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ON THE NUMERICAL SOLUTION OF AN ELLIPTIC PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS*

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Abstract. In this paper we consider a class of non-standard elliptic transmission problems in disjoint domains. As a model example, we consider an area consisting of two non-adjacent rectangles. In each subarea, a boundary-value problem of elliptic type is considered, where the interaction between their solutions is described by nonlocal integral conjugation conditions. An a priori estimate for its weak solution in an appropriate Sobolev-like space is proved. A finite difference scheme approximating this problem is proposed and analyzed. An estimate of the convergence rate, compatible with the smoothness of the input data, up to a slowly increasing logarithmic factor of the mesh size, is obtained.

Key words. transmission problem, boundary-value problem, nonlocal boundary condition, finite differences, Sobolev spaces, convergence rate estimates

AMS subject classifications. 65N12, 65N15

1. Introduction. It is well-known that transfer of energy or mass is fundamental for many biological, chemical, environmental, and industrial processes. The basic transport mechanisms of such processes are diffusion and bulk flow. Therefore, the corresponding flux has two components: a diffusive one and a convective one. Here we pay attention to diffusion in a two-dimensional domain with layers. Layers with material properties which significantly differ from those of the surrounding medium appear in a variety of applications. Layers can be structural, thermal, electromagnetic, optical, etc. Mathematical models of energy and mass transfer in domains with layers lead to so called transmission problems. We use a method proposed in [8] of modelling a thin layer as an interface. The interaction between solutions in subdomains is described by means of nonlocal integral conjugation conditions. In order to explain the method proposed in this paper for the mathematical modelling of layer phenomena, we consider a physical model.

Let us consider a simple physical model of a system $\Omega = \Omega^1 \cup \Omega^2$, where Ω^1 and Ω^2 are two disjoint, Lipschitz domains in $\mathbb{R}^N$, $N \geq 2$, surrounded by a transparent medium Ω^0. They represent conductive and opaque bodies with different material properties. We assume that all material are grey; see [5, 22]. Therefore, radiation only needs to be considered at the surfaces of the bodies Ω^1 and Ω^2.

For $\xi, \eta \in \mathbb{R}^N$ we define $[\xi, \eta]$ to be the line connecting the points ξ and η, i.e.,

$$[\xi, \eta] = \{\alpha\xi + (1-\alpha)\eta \,|\, \alpha \in [0,1]\}.$$

We also define, for $k = 1, 2$,

$$\Gamma^{k,R} = \{\xi \in \partial\Omega^k \,|\, \exists \eta \in \Omega^{3-k} : [\xi, \eta] \cap \Omega^k = \emptyset\},$$

and split $\partial\Omega^k \setminus \Gamma^{k,R}$ into two parts: $\Gamma^{k,D}$ and $\Gamma^{k,N}$. In such a way we have $\partial\Omega^k = \Gamma^{k,R} \cup \Gamma^{k,D} \cup \Gamma^{k,N}$, $k = 1, 2$. Nonstationary radiative heat transfer in the system of two disjoint

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bodies Ω^1 and Ω^2 (see Figure 1.1) can be described by the following equations [2]:

$$\frac{\partial u^k}{\partial t} - \sum_{j,l=1}^{N} \frac{\partial}{\partial \xi_j}\big(A^k_{jl}(\xi,t,u^k)\frac{\partial u^k}{\partial \xi_k}\big) = f^k(\xi,t), \quad \xi \in \Omega^k,\ t>0, \tag{1.1}$$

$$u^k(\xi,0) = u^k_0(\xi), \quad \xi \in \Omega^k, \tag{1.2}$$

$$u^k(\xi,t) = \varphi_k(\xi,t), \quad \xi \in \Gamma^{k,D},\ t>0, \tag{1.3}$$

$$\begin{aligned}&\sum_{j,l=1}^{N} A^k_{jl}(\xi,t,u^k)\frac{\partial u^k}{\partial \xi_l}\cos(\theta_j) + h_k(u^k(\xi,t)) = \\ &\int_{\partial\Omega^{3-k}} h_{3-k}(u^{3-k}(\eta,t))w(\eta,\xi,t)\,d\sigma(\eta) + g_k(\xi,t), \quad \xi \in \Gamma^{k,R}\cup\Gamma^{k,N},\ t>0,\end{aligned} \tag{1.4}$$

where θ_j is the angle between the unit outward normal vector $\nu(\xi)$ to $\partial\Omega$ at the point ξ, $d\sigma$ is the Lebesgue measure on $\partial\Omega$, $u^k(\xi,t)$ is the temperature of the body Ω^k, $A^k_{jl}(\xi,t,u^k)$ is the heat conductivity tensor, $h_k(u^k(\xi,t))$ is the surface radiation flux density,

$$\int_{\partial\Omega^{3-k}} h_{3-k}(u^{3-k}(\eta,t))w(\eta,\xi,t)\,d\sigma(\eta)$$

is the radiation flux density absorbed at the point ξ, and $k=1,2$. If the bodies are at rest, then the kernel w (the so called view factor) is independent of t and has the form

$$w(\eta,\xi) = \begin{cases} \frac{(\nu(\eta),\xi-\eta)(\nu(\xi),\eta-\xi)}{b_N|\eta-\xi|^{N+1}}, & [\xi,\eta]\cap\Omega=\emptyset,\\ 0, & [\xi,\eta]\cap\Omega\neq\emptyset,\end{cases} \tag{1.5}$$

where $b_N = \mathrm{meas}(S_{N-1})/(N-1)$, S_{N-1} is the $N-1$-dimensional unit sphere in $\mathbb{R}^N$, (ξ,η) is the standard Euclidean inner product in $\mathbb{R}^N$, and $|\xi| = (\xi,\xi)^{1/2}$ is the Euclidean norm. In the case of radiation of Stefan–Boltzmann type, the flux density term has the form $h_k(u^k) = \alpha_k|u^k|^3u^k$.

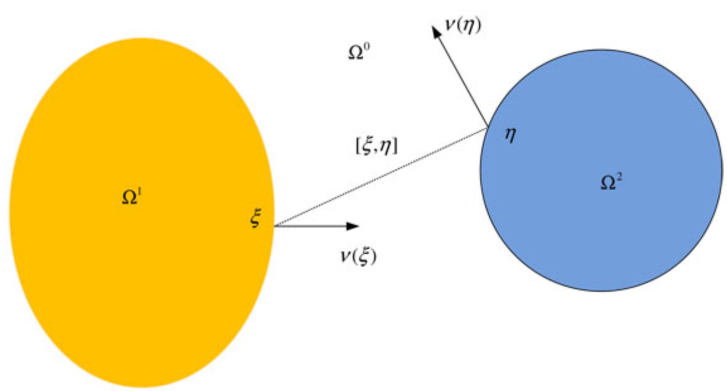

FIG. 1.1. *System of two bodies Ω^1 and Ω^2.*

Numerical methods for solving transmission problems of elliptic type were studied in many papers; see [10, 16, 7, 20, 24]. In particular, a one-dimensional elliptic problem in two

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disjoint interval was studied in [16, 7, 24]. As in [16], using the finite element method, a two-dimensional elliptic problem with Dirichlet boundary conditions was discussed in the paper [10]. The paper [20] marks the beginning of research on the transmission problem with Robin's boundary conditions.

The novel contribution of this paper is that the so called third boundary value problem (or Robin type boundary value problem) for elliptic type equations is considered. The existence and uniqueness of the weak solution will be analyzed in two cases, depending on whether the bilinear form is coercive or not. The problem is approximated by the finite difference method. In both cases, an estimate of the convergence rate is obtained.

2. Formulation of the problem. We begin by defining the domains Ω^1 and Ω^2 in the following way: $\Omega^1 = (a_1, b_1) \times (c, d)$, $\Omega^2 = (a_2, b_2) \times (c, d)$, with $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$ and $c < d$. We denote by $\Gamma^k = \partial\Omega^k = \cup_{i,j=1}^2 \Gamma_{ij}^k$ the boundaries of the considered subareas, where

$$\begin{aligned}
&\Gamma_{11}^1 = \{x = (x_1, x_2) \in \Gamma^1 \,|\, x_1 = a_1\}, && \Gamma_{12}^1 = \{x \in \Gamma^1 \,|\, x_1 = b_1\},\\
&\Gamma_{11}^2 = \{x = (x_1, x_2) \in \Gamma^2 \,|\, x_1 = a_2\} && \Gamma_{12}^2 = \{x \in \Gamma^2 \,|\, x_1 = b_2\},\\
&\Gamma_{21}^k = \{x \in \Gamma^k \,|\, x_2 = c\}, && \Gamma_{22}^k = \{x \in \Gamma^k \,|\, x_2 = d\}, \quad k = 1, 2.
\end{aligned}$$

As a model example,we consider the following boundary-value problem: find functions $u^1(x_1, x_2)$ and $u^2(x_1, x_2)$ that satisfy the system of elliptic equations

$$L^k u^k(x) = f^k(x_1, x_2), \quad x = (x_1, x_2) \in \Omega^k, \tag{2.1}$$

$$l^k u^k(x) = \begin{cases} r^k(x) u^{3-k}(x), & x \in \Gamma_{1,3-k}^k,\\ 0, & x \in \Gamma^k \setminus \Gamma_{1,3-k}^k, \end{cases} \tag{2.2}$$

where

$$L^k u^k(x) := -\sum_{i,j=1}^2 \frac{\partial}{\partial x_i}\left(p_{ij}^k(x) \frac{\partial u^k}{\partial x_j}(x)\right) + q^k(x) u^k(x), \tag{2.3}$$

$$l^k u^k(x) := \sum_{i,j=1}^2 p_{ij}^k(x) \frac{\partial u^k}{\partial x_j}(x) \cos(\theta_j^k) + \alpha^k(x) u^k(x), \tag{2.4}$$

$$\left(r^k u^{3-k}\right)(x) := \int_{\Gamma_{1,k}^{3-k}} \beta^k(x_2, x_2') u^{3-k}(x_1, x_2')\, \mathrm{d}x_2', \tag{2.5}$$

and θ_j^k is the angle between the unit outward normal vector ν^k to Γ^k $(k = 1, 2)$; see Figure 2.1.

Notice that the boundary condition (2.2) on $\Gamma^k \setminus \Gamma_{1,3-k}^k$ reduces to a homogeneous Robin boundary condition, while on $\Gamma_{1,3-k}^k$ it can be considered as a conjugation condition of nonlocal Robin–Dirichlet type. The boundary-value problem (2.1)–(2.5) reduces to a linearized stationary radiative heat transfer problem of the type (1.1)–(1.4) if we choose β_i in accordance with (1.5):

$$\beta_i(x_2, x_2') = \frac{\alpha_{3-i}(x_2')(a_2 - b_1)^2}{2[(a_2 - b_1)^2 + (x_2 - x_2')^2]^{3/2}}, \quad i = 1, 2.$$

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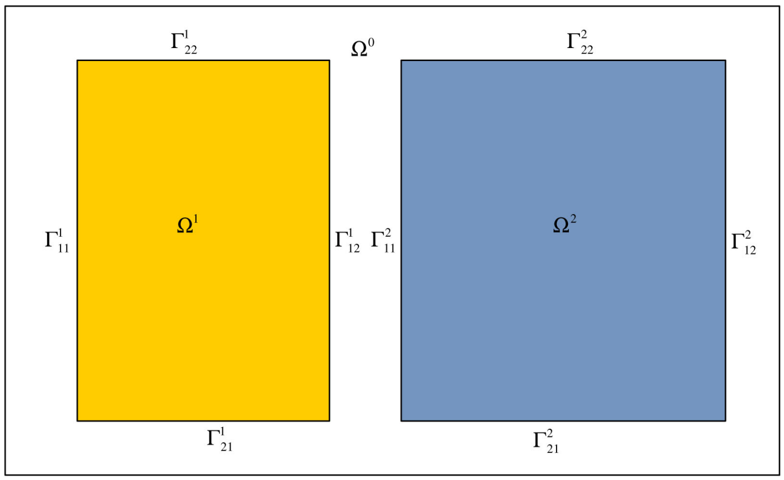

FIG. 2.1. *The rectangles* Ω^1 *and* Ω^2.

