$
$$\phi_3^{(2,4)}(\alpha) = \cos(\alpha)\sin(\alpha), \qquad \phi_4^{(2,4)}(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)}.$$

Hence, the sets $E_{i,j}$ are

$$E_{1,2} = \left\{\arctan\left(\frac{3}{2}\right)\right\}, \qquad E_{1,3} = \emptyset, \qquad E_{1,4} = \left\{\arcsin\left(\frac{2}{5}\right)\right\},$$
$$E_{2,3} = \emptyset, \qquad E_{2,4} = \left\{\arctan\left(\frac{\sqrt{-18+6\sqrt{109}}}{-3+\sqrt{109}}\right)\right\}, \qquad E_{3,4} = \emptyset.$$

In Tables 8.1, 8.2, and 8.3 below, we give some numerical results for the rational cubature formula of interpolatory type for different values α.

Consider the test function shown in Figure 8.1

$$h(t) = \exp\left(\frac{-t}{11t^2 + t + 50}\right).$$

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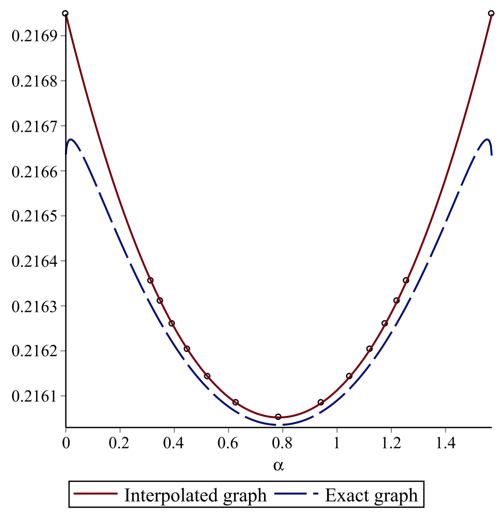

FIG. 8.1. *The exact solution at some discrete points of the integral are marked by circles.*

TABLE 8.1
Nodes of the rational cubature of interpolatory type formula for $k = 2$ and $m = 4$.

α	$\phi_1^{(2,4)}(\alpha)$	$\phi_2^{(2,4)}(\alpha)$	$\phi_3^{(2,4)}(\alpha)$	$\phi_4^{(2,4)}(\alpha)$
$2\pi/5$	0.7725424859373692	1.585094193825256	0.2938926261462368	0.3249196962329066
$7\pi/18$	0.8550503583141712	1.566154367976514	0.3213938048432695	0.3639702342662021
$3\pi/8$	0.9567085809127255	1.539799220852145	0.3535533905932741	0.4142135623730956
$5\pi/14$	1.084709347793895	1.501614779837365	0.3909157412340149	0.4815746188075286
$\pi/3$	1.250000000000000	1.443375672974064	0.4330127018922192	0.5773502691896256
$3\pi/10$	1.469463130731183	1.348361657291579	0.4755282581475768	0.7265425280053610
$\pi/4$	1.767766952966369	1.178511301977579	0.5000000000000000	1.
$\pi/5$	2.022542485937369	0.9796420871541220	0.4755282581475768	1.376381920471174
$\pi/6$	2.165063509461096	0.8333333333333333	0.4330127018922192	1.732050807568877
$\pi/7$	2.252422169756048	0.7231395651959305	0.3909157412340150	2.076521396572336
$\pi/8$	2.309698831278217	0.6378057206084830	0.3535533905932737	2.414213562373096
$\pi/9$	2.349231551964771	0.5700335722094479	0.3213938048432696	2.747477419454623
$\pi/10$	2.377641290737884	0.5150283239582458	0.2938926261462366	3.077683537175254

TABLE 8.2
Weights of the rational cubature of interpolatory type formula for $k = 2$ and $m = 4$.

