ON THE NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS ON EQUISPACED NODES[*]

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Dedicated to Lothar Reichel on the occasion of his 70th birthday.

Abstract. In the present paper, a Nyström-type method for second kind Volterra integral equations is introduced and studied. The method makes use of generalized Bernstein polynomials, defined for continuous functions and based on equally spaced points. Stability and convergence are studied in the space of continuous functions. Numerical tests illustrate the performance of the proposed approach.

Key words. Volterra integral equations, Nyström method, generalized Bernstein polynomials

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1. Introduction. This paper is concerned with the numerical treatment of the following kind of Volterra integral equations

$$(1.1) \qquad f(s) + \mu \int_0^s k(t,s) f(t)(s-t)^\alpha t^\beta dt = g(s), \quad s \in (0,1],$$

where f is the unknown function, g is defined on $[0,1]$, and the kernel k is defined on $\mathcal{D} = \{(t,s) : 0 < t < s \leq 1\}$, $\alpha, \beta > -1$, and $\mu \in \mathbb{R}$.

Several mathematical models arising in elasticity, scattering theory, seismology, heat conduction, fluid flow, chemical reactions, population dynamics, semi-conductors, and in other fields, involve Volterra equations of the type (1.1); see, e.g., [3, 6, 7, 8, 15, 20, 23, 30].

In light of such applications, a variety of numerical methods [4, 5, 9, 13, 17, 21, 31, 32, 33, 34] have been developed to approximate the solution f in suitable spaces both in the case when the kernel is sufficiently smooth and when it is weakly singular. In order for treating the case of functions presenting algebraic singularities at the endpoints of the interval $[0,1]$ and/or on the boundary of $\mathcal{D}$, weighted global approximation methods have been recently introduced and studied in [10, 14]. However, some of these efficient numerical approaches require the evaluation of the functions g and k at the zeros of orthogonal polynomials. Typically, the right-hand side g is known at a set of equispaced data points, provided by an instrument, and the kernel k, usually representing the response of the experimental equipment, is also known on uniform grids. Hence, in such cases, the mentioned methods are not reliable. On the other hand, methods based on piecewise polynomials can be used, but they produce a low degree of approximation or, more generally, show saturation phenomena.

In this paper, we propose a Nyström-type method based on the ℓ-th iterated Boolean sum of the classical Bernstein operator B_m [12, 22, 24]. For a fixed integer $\ell \in \mathbb{N}$,

$$B_{m,\ell} := I - (I - B_m)^\ell$$

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maps continuous functions f to polynomials of degree m, and $\{B_{m,\ell}(f)\}_m$ are the so-called generalized Bernstein polynomials of parameter ℓ. Like the classical polynomials $B_m(f)$, they require samples of f at $m + 1$ equidistant points and interpolate f at the endpoints of the interval. The main property of these operators, due to the presence of the parameter ℓ, is the following: as $m \to \infty$, $B_{m,\ell}(f)$ converges uniformly to f with a higher order of convergence than the classical Bernstein polynomial sequence. In particular, the saturation order is $\mathcal{O}(m^{-\ell})$ [22, 24] and the rate of convergence improves as the smoothness of f increases [16]. Here, we develop a Nyström method by employing generalized Bernstein polynomials to approximate the Volterra operator. Hence, we prove stability and convergence of the method in the space of continuous functions, and determine error estimates in suitable Zygmund-type subspaces. Furthermore, the theoretical error estimates are corroborated by means of several examples, exploiting the use of the parameter ℓ to speed up the rate of convergence. We point out that an analogous approach was proposed for Fredholm integral equations in [25]; see also [26].

The paper is organized as follows. In Section 2, we define the spaces in which we look for the solution of (1.1), we introduce the generalized Bernstein operators, and state their properties and convergence results. In Section 3, we propose a quadrature rule based on such polynomials and we give an estimate of the convergence error according to the smoothness properties of the kernel function. In Section 4, we present the Nyström method, whose numerical performance is demonstrated in Section 5 through four examples. Proofs of the main results are given in Section 6.