We assume that the following standard conditions of regularity and ellipticity are satisfied:

$$\begin{aligned} p^k_{ij} = p^k_{ji} \in L^\infty(\Omega^k), \qquad q^k \in L^\infty(\Omega^k), \\ \alpha^k \in L^\infty(\Gamma^k), \qquad \beta^k \in L^\infty(\Gamma^k_{1,3-k} \times \Gamma^{3-k}_{1,k}), \end{aligned} \tag{2.6}$$

$$c^k_0 \sum_{i=1}^2 \xi_i^2 \le \sum_{i,j=1}^2 p^k_{ij}\xi_i\xi_j \le c^k_1 \sum_{i=1}^2 \xi_i^2, \qquad \forall x \in \bar\Omega^k, \quad \forall \xi \in \mathbb{R}^2, \tag{2.7}$$

where we denote by C, c_i, and c^k_i, positive constants, independent of the solution of the boundary-value problem and the mesh sizes. In particular, C may take different values in different formulas.

3. Existence and uniqueness of weak solutions.

We introduce the product space

$$L = L^2(\Omega^1) \times L^2(\Omega^2) = \{v = (v^1, v^2) \,|\, v^k \in L^2(\Omega^k)\},$$

endowed with the inner product and associated norm

$$(u, v)_L = (u^1, v^1)_{L^2(\Omega^1)} + (u^2, v^2)_{L^2(\Omega^2)}, \quad \|v\|_{L^2(\Omega^k)} = (v, v)^{1/2}_{L^2(\Omega^k)},$$

where

$$(u^k, v^k)_{L^2(\Omega^k)} = \int\int_{\Omega^k} u^k v^k \, dxdy, \quad k = 1, 2.$$

We also define the spaces

$$H^s = \{v = (v^1, v^2) |\, v^k \in H^s(\Omega^k)\}, \quad s = 1, 2, ...,$$

endowed with the inner product and associated norm

$$(u, v)_{H^s} = (u^1, v^1)_{H^s(\Omega^1)} + (u^2, v^2)_{H^s(\Omega^2)}, \quad \|v\|_{H^s(\Omega^k)} = (v, v)^{1/2}_{H^s(\Omega^k)},$$

where $H^s(\Omega^k)$ are the standard Sobolev spaces [1]. Finally, setting $u = (u^1, u^2)$ and $v = (v^1, v^2)$, we define the bilinear form

$$\begin{aligned} a(u,v) = (Lu,v)_{L^2} = \sum_{k=1}^{2} \Bigg(\int\!\!\int_{\Omega^k} \Bigg(\sum_{i,j=1}^{2} p_{ij}^k \frac{\partial u^k}{\partial x_j} \frac{\partial v^k}{\partial x_i} + q^k u^k v^k \Bigg) \mathrm{d}x_1 \, \mathrm{d}x_2 \\ + \int_{\Gamma^k} \alpha^k u^k v^k \, \mathrm{d}\Gamma^k - \int_{\Gamma^k_{1,3-k}} \int_{\Gamma^{3-k}_{1,k}} \beta^k u^{3-k} v^k \, \mathrm{d}\Gamma^{3-k} \, \mathrm{d}\Gamma^k \Bigg). \end{aligned} \tag{3.1}$$

LEMMA 3.1. *Under conditions* (2.6) *the bilinear form* a *defined by* (3.1) *is bounded on* $H^1 \times H^1$. *If in addition conditions* (2.7) *are fulfilled, then this form satisfies Gårding's inequality on* H^1, *i.e., there exist positive constants* m *and* κ *such that*

$$a(u,u) + \kappa \|u\|_L^2 \geq m \|u\|_{H^1}^2, \quad \forall\, u \in H^1.$$

If β^k *are sufficiently small and* $\alpha^k > 0$ $(k = 1,2)$, *then the bilinear form A is coercive, i.e.,* $\kappa = 0$. *A sufficient condition for this to hold is that*

$$|\beta^1(x,x') + \beta^2(x',x)| \leq \frac{2\sqrt{\alpha^1(x)\alpha^2(x')}}{d-c}, \qquad \forall x \in \Gamma^1_{12}, \quad \forall x' \in \Gamma^2_{11}. \tag{3.2}$$

Proof. The proof is analogous to that of Lemma 3.8 in [14]. Boundedness of A follows from (2.6) and the trace theorem for Sobolev spaces:

$$\|u^k\|_{L^2(\partial\Omega^k)} \leq C \|u^k\|_{H^1(\Omega^k)}.$$

Gårding's inequality is then a consequence of (2.7), (3.1), the following multiplicative trace inequality (see Prop. 1.6.3 in [4])

$$\|u^k\|^2_{L^2(\partial\Omega^k)} \leq C \|u^k\|_{L^2(\Omega^k)} \|u^k\|_{H^1(\Omega^k)},$$

and the Cauchy–Schwarz and ϵ-inequalities, for a sufficiently small $\epsilon > 0$. ☐

THEOREM 3.2. *Let* $\alpha^k > 0$, $q^k \geq c_k > 0$, $f \in L^2(\Omega)$, *and let the assumptions* (2.6), (2.7), *and* (3.2) *hold. Then, the boundary-value problem* (2.1)–(2.5) *has a unique weak solution* $u \in H^1(\Omega)$, *and it depends continuously on* f.

Proof. The proof is an easy consequence of Lemma 3.1 and the Lax–Milgram Lemma; see Theorems 17.9 and 17.10 in [25]. ☐

4. Finite difference approximation. Let $n_k,\, n_3 \in \mathbb{N}$, $n_k, n_3 \geq 2$ and $h = (b_k - a_k)/n_k = (d-c)/n_3$, for $k = 1,2$. We consider the uniform meshes $\bar\omega^k$ with mesh size h on $\bar\Omega^k$. We also define the following meshes, for $i,j,k = 1,2$:

$$\begin{aligned} &\gamma^k = \bar\omega^k \cap \Gamma^k, \qquad \bar\gamma^k_{ij} = \bar\omega^k \cap \Gamma^k_{ij}, \qquad \gamma^k_{1j} = \{x \in \bar\gamma^k_{1j} \,:\, c < x_2 < d\}, \\ &\gamma^{k-}_{1j} = \{x \in \bar\gamma^k_{1j} \,:\, c \leq x_2 < d\}, \quad \gamma^{k+}_{1j} = \{x \in \bar\gamma^k_{1j} \,:\, c < x_2 \leq d\}, \quad \gamma^{k\star}_{1j} = \bar\gamma^k_{1j} \setminus \gamma^k_{1j}, \\ &\gamma^k_{2j} = \{x \in \bar\gamma^k_{2j} \,:\, a_k < x_1 < b_k\}, \qquad \gamma^{k-}_{2j} = \{x \in \bar\gamma^k_{2j} \,:\, a_k \leq x_1 < b_k\}, \\ &\gamma^{k+}_{2j} = \{x \in \bar\gamma^k_{2j} \,:\, a_k < x_1 \leq b_k\}, \quad \gamma^{k\star}_{2j} = \bar\gamma^k_{2j} \setminus \gamma^k_{2j}, \quad \gamma^k_\star = \gamma^k \setminus \{\cup_{i,j}\gamma^k_{ij}\}. \end{aligned}$$

We shall consider vector-functions of the form $v = (v^1, v^2)$, where v^k is a mesh function defined on $\overline{\omega}^k$, $k = 1,2$.

The relevant finite difference operators are defined in the usual manner [6, 17, 23]

$$v^k_{x_i} = \frac{(v^k)^{+i} - v^k}{h}, \qquad v^k_{\bar x_i} = \frac{v^k - (v^k)^{-i}}{h},$$

where $(v^k)^{\pm i}(x) = v^k(x \pm he_i)$ and e_i is the unit vector of the axis x_i, $i, k = 1, 2$.

We define the following discrete inner product on the space L^2

$$[v^k, w^k]_k = h^2 \sum_{x\in\omega^k} v^k(x)w^k(x) + \frac{h^2}{2} \sum_{x\in\gamma^k\setminus\gamma^k_\star} v^k(x)w^k(x) + \frac{h^2}{4} \sum_{x\in\gamma^k_\star} v^k(x)w^k(x),$$

and the inner products for the forward and backward differences on the vector space of all real-valued functions defined on $\overline{\omega}^1$ and $\overline{\omega}^2$. We give analogous definitions for the vector spaces of mesh functions on which the inner products and norms introduced below are defined

$$\begin{aligned}
[v^k, w^k)_{k,i} &= h^2 \sum_{x\in\omega^k\cup\gamma^k_{i1}} v^k(x)w^k(x) + \frac{h^2}{2} \sum_{x\in\gamma^{k-}_{3-i,1}\cup\gamma^{k-}_{3-i,2}} v^k(x)w^k(x),\\
(v^k, w^k]_{k,i} &= h^2 \sum_{x\in\omega^k\cup\gamma^k_{i2}} v^k(x)w^k(x) + \frac{h^2}{2} \sum_{x\in\gamma^{k+}_{3-i,1}\cup\gamma^{k+}_{3-i,2}} v^k(x)w^k(x),\\
[v^k, w^k)_k &= h^2 \sum_{x\in\omega^k\cup\gamma^{k-}_{11}\cup\gamma^{k-}_{21}} v^k(x)w^k(x),\\
(v^k, w^k]_k &= h^2 \sum_{x\in\omega^k\cup\gamma^{k+}_{12}\cup\gamma^{k+}_{22}} v^k(x)w^k(x),
\end{aligned}$$

with appropriate norms defined by

$$|[v^k]|_k^2 = [v^k, v^k]_k, \qquad |[v^k\|_{k,i}^2 = [v^k, v^k)_{k,i}, \qquad \|v^k]|_{k,i}^2 = (v^k, v^k]_{k,i},$$
$$|[v^k\|_k^2 = [v^k, v^k)_k, \qquad \|v^k]|_k^2 = (v^k, v^k]_k.$$

The discrete Sobolev norm is defined by

$$|[v^k]|^2_{H^1(\bar\omega^k)} = |[v^k]|_k^2 + |[v^k_{x_1}||^2_{k,1} + |[v^k_{x_2}\|^2_{k,2}.$$

We define the discrete analog of the norm on the space $\mathcal{C}$ of continuous functions by

$$|[v^k]|_{\mathcal{C}(\bar\omega^k)} = \max_{x\in\bar\omega^k} |v^k(x)|.$$

The discrete L^2 inner products and norms for functions defined on the boundary are

$$[v^k, w^k]_{\bar\gamma^k_{ij}} = h \sum_{x\in\gamma^k_{ij}} v^k(x)w^k(x) + \frac{h}{2} \sum_{x\in\gamma^{k\star}_{ij}} v^k(x)w^k(x), \qquad |[v^k]|^2_{\bar\gamma^k_{ij}} = [v^k, v^k]_{\bar\gamma^k_{ij}},$$
$$[v^k, w^k)_{\gamma^{k-}_{ij}} = h \sum_{x\in\gamma^{k-}_{ij}} v^k(x)w^k(x), \qquad |[v^k||^2_{\gamma^{k-}_{ij}} = [v^k, v^k)_{\gamma^{k-}_{ij}}.$$

Let us also introduce the fractional-order discrete Sobolev seminorm and norm by

$$\begin{aligned}
|v^k|^2_{H^{1/2}(\gamma^{k-}_{ij})} &= h^2 \sum_{x,x'\in\gamma^{k-}_{ij},x'\neq x} \left[\frac{v^k(x) - v^k(x')}{x_{3-i} - x'_{3-i}}\right]^2,\\
|[v^k\|^2_{H^{1/2}(\gamma^{k-}_{ij})} &= |v^k|^2_{H^{1/2}(\gamma^{k-}_{ij})} + |[v^k\|^2_{\gamma^{k-}_{ij}},\\
|[v^k\|^2_{\tilde H^{1/2}(\gamma^{k-}_{ij})} &= |v^k|^2_{H^{1/2}(\gamma^{k-}_{ij})} + h \sum_{x\in\gamma^{k-}_{ij}} \left(\frac{1}{x_{3-i} + h/2} + \frac{1}{l_{ki} - x_{3-i} - h/2}\right)|v^k(x)|^2,
\end{aligned}$$

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where $l_{k1} = d - c$ and $l_{k2} = b_k - a_k$.