α	$A_1^{(2,4)}(\alpha)$	$A_2^{(2,4)}(\alpha)$	$A_3^{(2,4)}(\alpha)$	$A_4^{(2,4)}(\alpha)$
$2\pi/5$	0.1269481628072058	0.07822335307097074	0.002883704888345295	0.01164559263480096
$7\pi/18$	0.1190703628569230	0.07626229373650935	-0.01103181355182133	0.03539997035970996
$3\pi/8$	0.1101589219926027	0.07398006199533127	-0.01689340114890447	0.05245523056229328
$5\pi/14$	0.09979539230698447	0.07132239598415320	-0.01549501584272252	0.06407804095290844
$\pi/3$	0.08611908515674661	0.06907954321955271	-0.008037564838628852	0.07253974986364871
$3\pi/10$	0.08514721873453091	0.04703952358592624	0.003329129522834805	0.08418494155804301
$\pi/4$	0.06092151833022646	-0.02878877470747727	0.01381756921473507	0.1737505005638428
$\pi/5$	0.03193229501456138	0.1478033561557251	0.01453300899619359	0.02543215323483874
$\pi/6$	-0.02656157338913685	0.1223710774821061	0.01023632933532651	0.1136549799730239
$\pi/7$	-0.2753272573156168	0.1171685881536023	0.005375191166698295	0.3724842913966512
$\pi/8$	0.8527878150884699	0.1148636519549771	0.001053898042592330	-0.7490045516847124
$\pi/9$	0.3335197073494495	0.1131394647093831	-0.002532193220097974	-0.2244261654374153
$\pi/10$	0.2556099202312826	0.1114752753575796	-0.005437360909425199	-0.1419470212781146

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TABLE 8.3
Relative errors of the rational symbolic cubature formula of interpolatory type for $k = 2$ and $m = 4$.

α	Relative errors
$2\pi/5$	$0.3337141745532643 \times 10^{-5}$
$7\pi/18$	$0.4773277080444892 \times 10^{-5}$
$3\pi/8$	$0.1123044535551117 \times 10^{-4}$
$5\pi/14$	$0.1484200472971130 \times 10^{-4}$
$\pi/3$	$0.1406198123521202 \times 10^{-4}$
$3\pi/10$	$0.7470816212517977 \times 10^{-5}$
$\pi/4$	$0.3784664333070251 \times 10^{-5}$
$\pi/5$	$0.6138534069087626 \times 10^{-5}$
$\pi/6$	$0.3963279913478425 \times 10^{-5}$
$\pi/7$	$0.2138585068475668 \times 10^{-4}$
$\pi/8$	$0.4268655890507296 \times 10^{-4}$
$\pi/9$	$0.6563048632875481 \times 10^{-4}$
$\pi/10$	$0.8884502411056956 \times 10^{-4}$

EXAMPLE 8.2. Choose $m = 2$. The orthogonal polynomials $\mathcal{V}_{k,2}(\alpha, z)$, $k = 0, 1, 2, 3, 4$, are