For $v = (v^1, v^2)$ and $w = (w^1, w^2)$ we define

$$[v, w] = [v^1, w^1]_1 + [v^2, w^2]_2, \qquad |[v]|^2 = [v, v], \qquad |[v]|^2_{H_h^1} = |[v^1]|^2_{H^1(\bar\omega^1)} + |[v^2]|^2_{H^1(\bar\omega^2)}.$$

For $f^k \in L^1((a_k, b_k); \mathcal{C}([c, d]))$ we also define the Steklov smoothing operators (see [9])

$$\begin{aligned}
T_{k1} f^k(x) &= T_{k1}^{\mp} f^k(x_1 \pm \frac{h}{2}, x_2) = \frac{1}{h}\int_{x_1-h/2}^{x_1+h/2} f^k(x_1', x_2)\,\mathrm{d}x_1',\\
T_{k2} f^k(x) &= T_{k2}^{\mp} f^k(x_1, x_2 \pm \frac{h}{2}) = \frac{1}{h}\int_{x_2-h/2}^{x_2+h/2} f^k(x_1', x_2)\,\mathrm{d}x_1',\\
T_{k1}^2 f^k(x) &= \frac{1}{h}\int_{x_1-h}^{x_1+h} \big(1 - \frac{|x_1 - x_1'|}{h}\big) f^k(x_1', x_2)\,\mathrm{d}x_1',\\
T_{11}^{2-} f^1(b_1, x_2) &= \frac{2}{h}\int_{b_1-h}^{b_1} \big(1 - \frac{b_1 - x_1'}{h}\big) f^1(x_1', x_2)\,\mathrm{d}x_1',\\
T_{21}^{2+} f^2(a_2, x_2) &= \frac{2}{h}\int_{a_2}^{a_2+h} \big(1 - \frac{x_1' - a_2}{h}\big) f^2(x_1', x_2)\,\mathrm{d}x_1',
\end{aligned}$$

with $k = 1, 2$, and similarly for the other mesh functions. These operators commute and transform derivatives into differences. For example,

$$T_{ki}^{+}\left(\frac{\partial u^k}{\partial x_i}\right) = u^k_{x_i}, \qquad T_{ki}^{-}\left(\frac{\partial u^k}{\partial x_i}\right) = u^k_{\bar{x}_i}, \qquad T_{ki}^{2}\left(\frac{\partial^2 u^k}{\partial x_i^2}\right) = u^k_{\bar{x}_i x_i}.$$

In the sequel, we shall assume that the weak solution of the problem (2.1)–(2.5) belongs to the Sobolev space H^s, $2 < s \le 3$, while the data satisfy the smoothness conditions

$$\begin{aligned}
&p_{ij}^k \in H^{s-1}(\Omega^k), \qquad \alpha^k \in H^{s-3/2}(\Gamma_{ij}^k), \quad \alpha^k \in C(\Gamma^k), \qquad \beta^k \in H^{s-1}(\Delta^k),\\
&f^k \in H^{s-2}(\Omega^k), \qquad q^k \in H^{s-2}(\Omega^k), \qquad k, i, j = 1, 2.
\end{aligned} \tag{4.1}$$

We denote

$$\tilde f^k = \begin{cases} T_{k1}^2 T_{k2}^2 f^k, & x \in \omega^k,\\ T_{ki}^{2\pm} T_{k,3-i}^2 f^k, & x \in \gamma_{i1}^k / x \in \gamma_{i2}^k,\\ T_{k1}^{2\pm} T_{k2}^{2\pm} f^k, & x \in \gamma_\star^k, \end{cases} \qquad \tilde q^k = \begin{cases} T_{k1}^2 T_{k2}^2 q^k, & x \in \omega^k,\\ T_{ki}^{2\pm} T_{k,3-i}^2 q^k, & x \in \gamma_{i1}^k / x \in \gamma_{i2}^k,\\ T_{k1}^{2\pm} T_{k2}^{2\pm} q^k, & x \in \gamma_\star^k, \end{cases}$$

$$\begin{aligned}
\tilde\alpha^k &= T_{k,3-i}^2 \alpha^k, \qquad x \in \gamma_{i1}^k \cup \gamma_{i2}^k, \qquad i = 1, 2,\\
\tilde\alpha_i^k &= T_{k,3-i}^{2\pm} \alpha^k, \qquad x \in \gamma_\star^k, \qquad i = 1, 2,
\end{aligned}$$

and

$$\tilde\beta^k = \begin{cases} T_{k2}^2 \beta^k, & x \in \gamma_{1,3-k}^k,\\ T_{k2}^{2\pm} \beta^k, & x \in \gamma_{1,3-k}^{k\star}. \end{cases}$$

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We approximate the bilinear form (3.1) by the following discrete bilinear form:
(4.2)

$$\begin{aligned}a_h(u,v) = [L_h u, v] = \sum_{k=1}^{2}\Big\{\frac{1}{2}\sum_{i=1}^{2}\Big[[p_{ii}^k u_{x_i}^k, v_{x_i}^k)_{k,i} + (p_{ii}^k u_{\bar{x}_i}^k, v_{\bar{x}_i}^k]_{k,i}\\ +[p_{i,3-i}^k u_{x_{3-i}}^k, v_{x_i}^k)_k + (p_{i,3-i}^k u_{\bar{x}_{3-i}}^k, v_{\bar{x}_i}^k]_k\Big] + [\tilde{q}^k u^k, v^k]_k + \sum_{i,j=1}^{2}\sum_{x\in\gamma_{ij}^k} h\tilde{\alpha}^k, u^k v^k\\ +\frac{1}{2}\sum_{x\in\gamma_\star^k} h(\tilde{\alpha}_1^k+\tilde{\alpha}_2^k)u^k v^k - h^2\sum_{x\in\gamma_{1,3-k}^k}\sum_{x'\in\gamma_{1,k}^{3-k}}\tilde{\beta}^k(x,x')u^{3-k}(x')v^k(x)\\ -\tfrac{h^2}{2}\textstyle\sum_{x\in\gamma_{1,3-k}^{k\star}}\sum_{x'\in\gamma_{1,k}^{(3-k)\star}}\tilde{\beta}^k(x,x')u^{3-k}(x')v^k(x)\Big\}.\end{aligned}$$

As $(Lu,v)_{L_2} = a(u,v) \sim a_h(u,v) = [L_h u, v]$, by performing a partial summation in formula (4.2), we conclude that the boundary value problem (2.1)–(2.5) can be approximated in the form

$$L_h^k v = \tilde{f}^k, \qquad x\in\bar{\omega}^k, \qquad k=1,2, \tag{4.3}$$

where $L_h^1 v$ is defined as

$$L_h^1 v = \begin{cases} -\frac{1}{2}\sum\limits_{i,j=1}^{2}\Big[\big(p_{ij}^1 v_{x_j}^1\big)_{\bar{x}_i} + \big(p_{ij}^1 v_{\bar{x}_j}^1\big)_{x_i}\Big] + \tilde{q}^1 v^1, & x\in\omega^1,\\ \frac{2}{h}\Big[-\frac{p_{11}^1+(p_{11}^1)^{+1}}{2}v_{x_1}^1 - p_{12}^1\frac{v_{x_2}^1+v_{\bar{x}_2}^1}{2} + \tilde{\alpha}^1 v^1\Big] - \big(p_{12}^1 v_{\bar{x}_2}^1\big)_{x_1} & \\ \quad -\big(p_{21}^1 v_{x_1}^1\big)_{\bar{x}_2} - \frac{1}{2}\big(p_{22}^1 v_{x_2}^1\big)_{\bar{x}_2} - \frac{1}{2}\big(p_{22}^1 v_{\bar{x}_2}^1\big)_{x_2} + \tilde{q}^1 v^1, & x\in\gamma_{11}^1,\\ \frac{2}{h}\Big[-\frac{p_{11}^1+(p_{11}^1)^{+1}}{2}v_{x_1}^1 - p_{12}^1 v_{x_2}^1 + \tilde{\alpha}_1^1 v^1\Big] & \\ \quad +\frac{2}{h}\Big[-p_{21}^1 v_{x_1}^1 - \frac{p_{22}^1+(p_{22}^1)^{+2}}{2}v_{x_2}^1 + \tilde{\alpha}_2^1 v^1\Big] + \tilde{q}^1 v^1, & x=(a_1,c),\\ \frac{2}{h}\Big[-\frac{p_{11}^1+(p_{11}^1)^{+1}}{2}v_{x_1}^1 - p_{12}^1 v_{\bar{x}_2}^1 + \tilde{\alpha}_1^1 v^1\Big] - 2\big(p_{12}^1 v_{\bar{x}_2}^1\big)_{x_1} & \\ \quad +\frac{2}{h}\Big[p_{21}^1 v_{x_1} + \frac{p_{22}^1+(p_{22}^1)^{-2}}{2}v_{\bar{x}_2}^1 + \tilde{\alpha}_2^1 v^1\Big] - 2\big(p_{21}^1 v_{x_1}^1\big)_{\bar{x}_2} + \tilde{q}^1 v^1, & x=(a_1,d),\\ \frac{2}{h}\Big[\frac{p_{11}^1+(p_{11}^1)^{-1}}{2}v_{\bar{x}_1}^1 + p_{12}^1\frac{v_{x_2}^1+v_{\bar{x}_2}^1}{2} + \tilde{\alpha}^1 v^1 & \\ \quad -[\tilde{\beta}^1(x,\cdot), v^2(\cdot)]_{\bar{\gamma}_{11}^2}\Big] - \big(p_{12}^1 v_{x_2}^1\big)_{\bar{x}_1} & \\ \quad -\big(p_{21}^1 v_{\bar{x}_1}^1\big)_{x_2} - \frac{1}{2}\big(p_{22}^1 v_{x_2}^1\big)_{\bar{x}_2} - \frac{1}{2}\big(p_{22}^1 v_{\bar{x}_2}^1\big)_{x_2} + \tilde{q}^1 v^1, & x\in\gamma_{12}^1,\\ \frac{2}{h}\Big[\frac{p_{11}^1+(p_{11}^1)^{-1}}{2}v_{\bar{x}_1}^1 + p_{12}^1 v_{x_2}^1 + \tilde{\alpha}_1^1 v^1 - [\tilde{\beta}^1(x,\cdot), v^2(\cdot)]_{\bar{\gamma}_{11}^2}\Big] & \\ \quad +\frac{2}{h}\Big[-p_{21}^1 v_{\bar{x}_1} - \frac{p_{22}^1+(p_{22}^1)^{+2}}{2}v_{x_2}^1 + \tilde{\alpha}_2^1 v^1\Big] & \\ \quad -2\big(p_{12}^1 v_{x_2}^1\big)_{\bar{x}_1} - 2\big(p_{21}^1 v_{\bar{x}_1}^1\big)_{x_2} + \tilde{q}^1 v^1, & x=(b_1,c),\\ \frac{2}{h}\Big[\frac{p_{11}^1+(p_{11}^1)^{-1}}{2}v_{\bar{x}_1}^1 + p_{12}^1 v_{\bar{x}_2}^1 + \tilde{\alpha}_1^1 v^1 - [\tilde{\beta}^1(x,\cdot), v^2(\cdot)]_{\bar{\gamma}_{11}^2}\Big] & \\ \quad +\frac{2}{h}\Big[p_{21}^1 v_{\bar{x}_1}^1 + \frac{p_{22}^1+(p_{22}^1)^{-2}}{2}v_{\bar{x}_2}^1 + \tilde{\alpha}_2^1 v^1\Big] + \tilde{q}^1 v^1, & x=(b_1,d),\\ \frac{2}{h}\Big[-\frac{p_{22}^1+(p_{22}^1)^{+2}}{2}v_{x_2}^1 - p_{21}^1\frac{v_{x_1}^1+v_{\bar{x}_1}^1}{2} + \tilde{\alpha}^1 v^1\Big] - \big(p_{21}^1 v_{\bar{x}_1}^1\big)_{x_2} & \\ \quad -\big(p_{12}^1 v_{x_2}^1\big)_{\bar{x}_1} - \frac{1}{2}\big(p_{11}^1 v_{x_1}^1\big)_{\bar{x}_1} - \frac{1}{2}\big(p_{11}^1 v_{\bar{x}_1}^1\big)_{x_1} + \tilde{q}^1 v^1, & x\in\gamma_{21}^1,\\ \frac{2}{h}\Big[\frac{p_{22}^1+(p_{22}^1)^{-2}}{2}v_{\bar{x}_2}^1 + p_{21}^1\frac{v_{x_1}^1+v_{\bar{x}_1}^1}{2} + \tilde{\alpha}^1 v^1\Big] - \big(p_{21}^1 v_{x_1}^1\big)_{\bar{x}_2} & \\ \quad -\big(p_{12}^1 v_{\bar{x}_2}^1\big)_{x_1} - \frac{1}{2}\big(p_{11}^1 v_{x_1}^1\big)_{\bar{x}_1} - \frac{1}{2}\big(p_{11}^1 v_{\bar{x}_1}^1\big)_{x_1} + \tilde{q}^1 v^1, & x\in\gamma_{22}^1,\end{cases}$$

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and $L_h^2 v$ is defined analogously. The finite difference scheme (FDS) (4.3) may be compactly presented as the operator-difference scheme

$$L_h v = \tilde{f}, \tag{4.4}$$

where $v = (v^1, v^2)$, $\tilde{f} = (\tilde{f}^1, \tilde{f}^2)$, and $L_h v = (L_h^1 v, L_h^2 v)$.

The following analogue of Lemma 3.1 holds.

LEMMA 4.1. *Under conditions (4.1), the bilinear form a_h defined by (4.2) is bounded on $H_h^1 \times H_h^1$. If in addition conditions (2.7) are fulfilled, then this form satisfies a discrete Gårding's inequality on H_h^1, i.e., there exist positive constants $\tilde{m}$ and $\tilde{\kappa}$ such that*

$$a_h(v, v) + \tilde{\kappa}\|v\|^2 \geq \tilde{m}\|v\|^2_{H_h^1}, \qquad \forall\, v \in H^1.$$

LEMMA 4.2. *Let p_{ij}^k, $\alpha^k > 0$ and β^k satisfy assumptions (4.1), let q^k satisfy assumption (2.7), and let conditions (3.2) be fulfilled. Then, for a sufficiently small mesh step h, there exist positive constants c_2 and c_3 such that*

$$c_2|[v]|^2_{H_h^1} \leq a_h(v, v) = [L_h v, v] \leq c_3|[v]|^2_{H_h^1}.$$

The proof is analogous to that of Lemma 3.1.

5. Error analysis and convergence rate estimate. Let $u = (u^1, u^2)$ be the solution of the boundary-value problem (BVP) (2.1)–(2.5), and let $v = (v^1, v^2)$ denote the solution of the FDS (4.4). The error $z = (z^1, z^2) = u - v$ satisfies the following conditions

$$L_h^k z = \psi^k, \quad x \in \bar{\omega}^k, \tag{5.1}$$

where

$$\psi^1 = \begin{cases} \displaystyle\sum_{i,j=1}^{2} \eta^1_{ij,\bar{x}_i} + \mu^1, & x \in \omega^1, \\ \frac{2}{h}\eta^1_{11} + \frac{2}{h}\eta^1_{12} + \tilde{\eta}^1_{21,\bar{x}_2} + \tilde{\eta}^1_{22,\bar{x}_2} + \frac{2}{h}\zeta^1 + \tilde{\mu}^1, & x \in \gamma^1_{11}, \\ \frac{2}{h}\left(\tilde{\eta}^1_{11} + \tilde{\eta}^1_{12} + \zeta^1_1 + \zeta^1_2\right) + \frac{2}{h}\left(\tilde{\eta}^1_{21} + \tilde{\eta}^1_{22} + \zeta^1_2\right) + \tilde{\tilde{\mu}}^1, & x = (a_1, c), \\ -\frac{2}{h}(\eta^1_{11})^{-1} - \frac{2}{h}(\eta^1_{12})^{-1} + \tilde{\eta}^1_{21,\bar{x}_2} + \tilde{\eta}^1_{22,\bar{x}_2} + \frac{2}{h}\zeta^1 + \frac{2}{h}\chi^1 + \tilde{\mu}^1, & x \in \gamma^1_{12}, \\ \frac{2}{h}\left[-(\tilde{\eta}^1_{11})^{-1} - (\tilde{\eta}^1_{12})^{-1} + \zeta^1_1 + \chi^1\right] + \frac{2}{h}\left(\tilde{\eta}^1_{21} + \tilde{\eta}^1_{22} + \zeta^1_2\right) + \tilde{\tilde{\mu}}^1, & x = (b_1, c), \\ \frac{2}{h}\eta^1_{21} + \frac{2}{h}\eta^1_{22} + \tilde{\eta}^1_{11,\bar{x}_1} + \tilde{\eta}^1_{12,\bar{x}_1} + \frac{2}{h}\zeta^1 + \tilde{\mu}^1, & x \in \gamma^1_{21}, \\ \frac{2}{h}\left[\tilde{\eta}^1_{11} + \tilde{\eta}^1_{12} + \zeta^1_1\right] + \frac{2}{h}\left(-(\tilde{\eta}^1_{21})^{-1} - (\tilde{\eta}^1_{22})^{-1} + \zeta^1_2\right) + \tilde{\tilde{\mu}}^1, & x = (a_1, d), \\ \frac{2}{h}\left[-(\eta^1_{21})^{-1} - (\eta^1_{22})^{-1} + \zeta^1\right] + \tilde{\eta}^1_{11,\bar{x}_1} + \tilde{\eta}^1_{12,\bar{x}_1} + \tilde{\mu}^1, & x \in \gamma^1_{22}, \\ \frac{2}{h}\left[-(\tilde{\eta}^1_{11})^{-1} - (\tilde{\eta}^1_{12})^{-1} + \zeta^1_1 + \chi^1\right] & \\ \quad + \frac{2}{h}\left[-(\tilde{\eta}^1_{21})^{-1} - (\tilde{\eta}^1_{22})^{-1} + \zeta^1_2\right] + \tilde{\tilde{\mu}}^1, & x = (b_1, d). \end{cases}$$

The mesh function ψ^2 is defined similarly. Furthermore,

$$\eta^k_{ij} = T^+_{ki}T^2_{k,3-i}\left(p^k_{ij}\frac{\partial u^k}{\partial x_j}\right) - \frac{1}{2}\left[p^k_{ij}u_{x_j} + (p^k_{ij})^{+i}(u^k_{\bar{x}_j})^{+i}\right], \qquad x \in \omega^k,$$

$$\tilde{\eta}^k_{ii} = T^+_{ki}T^{2\pm}_{k,3-i}\left(p^k_{ii}\frac{\partial u^k}{\partial x_i}\right) - \frac{p^k_{ii} + (p^k_{ii})^{+i}}{2}u^k_{x_i}, \qquad x \in \gamma^{k-}_{3-i,1}/x \in \gamma^{k-}_{3-i,2},$$

$$\tilde{\eta}^k_{i,3-i} = \begin{cases} T^+_{ki}T^{2+}_{k,3-i}\left(p^k_{i,3-i}\dfrac{\partial u^k}{\partial x_{3-i}}\right) - p^k_{i,3-i}u^k_{x_{3-i}}, & x \in \gamma^{k-}_{3-i,1}, \\ T^+_{ki}T^{2-}_{k,3-i}\left(p^k_{i,3-i}\dfrac{\partial u^k}{\partial x_{3-i}}\right) - (p^k_{i,3-i})^{+i}(u^k_{\bar{x}_{3-i}})^{+i}, & x \in \gamma^{k-}_{3-i,2}, \end{cases}$$

with

$$\begin{aligned}
\zeta^k &= (T_{ki}^2\alpha^k)u^k - T_{ki}^2(\alpha^k u^k), && x\in\gamma_{3-i,1}^k\cup\gamma_{3-i,2}^k,\\
\zeta_i^k &= (T_{ki}^{2\pm}\alpha^k)u^k - T_{ki}^{2\pm}(\alpha^k u^k), && x\in\gamma_\star^k,
\end{aligned}$$

and

$$\begin{aligned}
\mu^k &= (T_{ki}^2T_{k,3-i}^2q^k)u^k - T_{ki}^2T_{k,3-i}^2(q^k u^k), \qquad x\in\omega^k,\\
\tilde{\mu}^k &= (T_{k,3-i}^{2\pm}T_{ki}^2q^k)u^k - T_{k,3-i}^{2\pm}T_{ki}^2(q^k u^k), \qquad x\in\gamma_{3-i,1}^{k-}/x\in\gamma_{3-i,2}^{k-},\\
\tilde{\tilde{\mu}}^k &= (T_{k,3-i}^{2\pm}T_{ki}^{2\pm}q^k)u^k - T_{k,3-i}^{2\pm}T_{ki}^{2\pm}(q^k u^k), \qquad x\in\gamma_\star^k,\\
\chi^k &= \int_{\Gamma_{1k}^{3-k}} T_{k2}^2\beta^k(x,x')u^{3-k}(x')\mathrm{d}\Gamma_{1k}^{3-k} - h\sum_{x'\in\bar{\gamma}_{1k}^{3-k}} T_{k2}^2\beta^k(x,x')u^{3-k}(x')\\
&\quad -\frac{h}{2}\sum_{x'\in\bar{\gamma}_{1k}^{3-k}} T_{k2}^2\beta^k(x,x')u^{3-k}(x'), x\in\gamma_{1,3-k}^k,\\
\chi^k &= \int_{\Gamma_{1k}^{3-k}} T_{k2}^{2\pm}\beta^k(x,x')u^{3-k}(x')\,\mathrm{d}\Gamma_{1k}^{3-k} - h\sum_{x'\in\bar{\gamma}_{1k}^{3-k}} T_{k2}^{2\pm}\beta^k(x,x')u^{3-k}(x')\\
&\quad -\frac{h}{2}\sum_{x'\in\bar{\gamma}_{1k}^{3-k}} T_{k2}^{2\pm}\beta^k(x,x')u^{3-k}(x'), x\in\gamma_{1,3-k}^{k\star}.
\end{aligned}$$

We shall prove a suitable a priori estimate for the FDS (5.1). For this purpose, we need some auxiliary results.

LEMMA 5.1 ([13]). *The following inequality holds true:*

$$\left|[v^k, w_{x_{3-i}}^k)_{\gamma_{ij}^{k-}}\right| \le C|[v^k\|_{\ddot{H}^{1/2}(\gamma_{ij}^{k-})}|[w^k]|_{H^1(\bar{\omega}^k)}.$$

LEMMA 5.2 ([13]). *Let v^k be a mesh function on $\bar{\omega}^k$. Then,*

$$|[v^k]|_{C(\bar{\omega}^k)} \le C\sqrt{\log\frac{1}{h}}|[v^k]|_{H^1(\bar{\omega}^k)}.$$

Proof. We represent the function $v^k(x_1,x_2)$ in the form

$$v^k(x_1,x_2) = {\sum_{l=0}^{n_3}}' {\sum_{p=0}^{n_k}}' a_{pl}^k \cos\frac{p\pi x_1}{b_k-a_k}\cos\frac{l\pi x_2}{d-c} = {\sum_{l=0}^{n_3}}' C_l^k(x_1)\cos\frac{l\pi x_2}{d-c},$$

where

$${\sum_{l=0}^{n}}' b_l = b_0/2 + \sum_{l=1}^{n-1} b_l + b_n/2.$$

It follows immediately that

$$|v^k(x_1,x_2)| \le {\sum_{l=0}^{n_3}}' |C_l^k(x_1)| \le \left({\sum_{l=0}^{n_3}}' \sqrt{\lambda_l+1}(C_l^k(x_1))^2\right)^{1/2}\left({\sum_{l=0}^{n_3}}' \frac{1}{\sqrt{\lambda_l+1}}\right)^{1/2},$$

where $\lambda_l = \left(\frac{2}{h}\sin\frac{l\pi h}{2(d-c)}\right)^2$.