$$
\begin{aligned}
&\mathcal{V}_{0,2}(\alpha, z)\\
&\quad = -\frac{1}{48(\cos^4(\alpha)\sin^4(\alpha))}\Big(3\pi e^{-4}\big(2\cos^3(\alpha)\sin^3(\alpha) - 3\cos^2(\alpha)\sin^2(\alpha) + 1\big)\\
&\qquad\qquad + 3\pi e^{-4}(a-b)\cos^2(\alpha)\sin^2(\alpha)\times\\
&\qquad\qquad \Big((a-b)\cos^2(\alpha)\sin^2(\alpha) + 2(\cos^3(\alpha)+\sin^3(\alpha))\Big)\Big)z^2\\
&\qquad + \frac{1}{12(\cos^3(\alpha)\sin^3(\alpha))}\Big(3(a-b)\,\mathrm{BesselK}(0,2)\cos^2(\alpha)\sin^2(\alpha)\\
&\qquad\qquad + 3\,\mathrm{BesselK}(0,2)(\cos^3(\alpha)+\sin^3(\alpha))\\
&\qquad\qquad - 2(a-b)\,\mathrm{BesselK}(1,2)\Big)z\\
&\qquad - \frac{1}{8(\cos^2(\alpha)\sin^2(\alpha))}\Big(-\pi e^{-4}(a-b)^2\cos^2(\alpha)\sin^2(\alpha)\\
&\qquad\qquad + \pi e^{-4}(a-b)(\cos^2(\alpha)\sin(\alpha) + \cos(\alpha)\sin^2(\alpha))\\
&\qquad\qquad - (\cos(\alpha)+\sin(\alpha)) + 8\,\mathrm{BesselK}(0,2)^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{V}_{1,2}(\alpha, z)\\
&\quad = -\frac{1}{8(\cos^2(\alpha)\sin^2(\alpha))}\Big(8\,\mathrm{BesselK}(0,2)^2 - \pi e^{-4}(a-b)^2\cos^2(\alpha)\sin^2(\alpha)\\
&\qquad\qquad - \pi e^{-4}(a-b)(\cos^3(\alpha)+\sin^3(\alpha))\Big)z^2\\
&\qquad + \frac{\sqrt{\pi}e^{-2}}{8(\cos^2(\alpha)\sin^2(\alpha))}\Big(-\mathrm{BesselK}(1,2) + \pi(\cos^3(\alpha)+\sin^3(\alpha))
\end{aligned}
$$

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$$
\begin{aligned}
&- \pi(a-b)\,\mathrm{BesselK}(1,2)\cos^2(\alpha)\sin^2(\alpha)\\
&+ 4(a-b)\,\mathrm{BesselK}(0,2)\cos(\alpha)\sin(\alpha)\Big)z\\
&- \frac{1}{4(\cos(\alpha)\sin(\alpha))}\Big(-2\,\mathrm{BesselK}(1,2)\,\mathrm{BesselK}(0,2)\\
&+ \pi e^{-4}(a-b)^2\cos(\alpha)\sin(\alpha)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{V}_{2,2}(\alpha,z)\\
&= -\frac{\pi}{4\cos(\alpha)\sin(\alpha)}\Big(-2\,\mathrm{BesselK}(1,2)\,\mathrm{BesselK}(0,2)\\
&\quad + e^{-4}(a-b)^2\cos(\alpha)\sin(\alpha)\Big)z^2\\
&\quad + \frac{\sqrt{\pi}e^{-2}}{4\cos(\alpha)\sin(\alpha)}\Big(\pi(a-b)\,\mathrm{BesselK}(1,2)\cos(\alpha)\sin(\alpha)\\
&\quad - 3\,\mathrm{BesselK}(0,2)(\cos(\alpha)+\sin(\alpha))\Big)z\\
&\quad - \frac{\pi}{8}\Big(2\pi\,\mathrm{BesselK}(0,2)^2 - \pi e^{-4}(a-b)\big((\cos^3(\alpha)+\sin^3(\alpha))\\
&\quad + (a-b)\cos^2(\alpha)\sin^2(\alpha)\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{V}_{3,2}(\alpha,z)\\
&= -\frac{\pi}{8}\Big(2\pi\,\mathrm{BesselK}(1,2)^2 - 3e^{-4}(a-b)(\cos(\alpha)\sin(\alpha))\Big)z^2\\
&\quad + \frac{\sqrt{\pi}e^{-2}}{8}\Big(-(a-b)(\mathrm{BesselK}(1,2)+\mathrm{BesselK}(0,2))(\pi+4\cos(\alpha)\sin(\alpha))\\
&\quad + 3\pi\,\mathrm{BesselK}(1,2)(\cos(\alpha)+\sin(\alpha))\Big)z\\
&\quad - \frac{\pi}{16}\Big(9e^{-4} - 2\pi(\mathrm{BesselK}(1,2)^2+\mathrm{BesselK}(1,2))\\
&\quad + \big(18e^{-4} - 8\,\mathrm{BesselK}(1,2)^2 - 8\,\mathrm{BesselK}(1,2)\,\mathrm{BesselK}(0,2)\big)\times\\
&\quad \cos(\alpha)\sin(\alpha)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{V}_{4,2}(\alpha,z)\\
&= -\frac{\pi}{16}\Big(9e^{-4} - 2\pi(\mathrm{BesselK}(1,2)^2+\mathrm{BesselK}(1,2))\\
&\quad + \big(18e^{-4} - 8\,\mathrm{BesselK}(1,2)^2 - 8\,\mathrm{BesselK}(1,2)\,\mathrm{BesselK}(0,2)\big)\times\\
&\quad \cos(\alpha)\sin(\alpha)\Big)gz^2
\end{aligned}
$$