Further $\sqrt{\lambda_l+1} \geq \sqrt{\lambda_l} = \frac{2}{h}\sin\frac{l\pi h}{2(d-c)} \geq \frac{2l}{d-c}$ $(l=0,1,\ldots,n_3)$, from which

$$\begin{aligned}\sum_{l=0}^{n_3}{}' \frac{1}{\sqrt{\lambda_l+1}} &\leq \sum_{l=0}^{n_3} \frac{1}{\sqrt{\lambda_l+1}} = 1+\sum_{l=1}^{n_3}\frac{1}{\sqrt{\lambda_l+1}} \\ &\leq 1+\sum_{l=1}^{n_3}\frac{1}{\sqrt{\lambda_l}} \leq 1+\sum_{l=1}^{n_3}\frac{d-c}{2l},\end{aligned}$$

and

$$1+\sum_{l=1}^{n_3}\frac{d-c}{2l} \asymp \log\frac{n_3}{d-c} = \log\frac{1}{h}.$$

Let us now estimate the sum $S_l = \sum_{l=0}^{n_3}{}' \sqrt{\lambda_l+1}(C_l^k(x_1))^2$. Using the inequality [3]

$$\begin{aligned}\max_{x_1}\left|C^k(x_1)\right|^2 \leq & \varepsilon_l h \sum_{m=0}^{n_k-1}(C^k_{l,x_1}(a_k+mh))^2 \\ &+\left(\frac{1}{\varepsilon_l}+\frac{1}{b_k-a_k}\right) h \sum_{m=0}^{n_k}{}'(C_l^k(a_k+mh))^2,\end{aligned}$$

where $C^k(x_1)$ is a function defined on the mesh for $x_1 \in \{a_k, a_k+h, \ldots, a_k+hn_k\}$ and $\varepsilon > 0$, we obtain

$$\begin{aligned}\sum_{l=0}^{n_3}{}' \sqrt{\lambda_l+1}(C_l^k(x_1))^2 \leq \sum_{l=0}^{n_3}{}' \sqrt{\lambda_l+1}\Bigg\{&\varepsilon_l h \sum_{m=0}^{n_k-1}(C^k_{l,x_1}(a_k+mh))^2 \\ &+\left(\frac{1}{\varepsilon_l}+\frac{1}{b_k-a_k}\right) h \sum_{m=0}^{n_k}{}'(C_l^k(a_k+mh))^2\Bigg\}.\end{aligned}$$

We shall choose ε_l from the condition $\frac{1}{\varepsilon_l}+\frac{1}{b_k-a_k} = (\lambda_l+1)\varepsilon_l$. A quadratic equation with two solutions is obtained, from which we choose the positive one $\varepsilon_l = \frac{1+\sqrt{1+4(\lambda_l+1)(b_k-a_k)^2}}{2(\lambda_l+1)(b_k-a_k)}$. This leads to

$$\begin{aligned}S_l \leq \sum_{l=0}^{n_3}{}' \frac{1+\sqrt{1+4(\lambda_l+1)(b_k-a_k)^2}}{2\sqrt{\lambda_l+1}(b_k-a_k)}\Bigg\{&h\sum_{m=0}^{n_k-1}(C^k_{l,x_1}(a_k+mh))^2 \\ &+(\lambda_l+1)h\sum_{m=0}^{n_k}{}'(C_l^k(a_k+mh))^2\Bigg\}.\end{aligned}$$

Note that the quotient $\frac{1+\sqrt{1+4(\lambda_l+1)(b_k-a_k)^2}}{2\sqrt{\lambda_l+1}(b_k-a_k)}$ decreases when l increases, so that

$$\begin{aligned}\frac{1+\sqrt{1+4(\lambda_l+1)(b_k-a_k)^2}}{2\sqrt{\lambda_l+1}(b_k-a_k)} &\leq \frac{1+\sqrt{1+4(\lambda_0+1)(b_k-a_k)^2}}{2\sqrt{\lambda_0+1}(b_k-a_k)} \\ &= \frac{1+\sqrt{1+4(b_k-a_k)^2}}{2(b_k-a_k)}.\end{aligned}$$

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We therefore deduce that

$$\begin{aligned}\sum_{l=0}^{n_3}{}' \sqrt{\lambda_l+1}(C_l^k(x_1))^2 \leq \frac{1+\sqrt{1+4(b_k-a_k)^2}}{2(b_k-a_k)}&\sum_{l=0}^{n_3}{}'\Big\{h_k\sum_{m=0}^{n_k-1}(C_{l,x_1}^k(a_k+mh))^2\\ &+(\lambda_l+1)h\sum_{m=0}^{n_k}{}'(C_l^k(a_k+mh))^2\Big\}\\ =\frac{1+\sqrt{1+4(b_k-a_k)^2}}{(b_k-a_k)(d-c)}&\left(|[v_{x_1}^k\|_1^2+|[v_{x_2}^k\|_2^2+|[v^k]|^2\right)\\ =\frac{1+\sqrt{1+4(b_k-a_k)^2}}{(b_k-a_k)(d-c)}&|[v^k]|^2_{H^1(\bar\omega^k)}.\end{aligned}$$

The required result follows from the obtained inequalities. □

Let us rearrange the terms in the truncation error ψ in the form

$$\tilde\eta_{ij}^k=\eta_{ij}^k+\bar\eta_{ij}^k,\qquad \tilde\mu^k=\mu^k+\mu^{\star k},\qquad \tilde{\tilde\mu}^k=\mu^k+\mu^{\star\star k},$$

where

$$\begin{aligned}\bar\eta_{ii}^k&=\pm\frac{h}{3}T_{ki}^+\Big(\frac{\partial}{\partial x_{3-i}}\Big(p_{ii}^k\frac{\partial u^k}{\partial x_i}\Big)\Big),\qquad x\in\gamma_{3-i,1}^k/x\in\gamma_{3-i,2}^k,\\ \bar\eta_{i,3-i}^k&=\pm\frac{h}{3}T_{ki}^+\Big(\frac{\partial}{\partial x_{3-i}}\Big(p_{i,3-i}^k\frac{\partial u^k}{\partial x_{3-i}}\Big)\Big)\mp\frac{h}{2}T_{ki}^+\Big(p_{i,3-i}^k\frac{\partial^2u^k}{\partial x_{3-i}^2}\Big)\\ &\quad+\frac{h}{2}T_{ki}^+\Big(\frac{\partial}{\partial x_i}\Big(p_{i,3-i}^k\frac{\partial u^k}{\partial x_{3-i}}\Big)\Big),\qquad x\in\gamma_{3-i,1}^k/x\in\gamma_{3-i,2}^k,\\ \mu^{\star k}&=\pm\frac{h}{3}\Big(T_{k,3-i}^{2\pm}T_{ki}^2q^k\Big)\Big(T_{ki}^2\frac{\partial u^k}{\partial x_{3-i}}\Big),\qquad x\in\gamma_{3-i,1}^k/x\in\gamma_{3-i,2}^k,\\ \mu^{\star\star k}&=\pm\frac{h}{3}\Big(T_{ki}^{2\pm}T_{k,3-i}^{2\pm}q^k\Big)\Big(T_{k,3-i}^{2\pm}\frac{\partial u^k}{\partial x_i}\Big)\pm\frac{h}{3}\Big(T_{k,3-i}^{2\pm}T_{ki}^{2\pm}q^k\Big)\Big(T_{ki}^{2\pm}\frac{\partial u^k}{\partial x_{3-i}}\Big),\quad x\in\gamma_\star^k.\end{aligned}$$

THEOREM 5.3. *The finite difference scheme (5.1) is stable, in the sense that the following a priori estimate holds*

$$\begin{aligned}|[z]|_{H_h^1}\leq &C\sum_{k=1}^2\Bigg\{\sum_{i,j=1}^2\Big(|[\eta_{ij}^k\|_{k,i}+\|\zeta^k\|_{\gamma_{ij}^k}+h\|\mu^{\star k}\|_{\gamma_{ij}^k}\Big)\\ &+|[\mu^k]|_k+|[\chi^k]|_{\bar\gamma_{1,3-k}^k}+h\sum_{i,j,l=1}^2|[\bar\eta_{ij}^k\|_{\ddot H^{1/2}(\gamma_{3-i,l}^{k-})}\\ &+h\sqrt{\log\frac1h}\sum_{x\in\gamma_\star^k}\Big(\sum_{i=1}^2|\zeta_i^k|+h|\mu^{\star\star k}|\Big)\Bigg\}.\end{aligned}\tag{5.2}$$

Proof. Multiplying equation (5.1) by z and summing over the mesh, one obtains

$$[L_h^kz,z]_k=[\psi^k,z]_k.$$

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Let us set $k=1$. Then,

$$\begin{aligned}
[\psi^1,z^1]_1 =& -h^2\sum_{i=1}^{2}\sum_{x\in\omega^1\cup\gamma_{11}^1\cup\gamma_{21}^1}\Big(\eta_{i1}^1+\eta_{i2}^1\Big)z_{x_i}^1\\
&-\frac{h^2}{2}\sum_{i=1}^{2}\sum_{x\in\gamma_{3-i,1}^{1-}\cup\gamma_{3-i,2}^{1-}}\Big(\eta_{i1}^1+\eta_{i2}^1\Big)z_{x_i}^1\\
&-\frac{h^2}{2}\sum_{i=1}^{2}\sum_{x\in\gamma_{3-i,1}^{1-}\cup\gamma_{3-i,2}^{1-}}\Big(\bar\eta_{i1}^1+\bar\eta_{i2}^1\Big)z_{x_i}^1\\
&+h^2\sum_{x\in\omega^1}\mu^1z^1+\frac{h^2}{2}\sum_{x\in\gamma^1\setminus\gamma_\star^1}\mu^1z^1+\frac{h^2}{4}\sum_{x\in\gamma_\star^1}\mu^1z^1\\
&+\frac{h^2}{2}\sum_{x\in\gamma^1\setminus\gamma_\star^1}\mu^{\star1}z^1+\frac{h^2}{4}\sum_{x\in\gamma_\star^1}\mu^{\star\star1}z^1+h\sum_{x\in\gamma_{11}^1\cup\gamma_{12}^1}\zeta^1z^1\\
&+h\sum_{x\in\gamma_{21}^1\cup\gamma_{22}^1}\zeta^1z^1+\frac{h}{2}\sum_{x\in\gamma_\star^1}\zeta_1^1z^1+\frac{h}{2}\sum_{x\in\gamma_\star^1}\zeta_2^1z^1+[\chi^1,z^1]_{\bar\gamma_{12}^1}.
\end{aligned}$$

The following estimates hold:

$$\begin{aligned}
-h^2\sum_{i=1}^{2}\sum_{x\in\omega^1\cup\gamma_{11}^1\cup\gamma_{21}^1}\Big(\eta_{i1}^1+\eta_{i2}^1\Big)z_{x_i}^1-\frac{h^2}{2}\sum_{i=1}^{2}\sum_{x\in\gamma_{3-i,1}^{1-}\cup\gamma_{3-i,2}^{1-}}\Big(\eta_{i1}^1+\eta_{i2}^1\Big)z_{x_i}^1&\\
\le\sum_{i,j=1}^{2}|[\eta_{ij}^1\|_{1,i}|[z^1]|_{H^1(\bar\omega^1)},&\\
h^2\sum_{x\in\omega^1}\mu^1z^1+\frac{h^2}{2}\sum_{x\in\gamma^1\setminus\gamma_\star^1}\mu^1z^1+\frac{h^2}{4}\sum_{x\in\gamma_\star^1}\mu^1z^1\le|[\mu^1]|_k|[z^1]|_{H^1(\bar\omega^1)},&
\end{aligned}$$

and

$$\begin{aligned}
[\chi^1,z^1]_{\bar\gamma_{12}^1}=&h\sum_{\gamma_{12}^1}\chi^1z^1+\frac{h}{2}\sum_{\gamma_{12}^{1\star}}\chi^1z^1\\
\le&\Big(h\sum_{\gamma_{12}^1}\big(\chi^1\big)^2\Big)^{1/2}\Big(h\sum_{\gamma_{12}^1}\big(z^1\big)^2\Big)^{1/2}+\Big(\frac{h}{2}\sum_{\gamma_{12}^{1\star}}\big(\chi^1\big)^2\Big)^{1/2}\Big(\frac{h}{2}\sum_{\gamma_{12}^{1\star}}\big(z^1\big)^2\Big)^{1/2}\\
\le&\Big\{\Big(h\sum_{\gamma_{12}^1}\big(\chi^1\big)^2\Big)^{1/2}+\Big(\frac{h}{2}\sum_{\gamma_{12}^{1\star}}\big(\chi^1\big)^2\Big)^{1/2}\Big\}\\
&\times\Big\{\Big(h\sum_{\gamma_{12}^1}\big(z^1\big)^2\Big)^{1/2}+\Big(\frac{h}{2}\sum_{\gamma_{12}^{1\star}}\big(z^1\big)^2\Big)^{1/2}\Big\}\\
\le&4C_1\Big\{h\sum_{\gamma_{12}^1}\big(\chi^1\big)^2+\frac{h}{2}\sum_{\gamma_{12}^{1\star}}\big(\chi^1\big)^2\Big\}^{1/2}\Big\{h\sum_{\gamma_{12}^1}\big(z^1\big)^2+\frac{h}{2}\sum_{\gamma_{12}^{1\star}}\big(z^1\big)^2\Big\}^{1/2}\\
\le&C|[\chi^1]|_{\bar\gamma_{12}^1}|[z^1]|_{H^1(\bar\omega^1)}.
\end{aligned}$$