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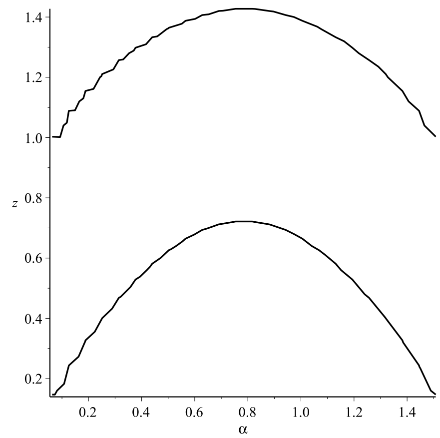

FIG. 8.2. *Zeros of* $\mathcal{V}_{2,2}(\alpha, z)$.

TABLE 8.4
Nodes and weights of the rational symbolic Gaussian cubature formula for $k = m = 2$.

α	$\phi_1^{(2,2)}(\alpha)$	$\phi_2^{(2,2)}(\alpha)$	$A_1^{(2,2)}(\alpha)$	$A_2^{(2,2)}(\alpha)$
$\pi/4$	0.7225719582946155	1.429405491101398	0.08434109400766578	0.1353597193936567
$\pi/5$	0.6915792626139660	1.408759908053796	0.08116737957001600	0.1385334338313066
$\pi/6$	0.6385079834998610	1.373747128437559	0.07624226326755025	0.1434585501337722
$\pi/7$	0.5865874591403105	1.339495732441536	0.07190707354034310	0.1477937398609794
$\pi/8$	0.5407129917175890	1.308932076311594	0.06835818907780278	0.1513426243235197
$\pi/9$	0.5011576642293720	1.282182473210330	0.06543693857056758	0.1542638748307549
$\pi/10$	0.4671294602305996	1.258782252613701	0.06298199311047522	0.1567188202908472

$$
\begin{aligned}
&+ \frac{\sqrt{\pi}e^{-2}}{48}\Bigg(9\pi(\cos(\alpha)+\sin(\alpha)) \\
&\qquad + 36(\cos^2(\alpha)\sin(\alpha)+\cos(\alpha)\sin^2(\alpha))\,\mathrm{BesselK}(0,2) \\
&\qquad + \mathrm{BesselK}(1,2)(13\pi(\cos^3(\alpha)+\sin^3(\alpha)) - 30\pi(\cos(\alpha)+\sin(\alpha)))\Bigg)z \\
&+ \frac{13\pi e^{-4}}{16} - \big(\mathrm{BesselK}(1,2)+\mathrm{BesselK}(0,2)\big)^2 \times \\
&\qquad\qquad \big(\frac{\pi^2}{16} + \frac{\pi}{2}\cos(\alpha)\sin(\alpha) + \cos^2(\alpha)\sin^2(\alpha)\big) \\
&+ \frac{\pi e^{-4}}{16}\left(13\cos^2(\alpha)\sin^2(\alpha) + \frac{65}{2}\cos(\alpha)\sin(\alpha)\right),
\end{aligned}
$$

where we set $a = \ln\left(\frac{1+\cos(\alpha)}{\sin(\alpha)}\right)$ and $b = \ln\left(\frac{1-\sin(\alpha)}{\cos(\alpha)}\right)$.