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Applying Lemma 5.2 we obtain

$$-\frac{h^2}{2}\sum_{i=1}^{2}\sum_{x\in\gamma_{3-i,1}^{1-}\cup\gamma_{3-i,2}^{1-}}\left(\bar\eta_{i1}^1+\bar\eta_{i2}^1\right)z_{x_i}^1 \le Ch[\bar\eta_{ij}^1, z_{x_i}^1)_{\gamma_{3-i,l}^{1-}}$$
$$\le Ch|[\bar\eta_{ij}^1\|_{\ddot H^{1/2}(\gamma_{3-i,l}^{1-})}|[z^1]|_{H^1(\bar\omega^1)}.$$

Furthermore, by applying Lemma 5.1,

$$\frac{h^2}{2}\sum_{x\in\gamma^1\setminus\gamma_\star^1}\mu^{\star 1}z^1+\frac{h^2}{4}\sum_{x\in\gamma_\star^1}\mu^{\star\star 1}z^1 \le 2Ch\|\mu^{\star 1}\|_{\gamma_{ij}^k}|[z^1]|_{H^1(\bar\omega^1)}$$
$$+Ch^2\sqrt{\log\frac{1}{h}}\sum_{x\in\gamma_\star^1}|\mu^{\star\star 1}||[z^1]|_{H^1(\bar\omega^1)},$$

and

$$h\sum_{x\in\gamma_{11}^1\cup\gamma_{12}^1}\zeta^1 z^1+h\sum_{x\in\gamma_{21}^1\cup\gamma_{22}^1}\zeta^1 z^1+\frac{h}{2}\sum_{x\in\gamma_\star^1}\zeta_1^1 z^1+\frac{h}{2}\sum_{x\in\gamma_\star^1}\zeta_2^1 z^1$$
$$\le 4C_1\|\zeta^1\|_{\gamma_{ij}^1}|[z^1]|_{H^1(\bar\omega^1)}+C\frac{h}{2}\sqrt{\log\frac{1}{h}}\sum_{x\in\gamma_\star^1}|\zeta_1^1||[z^1]|_{H^1(\bar\omega^1)}$$
$$+C\frac{h}{2}\sqrt{\log\frac{1}{h}}\sum_{x\in\gamma_\star^1}|\zeta_2^1||[z^1]|_{H^1(\bar\omega^1)}.$$

The proof is similar for $k=2$. ∎

THEOREM 5.4. *Let the assumptions of Lemma 4.2 hold. Then the solution of the FDS (4.3) converges to the solution of BVP (2.1)–(2.5), and the following convergence rate estimates hold*

$$(5.3)\quad \begin{aligned}|[u-v]|_{H_h^1} \le& Ch^{s-1}\sqrt{\log\frac{1}{h}}\Big(1+\max_{i,j,k}\|p_{ij}^k\|_{H^{s-1}(\Omega^k)}+\max_k\|q^k\|_{H^{s-2}(\Omega^k)}\\ &+\max_{i,j,k}\|\alpha^k\|_{H^{s-3/2}(\Gamma_{ij}^k)}+\max_k\|\beta^k\|_{H^{s-1}(\Gamma_{1,3-k}^k\times\Gamma_{1,k}^{3-k})}\Big)\|u\|_{H^s},\end{aligned}$$

for $2.5<s<3$, *and*

$$(5.4)\quad \begin{aligned}|[u-v]|_{H_h^1} \le& Ch^2\left(\log\frac{1}{h}\right)^{3/2}\Big(1+\max_{i,j,k}\|p_{ij}^k\|_{H^2(\Omega^k)}+\max_k\|q^k\|_{H^1(\Omega^k)}\\ &+\max_{i,j,k}\|\alpha^k\|_{H^{3/2}(\Gamma_{ij}^k)}+\max_k\|\beta^k\|_{H^2(\Gamma_{1,3-k}^k\times\Gamma_{1,k}^{3-k})}\Big)\|u\|_{H^3},\end{aligned}$$

when $s=3$.

Proof. The terms η_{ij}^k and μ^k at the internal nodes of the mesh $\bar\omega^k$ can be estimated in the same manner as in the case of the Dirichlet BVP [9, 14]

$$h^2\sum_{x\in\omega^k\cup\gamma_{i1}^k}\left(\eta_{ij}^k\right)^2 \le Ch^{2s-2}\|p_{ij}^k\|_{H^{s-1}(\Omega^k)}^2\|u^k\|_{H^s(\Omega^k)}^2,\quad 2<s\le 3.$$

An analogous result at the boundary nodes is obtained in [11]

$$h^2 \sum_{x\in\gamma^{k-}_{3-i,1}\cup\gamma^{k-}_{3-i,2}} \left(\eta^k_{ij}\right)^2 \le Ch^{2s-2}\|p^k_{ij}\|^2_{H^{s-1}(\Omega^k)}\|u^k\|^2_{H^s(\Omega^k)}, \quad 2.5 < s \le 3.$$

From these inequalities it follows that

$$|[\eta^k_{ij}\|_{k,i} \le Ch^{s-1}\|p^k_{ij}\|_{H^{s-1}(\Omega^k)}\|u^k\|_{H^s(\Omega^k)}, \quad 2.5 < s \le 3,$$

and

$$|[\mu^k\|_{k,i} \le Ch^{s-1}\|q^k\|_{H^{s-2}(\Omega^k)}\|u^k\|_{H^s(\Omega^k)}, \quad 2.5 < s \le 3.$$

Now it is necessary to estimate the term $\|\mu^{\star k}\|_{\gamma^k_{ij}}$. It is sufficient to derive an estimate for $i = j = 1$. For the other boundaries, the estimate can be obtained in the same way.

For $x = (a_k, x_2) \in \gamma^k_{11}$, we have

$$\begin{aligned}\mu^{\star k}(a_k, x_2) &= \frac{h}{3}\Big(T^{2+}_{k1}T^2_{k2}q^k\Big)\Big(T^2_2\frac{\partial u^k}{\partial x_1}\Big)\\ &= \frac{h}{3}\Bigg[\frac{2}{h}\int\limits_{a_k}^{a_k+h}\Big(1-\frac{x'_1-a_k}{h}\Big)\frac{1}{h}\int\limits_{x_2-h}^{x_2+h}\Big(1-\frac{|x_2-x'_2|}{h}\Big)q^k(x'_1,x'_2)\mathrm{d}x'_1\,\mathrm{d}x'_2\Bigg]\\ &\qquad\cdot\frac{1}{h}\int\limits_{x_2-h}^{x_2+h}\Big(1-\frac{|x_2-x'_2|}{h}\Big)\frac{\partial u^k}{\partial x_1}(x_1,x'_2)\,\mathrm{d}x'_2.\end{aligned}$$

Using the Cauchy–Schwarz inequality and bounding the last term by its maximum, we obtain

$$|\mu^{\star k}(a_k, x_2)| \le C\|q^k\|_{L^2((a_k,a_k+h)\times(x_2-h,x_2+h))}\|u\|_{C^1(\bar\Omega^k)}.$$

Summing over the mesh γ^k_{11} leads to

$$\|\mu^{\star k}\|_{\gamma^k_{11}} \le Ch^{1/2}\|q^k\|_{L^2((a_k,a_k+h)\times(c,d))}\|u^k\|_{C^1(\bar\Omega^k)}.$$

Using the known inequality [14]

$$\|f\|_{L^2(0,h)} \le Ch^{1/2}\|f\|_{H^r(0,1)}, \qquad 0 < h < 1, \qquad r > 1/2,$$

and the embedding theorem

$$\|u^k\|_{C^1(\bar\Omega^k)} \le C\|u^k\|_{H^s(\Omega^k)}, \qquad s > 2,$$

gives

$$\|\mu^{\star k}\|_{\gamma^k_{11}} \le Ch\|q^k\|_{H^r(\Omega^k)}\|u^k\|_{H^s(\Omega^k)}, \qquad r > 1/2, \qquad s > 2.$$

From this, by putting $r = s - 2$, we have

$$\|\mu^{\star k}\|_{\gamma^k_{ij}} \le Ch\|q^k\|_{W^{s-2}_2(\Omega^k)}\|u^k\|_{W^s_2(\Omega^k)}, \quad 2.5 < s \le 3. \tag{5.5}$$

In a similar way we can estimate the term $|\mu^{\star\star k}|$.

Next, we consider the lower left boundary point (a_k, c)

$$\mu^{\star\star k}(a_k,c)=\frac{h}{3}\Big(T_{k1}^{2+}T_{k2}^{2+}q^k\Big)\Big(T_{k2}^{2+}\frac{\partial u^k}{\partial x_1}\Big)+\frac{h}{3}\Big(T_{k1}^{2+}T_{k2}^{2+}q\Big)\Big(T_{k1}^{2+}\frac{\partial u^k}{\partial x_2}\Big).$$

and obtain, as in the previous case,

$$|\mu^{\star\star k}(a_k,c)|\le C\|q^k\|_{L^2((a_k,a_k+h)\times(c,c+h))}\|u^k\|_{C^1(\bar\Omega^k)}.$$

Further,

$$\|q^k\|_{L^2((a_k,a_k+h)\times(c,c+h))}\le\|q^k\|_{L^2(\Omega^k)}\le\|q^k\|_{H^{s-2}(\Omega^k)},\quad s>2,$$

and

$$\|u^k\|_{C^1(\bar\Omega^k)}\le C\|u^k\|_{H^s(\Omega^k)},\quad s>2.$$

Therefore,

$$|\mu^{\star\star k}|\le C\|q^k\|_{H^{s-2}(\Omega^k)}\|u\|_{H^s(\Omega^k)},\qquad x\in\gamma_\star^k,\quad s>2. \tag{5.6}$$

Let us now estimate the term

$$\zeta^k=(T_{ki}^2\alpha^k)u^k-T_{ki}^2(\alpha^k u^k),$$

which we will present in the form

$$\zeta^k=\zeta_{10}^k+\zeta_{20}^k,\qquad x\in\gamma_{3-i,1}^k\cup\gamma_{3-i,2}^k,$$

where

$$\zeta_{10}^k=(T_{ki}^2\alpha^k)u^k-(T_{ki}^2\alpha^k)(T_{ki}^2u^k)$$

and

$$\zeta_{20}^k=(T_{ki}^2\alpha^k)(T_{ki}^2u^k)-T_{ki}^2(\alpha^k u^k).$$

For the estimate of ζ_{10}^k, we will focus on the case $i=2$. Using the representation

$$\begin{aligned}\zeta_{10}^k(a_k,x_2)&=(T_{k2}^2\alpha^k)\bigg(u^k(a_k,x_2)-\frac{1}{h}\int\limits_{x_2-h}^{x_2+h}\Big(1-\frac{|x_2-x_2'|}{h}\Big)u^k(a_k,x_2')\,\mathrm{d}x_2'\bigg)\\&=(T_{k2}^2\alpha^k)\frac{1}{h}\int\limits_{x_2-h}^{x_2+h}\Big(1-\frac{|x_2-x_2'|}{h}\Big)(u^k(a_k,x_2)-u^k(a_k,x_2'))\,\mathrm{d}x_2'\\&=(T_{k2}^2\alpha^k)\frac{1}{h}\int\limits_{x_2-h}^{x_2+h}\Big(1-\frac{|x_2-x_2'|}{h}\Big)\int\limits_{x_2'}^{x_2}\frac{\partial u^k}{\partial x_2}(a_k,x_2'')\,\mathrm{d}x_2'\,\mathrm{d}x_2'',\end{aligned}$$

we can write

$$\begin{aligned}\big|\zeta_{10}^k(a_k,x_2)\big|&\le C\|\alpha^k\|_{C(\Gamma_{11}^k)}\int\limits_c^d\frac{\partial u^k}{\partial x_2}(a_k,x_2'')\,\mathrm{d}x_2'\,\mathrm{d}x_2''\le C\|\alpha^k\|_{C(\Gamma_{11}^k)}\Big\|\frac{\partial u^k}{\partial x_2}\Big\|_{L_2(c,d)}\\&\le Ch^{1/2}\|\alpha^k\|_{C(\Gamma_{11}^k)}\Big\|\frac{\partial u^k}{\partial x_2}\Big\|_{W_2^r(c,d)}\le Ch^{1/2}\|\alpha^k\|_{C(\Gamma_{11}^k)}\Big\|\frac{\partial u^k}{\partial x_2}\Big\|_{W_2^{r+1/2}(\Omega^k)}\\&\le Ch^{1/2}\|\alpha^k\|_{C(\Gamma_{11}^k)}\big\|u^k\big\|_{W_2^{r+1+1/2}(\Omega^k)}\le Ch^{1/2}\|\alpha^k\|_{C(\Gamma_{11}^k)}\big\|u^k\big\|_{W_2^{2r+1}(\Omega^k)},\end{aligned}$$

for $r > 1/2$. The embedding $W_2^{s-3/2} \subseteq \mathcal{C}$ holds for $s > 2.5$, $s = 2r+1$. Moreover,

$$\|\zeta_{10}^k\|_{\gamma_{11}^k} \leq Ch^{s-1}\|\alpha^k\|_{W_2^{s-3/2}(\Gamma_{11}^k)}\|u^k\|_{W_2^s(\Omega^k)}, \qquad 2.5 < s \leq 3,$$

and ζ_{20}^k is a bounded bilinear functional of $(\alpha^k, u^k) \in W_q^r(\Gamma_{3-i,j}^k) \times W^p_{\frac{2q}{q-2}}(\Gamma_{3-i,j}^k)$, which vanishes when u^k or α^k constants. Using the bilinear version of the Bramble–Hilbert Lemma [14], after summation over the mesh $\gamma_{3-i,j}^k$ we obtain

$$\|\zeta_{20}^k\|_{\gamma_{3-i,j}^k} \leq Ch^{r+p}\|\alpha^k\|_{W_q^r(\Gamma_{3-i,j}^k)}\|u^k\|_{W^p_{\frac{2q}{q-2}}(\Gamma_{3-i,j}^k)}, \quad 0 < r \leq 1,\ 0 < p \leq 1,\ q > 2.$$

For $0 \leq r \leq 1$ and $1 - \frac{1}{q} \leq p \leq 1$, the following embeddings hold

$$W_2^{r+\frac{1}{2}-\frac{1}{q}}(\Gamma_{3-i,j}^k) \subseteq W_q^r(\Gamma_{3-i,j}^k) \quad \text{and} \quad W_2^{p+\frac{1}{q}}(\Gamma_{3-i,j}^k) \subseteq W^p_{\frac{2q}{q-2}}(\Gamma_{3-i,j}^k).$$

It follows that

$$\|\zeta_{20}^k\|_{\gamma_{3-i,j}^k} \leq Ch^{r+p}\|\alpha^k\|_{W_2^{r+\frac{1}{2}-\frac{1}{q}}(\Gamma_{3-i,j}^k)}\|u^k\|_{W_2^{p+\frac{1}{q}}(\Gamma_{3-i,j}^k)}.$$

Using embeddings and trace theorems [18] we obtain

$$\begin{aligned}\|\zeta_{20}^k\|_{\gamma_{3-i,j}^k} &\leq Ch^{r+p}\|\alpha^k\|_{W_2^{r+\frac{1}{2}-\frac{1}{q}}(\Gamma_{3-i,j}^k)}\|u^k\|_{W_2^{p+\frac{1}{q}+\frac{1}{2}}(\Omega^k)} \\ &\leq Ch^{r+p}\|\alpha^k\|_{W_2^{r+p-\frac{1}{2}}(\Gamma_{3-i,j}^k)}\|u^k\|_{W_2^{p+r+1}(\Omega^k)}.\end{aligned}$$

Settings $r + p = s - 1$, produces

$$\|\zeta_{20}^k\|_{\gamma_{3-i,j}^k} \leq Ch^{s-1}\|\alpha^k\|_{W_2^{s-\frac{3}{2}}(\Gamma_{3-i,j}^k)}\|u^k\|_{W_2^s(\Omega^k)}, \quad 2 < s \leq 3,$$

and summing the inequalities for ζ_{10}^k and ζ_{20}^k leads to

$$\|\zeta^k\|_{\gamma_{3-i,j}^k} \leq Ch^{s-1}\|\alpha^k\|_{W_2^{s-3/2}(\Gamma_{3-i,j}^k)}\|u^k\|_{W_2^s(\Omega^k)}, \quad 2.5 < s \leq 3. \tag{5.7}$$

The term ζ_i^k can be represented as

$$\zeta_i^k = (T_{ki}^{2\pm}\alpha^k)u^k - T_{ki}^{2\pm}(\alpha^k u^k) = \zeta_{i1}^k + \zeta_{i2}^k, \qquad x \in \gamma_\star^k,$$

where

$$\zeta_{i1}^k = (T_{ki}^{2\pm}\alpha^k)(u^k - (T_{ki}^{2\pm}u^k))$$

and

$$\zeta_{i2}^k = (T_{ki}^{2\pm}\alpha^k)(T_{ki}^{2\pm}u^k) - T_{ki}^{2\pm}(\alpha^k u^k).$$

Let us now consider the lower left point. Then,

$$
\begin{aligned}
|\zeta_{i1}^k| &= \left| (T_{ki}^{2\pm}\alpha^k)\frac{2}{h}\int\limits_{a_k}^{a_k+h}\left(1-\frac{x_1'-a_k}{h}\right)\Big(u^k(a_k,c)-u^k(x_1',c)\Big)\,\mathrm{d}x_1'\right| \\
&= \left| (T_{ki}^{2\pm}\alpha^k)\frac{2}{h}\int\limits_{a_k}^{a_k+h}\left(1-\frac{x_1'-a_k}{h}\right)\int\limits_{a_k}^{x_1'}\frac{\partial u^k}{\partial x_1}(x_1'',c)\,\mathrm{d}x_1'\,\mathrm{d}x_1''\right| \\
&\le Ch^{1/2}\|\alpha^k\|_{C(\Gamma_{11}^k)}\left\|\frac{\partial u^k}{\partial x_1}\right\|_{L_2(a_k,a_k+h)} \le Ch\|\alpha^k\|_{C(\Gamma_{11}^k)}\left\|\frac{\partial u^k}{\partial x_1}\right\|_{W_2^r(a_k,b_k)} \\
&\le Ch\|\alpha^k\|_{C(\Gamma_{11}^k)}\left\|\frac{\partial u^k}{\partial x_1}\right\|_{W_2^{r+1/2}(\Omega^k)} \le Ch\|\alpha^k\|_{W_2^{2r-1/2}(\Gamma_{11}^k)}\|u^k\|_{W_2^{r+1/2+1}(\Omega^k)} \\
&\le Ch\|\alpha^k\|_{W_2^{2r-1/2}(\Gamma_{11}^k)}\|u^k\|_{W_2^{2r+1}(\Omega^k)}, \quad r>1/2,
\end{aligned}
$$

and, setting $2r+1=s$,

$$
|\zeta_{i1}^k| \le Ch\|\alpha^k\|_{W_2^{s-3/2}(\Gamma_{3-i,k}^k)}\|u^k\|_{W_2^s(\Omega^k)}, \quad s>2, \qquad x\in\gamma_\star^k.
$$

Now, ζ_{i2}^k is bounded bilinear functional of $(\alpha^k,u^k)\in W_q^r(\Gamma_{3-i,j}^k)\times W^p_{\frac{2q}{q-2}}(\Gamma_{3-i,j}^k)$ which vanishes when u^k or α^k are constants. Hence,

$$
|\zeta_{i2}^k| \le Ch\|\alpha^k\|_{W_2^{s-3/2}(\Gamma_{3-i,k}^k)}\|u^k\|_{W_2^s(\Omega^k)}, \quad s>2, \qquad x\in\gamma_\star^k,
$$

that is,

$$
|\zeta_{i}^k| \le Ch\|\alpha^k\|_{W_2^{s-3/2}(\Gamma_{3-i,k}^k)}\|u^k\|_{W_2^s(\Omega^k)}, \quad s>2, \qquad x\in\gamma_\star^k. \tag{5.8}
$$

The term χ^k can be estimated as follows. Let us denote

$$
I_1 = I_1(g) = \int_0^h g(x)\,\mathrm{d}x - \frac{h}{2}[g(0)+g(h)].
$$

For $r>0.5$, $I_1(g)$ is a bounded linear functional of $g\in H^r(0,h)$ which vanishes when $g(x)=1$ and $g(x)=x$. Using the Bramble–Hilbert Lemma [9] we obtain

$$
|I_1| \le Ch^{r+1/2}|g|_{H^r(0,h)}, \qquad 0.5<r\le 2,
$$

whereby it follows that

$$
\left|\int_0^1 g(x)\,\mathrm{d}x - h\left[\frac{g(0)}{2}+\sum_{i=1}^{n-1}g(ih)+\frac{g(1)}{2}\right]\right| \le Ch^r|g|_{H^r(0,1)}, \qquad 0.5<r\le 2.
$$

From this inequality, using properties of multipliers in Sobolev spaces [19], we immediately obtain that

$$
\begin{aligned}
|\chi^k(x)| &\le Ch^r\left\|T_{k2}^2\beta^k(x,\cdot)u^{3-k}(\cdot)\right\|_{H^r(\Gamma_{1k}^{3-k})} \\
&\le Ch^r\left\|T_{k2}^2\beta^k(x,\cdot)\right\|_{H^r(\Gamma_{1k}^{3-k})}\left\|u^{3-k}(\cdot)\right\|_{H^r(\Gamma_{1k}^{3-k})}, \qquad 1<r\le 2,
\end{aligned}
$$

when $x \in \gamma_{1,3-k}^{k}$, while for $x \in \gamma_{1,3-k}^{k\star}$ analogous inequalities hold. After summation over the mesh $\bar{\gamma}_{1,3-k}^{k}$ we obtain

$$|[\chi^k]|_{\bar{\gamma}_{1,3-k}^{k}} \le Ch^r \|\beta^k\|_{H^r(\Gamma_{1,3-k}^k \times \Gamma_{1,k}^{3-k})} \|u^{3-k}\|_{H^r(\Gamma_{1k}^{3-k})}, \qquad 1 < r \le 2.$$

Finally, using the trace theorem for anisotropic Sobolev spaces [18] and denoting $r = s - 1$,

$$|[\chi^k]|_{\bar{\gamma}_{1,3-k}^{k}} \le Ch^{s-1} \|\beta^k\|_{H^{s-1}(\Gamma_{1,3-k}^k \times \Gamma_{1,k}^{3-k})} \|u^{3-k}\|_{H^s(\Omega^{3-k})}, \qquad 2 < s \le 3.$$

Let us now estimate $\|\bar{\eta}_{ij}^k\|_{\ddot{H}^{1/2}(\gamma_{3-i,l}^{k-})}$. Using Lemma 4.6 from [11] we obtain

$$\begin{aligned} |\bar{\eta}_{ii}^k|_{W_2^{1/2}(\gamma_{3-i,k}^{k-})} &\le Ch^{r+1/2} \left| \left(\frac{\partial}{\partial x_{3-i}} \left(p_{ii}^k \frac{\partial u^k}{\partial x_i} \right) \right) \right|_{W_2^r(\Gamma_{3-i,k}^k)} \\ &\le Ch^{r+1/2} \left\| \left(\frac{\partial}{\partial x_{3-i}} \left(p_{ii}^k \frac{\partial u^k}{\partial x_i} \right) \right) \right\|_{W_2^{r+1/2}(\Omega^k)}, \qquad 0 < r \le 0.5. \end{aligned}$$