In Tables 8.4, 8.5, and 8.6 below, we give some numerical results for the rational symbolic Gaussian cubature formula for different values of α. We present in Figure 8.2 the zeros of the second-degree orthogonal polynomials $\mathcal{V}_{2,2}(\alpha, z)$ as functions of the parameter α. Note that every zero (α, z) with $\alpha \in [0, \frac{\pi}{2}]$ lies on two trajectories in the domain Ω which are symmetrical with respect to the axis $\alpha = \frac{\pi}{4}$.

Table 8.7 presents the relative errors between the exact values and the rational symbolic Gaussian cubature formula for $k = 4$ and $m = 3$.

TABLE 8.5
Nodes of the rational symbolic Gaussian cubature formula for $k = 4$ and $m = 3$.

α	$\phi_1^{(4,3)}(\alpha)$	$\phi_2^{(4,3)}(\alpha)$	$\phi_3^{(4,3)}(\alpha)$
$\pi/4$	0.6161503629514887	1.114395435588610	1.894548292262727
$\pi/5$	0.5828998825097352	1.087933941318848	1.877492354226239
$\pi/6$	0.5300275705823403	1.046060130006930	1.850561357129055
$\pi/7$	0.4819162238424462	1.007251738910978	1.825067008201785
$\pi/8$	0.4414376479005542	0.9736169068022864	1.802279882258584
$\pi/9$	0.4075895621605507	0.9446284412807688	1.782070638966526
$\pi/10$	0.3790243205296104	0.9194803033446130	1.764127862184243

TABLE 8.6
Weights of the rational symbolic Gaussian cubature formula for $k = 4$ and $m = 3$.

α	$A_1^{(4,3)}(\alpha)$	$A_2^{(4,3)}(\alpha)$	$A_3^{(4,3)}(\alpha)$
$\pi/4$	0.03842739992165646	0.1444356138009614	0.03683779967860888
$\pi/5$	0.03599902677130752	0.1451298630219150	0.03857192360807164
$\pi/6$	0.03318706537976423	0.1454169680334712	0.04109677998799345
$\pi/7$	0.03146152562857054	0.1449743804660527	0.04326490730671018
$\pi/8$	0.03042637042990856	0.1442051754625434	0.04506926750890688
$\pi/9$	0.02974536776516242	0.1433522985702679	0.04660314706596627
$\pi/10$	0.02924286194200602	0.1425213620560342	0.4793658940323795

9. Conclusion. In this work, we have presented rational symbolic cubature rules over the first quadrant in a Cartesian plane. More precisely, from the connection between the theory of two-point Padé approximants and the symbolic Gaussian cubature formulas corresponding to a bivariate Stieltjes function, we have constructed some cubature formulas that integrate exactly a combination of bivariate homogeneous polynomials and some specific bivariate rational functions. In order to illustrate the main idea of the work, some examples have been presented.

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TABLE 8.7
Relative errors of the rational symbolic Gaussian cubature formula for $k = 4$ and $m = 3$.

α	Relative errors
$\pi/4$	$0.2579160251022750 \times 10^{-11}$
$\pi/5$	$0.2619972007496391 \times 10^{-11}$
$\pi/6$	$0.2503591428055462 \times 10^{-11}$
$\pi/7$	$0.2133636277934934 \times 10^{-11}$
$\pi/8$	$0.1551488963016243 \times 10^{-11}$
$\pi/9$	$0.8148899752314427 \times 10^{-8}$
$\pi/10$	$0.2691177976984070 \times 10^{-9}$

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