Using the inequality [21]

$$\|F\|_{L_2(0,\varepsilon)} \le C \begin{cases} \varepsilon^r \|F\|_{W_2^r(0,1)}, & 0 < r < 0.5, \\ \varepsilon^{1/2} \log \frac{1}{\varepsilon} \|F\|_{W_2^{1/2}(0,1)}, & r = 0.5, \\ \varepsilon^{1/2} \|F\|_{W_2^{1/2}(0,1)}, & r > 0.5, \end{cases}$$

where $0 < \varepsilon < 1$, we have

$$\begin{aligned} h \sum_{x \in \gamma_{3-i,k}^{k-}} & \left(\frac{1}{x_{3-i} + h/2} + \frac{1}{1 - x_{3-i} - h/2} \right) (\bar{\eta}_{ii}^k)^2 \\ &\le Ch^{2r+1} \log \frac{1}{h} \left\| \left(\frac{\partial}{\partial x_{3-i}} \left(p_{ii}^k \frac{\partial u^k}{\partial x_i} \right) \right) \right\|_{W_2^r(\Gamma_{3-i,k}^k)}^2 \\ &\le Ch^{2r+1} \log \frac{1}{h} \left\| \left(\frac{\partial}{\partial x_{3-i}} \left(p_{ii}^k \frac{\partial u^k}{\partial x_i} \right) \right) \right\|_{W_2^{r+1/2}(\Omega^k)}^2, \qquad 0 < r < 0.5, \end{aligned}$$

and

$$\begin{aligned} h \sum_{x \in \gamma_{3-i,k}^{k-}} & \left(\frac{1}{x_{3-i} + h/2} + \frac{1}{1 - x_{3-i} - h/2} \right) (\bar{\eta}_{ii}^k)^2 \\ &\le Ch^2 \log^3 \frac{1}{h} \left\| \left(\frac{\partial}{\partial x_{3-i}} \left(p_{ii}^k \frac{\partial u^k}{\partial x_i} \right) \right) \right\|_{W_2^1(\Omega^k)}^2. \end{aligned}$$

Setting $r + 2.5 = s$ and using properties of multipliers in Sobolev spaces, we immediately obtain that

$$\|\bar{\eta}_{ii}^k\|_{\ddot{W}_2^{1/2}(\gamma_{3-i,k}^{k-}))} \le Ch^{s-2} \sqrt{\log \tfrac{1}{h}} \|p_{ii}^k\|_{W_2^{s-1}(\Omega^k)} \|u^k\|_{W_2^s(\Omega^k)}, \tag{5.9}$$

for $2.5 < s < 3$, and

$$\|\bar{\eta}_{ii}^k\|_{\ddot{W}_2^{1/2}(\gamma_{3-i,k}^{k-}))} \le Ch \left(\log \tfrac{1}{h} \right)^{3/2} \|p_{ii}^k\|_{W_2^2(\Omega^k)} \|u\|_{W_2^3(\Omega^k)}. \tag{5.10}$$

This completes the proof of the theorem. ◻

Convergence rate estimates of the form

$$\|u-v\|_{H^m(\Omega_h)} \le C h^{s-m} \|u\|_{H^s(\Omega)}, \qquad m < s \le m+r,$$

are often called "compatible with the smoothness of the solution"; see, e.g., [9]. Here u is the solution of the boundary-value problem defined in the domain Ω, v is the solution of the corresponding finite difference scheme defined on the mesh $\Omega_h \subset \Omega$, h is the discretization parameter (mesh size), r is a given constant (the highest possible order of convergence), $H^s(\Omega)$ is a Sobolev space, and $H^m(\Omega_h)$ is a discrete Sobolev space of mesh functions. In such a manner, the error bounds obtained in Theorem 5.3 are compatible with the smoothness of the solution up to a slowly increasing logarithmic factor of the mesh size.

6. The case of non-coercive operator. Let us now consider the case when the coerciveness condition (3.2) is not satisfied. For the sake of simplicity we assume that

$$\beta^1(x_2, x_2') = \beta^2(x_2', x_2). \tag{6.1}$$

Hence, the operator L is selfadjoint. We deduce that the operator $L+\kappa I$ is selfadjoint and positive definite, so all its eigenvalues are real and positive and their only point of accumulation is $+\infty$. It follows that all eigenvalues of the operator L are real, larger than $-\kappa$, and their only point of accumulation is $+\infty$. Therefore, the operator L has a finite number of negative eigenvalues. Denoting the eigenvalues of L by λ_i, $i = 1, 2, \dots$, we conclude that there is an index l such that

$$-\kappa < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_l < 0 < \lambda_{l+1} \le \cdots .$$

The corresponding eigenvectors $u_i = (u_i^1, u_i^2)$ are orthogonal in the inner product $(\cdot,\cdot)_{L^2}$. Let us assume that 0 is not an eigenvalue of L and define the norm $\|u\|_{L+\kappa I} := ((L+\kappa I)u, u)^{1/2}$. Then, the inverse operator L^{-1} exists and applying it to the right-hand side of (2.1)

$$u = L^{-1} f,$$

one has

$$\|u\|_{L+\kappa I} = \|L^{-1} f\|_{L+\kappa I}.$$

Using Parseval's equality, we immediately obtain

$$\begin{aligned}\|u\|_{L+\kappa I} &= \left\{ \sum_{i=1}^{\infty} (\lambda_i+\kappa) \left(\frac{f_i}{\lambda_i}\right)^2 \right\}^{1/2} = \left\{ \sum_{i=1}^{\infty} \left(\frac{\lambda_i+\kappa}{\lambda_i}\right)^2 \frac{f_i^2}{\lambda_i+\kappa} \right\}^{1/2} \\ &\le \max_i \left|\frac{\lambda_i+\kappa}{\lambda_i}\right| \left(\sum_{i=1}^{\infty} \frac{f_i^2}{\lambda_i+\kappa} \right)^{1/2} = \max_i \left|\frac{\lambda_i+\kappa}{\lambda_i}\right| \|f\|_{(L+\kappa I)^{-1}},\end{aligned}$$

where the f_i are the Fourier coefficients of f: $f_i = (f, u_i)$, $i = 1, 2, \dots$.

Let us prove that the quotient $\left|\frac{\lambda_i+\kappa}{\lambda_i}\right|$ is bounded. Indeed, for $1 \le i \le l$ we have

$$\left|\frac{\lambda_i+\kappa}{\lambda_i}\right| = \frac{\lambda_i+\kappa}{|\lambda_i|} = \frac{\kappa}{|\lambda_i|} - 1 \le \frac{\kappa}{|\lambda_l|} - 1,$$

while for $i \ge l+1$

$$\left|\frac{\lambda_i+\kappa}{\lambda_i}\right| = \frac{\lambda_i+\kappa}{\lambda_i} = \frac{\kappa}{\lambda_i} + 1 \le \frac{\kappa}{\lambda_{l+1}} + 1.$$

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The previous relations yields the following a priori estimate

$$\|u\|_{L+\kappa I} \le c_4 \|f\|_{(L+\kappa I)^{-1}}, \tag{6.2}$$

where

$$c_4 = \max\left\{\frac{\kappa}{|\lambda_l|} - 1, \frac{\kappa}{\lambda_{l+1}} + 1\right\}.$$

Note that the constant c_4 behaves as $\frac{\text{const.}}{\min_i |\lambda_i|}$ and tends to infinity as the smallest eigenvalue in modulus approaches zero.

In the discrete case similar results are obtained. Therefore,

$$\|v\|_{L_h+\tilde{\kappa} I_h} \le c_5 \|\tilde{f}\|_{(L_h+\tilde{\kappa} I_h)^{-1}}, \tag{6.3}$$

where

$$c_5 = \max\left\{\frac{\tilde{\kappa}}{|\lambda_l^h|} - 1, \frac{\tilde{\kappa}}{\lambda_{l+1}^h} + 1\right\}.$$

For the operator $L_h + \tilde{\kappa} I_h$ the corresponding bilinear form is

$$\tilde{a}_h(v, w) = [(L_h + \tilde{\kappa} I_h)v, w] = a_h(v, w) + \tilde{\kappa}[v, w].$$

Equation (4.3) and Lemma 3.1 imply

$$c_6 |[v]|^2_{H_h^1} \le |[v]|^2_{L_h+\tilde{\kappa} I_h} \le c_7 |[v]|^2_{H_h^1}, \tag{6.4}$$

where $c_6 = \tilde{m}$ i $c_7 = c_3 + \tilde{\kappa}$.

Applying (6.3) to (5.1) and using (6.4) we can write

$$|[z]|_{H_h^1} \le \frac{c_5}{\sqrt{c_6}} |[\psi]|_{(L_h+\tilde{\kappa} I_h)^{-1}} = \frac{c_5}{\sqrt{c_6}} \sup_{w \ne 0} \frac{[\psi, w]}{|[w]|_{L_h+\tilde{\kappa} I_h}},$$

whereby, in the same manner as in the proof of Theorem 5.3, one obtains an a priori estimate of the form (5.2). Notice that the constant C in this estimate now depends on c_5.

In such a manner, we have proved the following assertion.

THEOREM 6.1. *Let the conditions* (4.1) *and* (6.1) *hold and let 0 neither be an eigenvalue of the problem* (2.1) *nor of the difference problem* (4.3)*. Then, the solution of the finite difference scheme* (4.3) *converges to the solution of the boundary-value problem* (2.1)–(2.5) *and the error bounds* (5.3) *and* (5.4) *hold.*

7. Numerical example. The following numerical experiment aims at assessing the accuracy of the finite difference. The test example is the problem (2.1)–(2.5) with $p_{ij}^k = 1$, $q^k = 0$, $\Omega^1 = (-3, -1) \times (-3, 3)$, and $\Omega^2 = (1, 3) \times (-3, 3)$:

$$\begin{aligned}
-\frac{\partial^2 u^1}{\partial x_1^2} - \frac{\partial^2 u^1}{\partial x_2^2} &= 2(9 - x_1^2) + 2(9 - x_2^2), \qquad x \in \Omega^1,\\
-\frac{\partial^2 u^2}{\partial x_1^2} - \frac{\partial^2 u^2}{\partial x_2^2} &= 2(9 - x_1^2) + 2(9 - x_2^2), \qquad x \in \Omega^2,\\
\alpha^k = 2, \qquad \beta^k(x_2, x_2') &= \frac{1}{16}(9 - x_2^2), \qquad k = 1, 2,
\end{aligned}$$

TABLE 7.1
The experimental error results and the temporal convergence orders.

h	$\|e(h)\|_{C(\bar\omega_h)}$	$R_{C(\bar\omega_h)}$	$\|e(h)\|_{L^2(\bar\omega_h)}$	$R_{L^2(\bar\omega_h)}$	$\|e(h)\|_{H^1(\bar\omega_h)}$	$R_{H^1(\bar\omega_h)}$
2^{-1}	2.5149e+00	1.95	5.8566e+00	2.08	6.4597e+00	2.18
2^{-2}	6.5092e-01	1.98	1.3813e+00	2.05	1.4234e+00	2.09
2^{-3}	1.6419e-01	1.99	3.3148e-01	2.03	3.3423e-01	2.04
2^{-4}	4.1140e-02	1.99	8.0938e-02	2.01	8.1113e-02	2.02
2^{-5}	1.0291e-02	1.99	1.9981e-02	2.00	1.9992e-02	2.01
2^{-6}	2.5731e-03	1.99	4.9629e-03	2.00	4.9636e-03	2.00
2^{-7}	6.4330e-04		1.2366e-03		1.2367e-03	

with exact solution

$$u(x,y) = (9 - x_1^2)(9 - x_2^2), \qquad x \in \Omega^1 \cup \Omega^2.$$

The problem (2.1)–(2.5) is approximated by the finite-difference scheme (4.3). Since the input data are smooth functions, Steklov averaging is not applied. The errors were estimated in the norms $\|\cdot\|_{\mathcal{C}(\bar\omega_h)}$, $\|\cdot\|_{L^2(\bar\omega_h)}$, and $\|\cdot\|_{H^1(\bar\omega_h)}$. The convergence order was estimated in the spaces $\mathcal{C}(\bar\omega_h)$, $L^2(\bar\omega_h)$, and $H^1(\bar\omega_h)$. The numerical experiment confirms the theoretical results; see Table 7.1 and Figure 7.1.

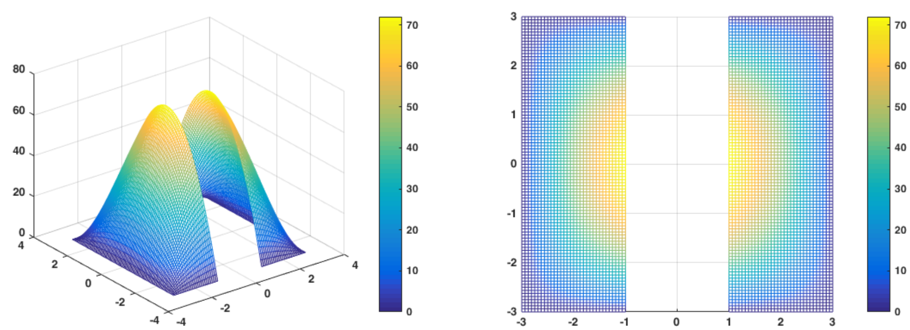

FIG. 7.1. *Graphics of the approximation with step $h = 2^{-4}$.*

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