2. Preliminaries. In the sequel, $\mathcal{C}$ denotes a positive constant having different meanings in different formulas. We write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots$, and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. If $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a constant $\mathcal{C} \neq \mathcal{C}(A, B)$ such that $\mathcal{C}^{-1}B \leq A \leq \mathcal{C}B$.

For any bivariate function $h(x, y)$, we denote by h_x (resp. h_y) the function h seen as an univariate function in the variable y (resp. x).

For a given integer m, we use the notation $N_0^m := \{0, 1, 2, \ldots, m\}$.

2.1. Function spaces. Let $C^0 := C^0([0, 1])$ be the space of all continuous functions on $[0, 1]$ equipped with the uniform norm

$$\|f\| = \sup_{x \in [0,1]} |f(x)|,$$

and for each $f \in C^0$, let us introduce the r-th Ditzian-Totik modulus of smoothness [11]

$$\omega_\varphi^r(f, t) = \sup_{0 < h \leq t} \|\Delta_{h\varphi}^r f\|, \qquad r \in \mathbb{N},$$

where $\varphi(x) = \sqrt{x(1-x)}$ and $\Delta_{h\varphi}^r$ denotes the finite differences of order r with variable step size given by

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + (r - 2k)\frac{h}{2}\varphi(x)\right).$$

Let us note that, because of the variability of the step size, the above finite differences cannot be defined through a recurrence relation; see [11, pp. 37–38].

It is well-known that the modulus of smoothness is connected to the error of best polynomial approximation $E_m(f)$ of $f \in C^0$, defined as

$$E_m(f) = \inf_{P \in \mathbb{P}_m} \|f - P\|,$$

where $\mathbb{P}_m$ is the set of all algebraic polynomials of degree at most m.

Indeed, one has [11, Theorem 7.2.1 and Theorem 7.2.4]

$$E_m(f) \leq \mathcal{C}\, \omega_\varphi^r\left(f, \frac{1}{m}\right), \quad \forall r < m, \quad \mathcal{C} \neq \mathcal{C}(m,f),$$

and

$$\omega_\varphi^r(f,t) \leq \mathcal{C}\, t^r \sum_{k=0}^{[\frac{1}{t}]} (1+k)^{r-1} E_k(f), \quad \mathcal{C} \neq \mathcal{C}(t,f).$$

For our aims, let us also define the Hölder-Zygmund-type space as

$$Z_\lambda = \left\{ f \in C^0 : \sup_{t>0} \frac{\omega_\varphi^r(f,t)}{t^\lambda} < \infty, \, r > \lambda \right\},$$

endowed with the norm

$$\|f\|_{Z_\lambda} = \|f\| + \sup_{t>0} \frac{\omega_\varphi^r(f,t)}{t^\lambda}.$$

It is proved in [11, Theorem 2.1] that for each $\lambda > 0$ one has

$$\|f\|_{Z_\lambda} \sim \|f\| + \sup_{m>0}(m+1)^\lambda E_m(f),$$

from which we deduce the following relations that characterize any function $f \in Z_\lambda$

$$f \in Z_\lambda \iff E_m(f) = \mathcal{O}\left(\frac{1}{m^\lambda}\right),$$

and

(2.1) $$\omega_\varphi^r(f,t) \leq \mathcal{C}\, t^\lambda \|f\|_{Z_\lambda}, \quad \mathcal{C} \neq \mathcal{C}(f,t).$$

2.2. Generalized Bernstein polynomials. In this paragraph, we recall the definition and main properties of the so-called generalized Bernstein operators, introduced and studied in [12, 22, 24, 26]. In the following, we will refer to them as GB operators.

For any $f \in C^0$, the m-th Bernstein polynomial $B_m(f)$ is defined as

$$B_m(f,x) = \sum_{k=0}^m p_{m,k}(x) f\left(\frac{k}{m}\right), \qquad x \in [0,1],$$

where

(2.2) $$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \qquad k \in N_0^m,$$

satisfying the recurrence relation

$$p_{m,k}(x) = (1-x) p_{m-1,k}(x) + x\, p_{m-1,k-1}(x).$$

Then, for a given $\ell \in \mathbb{N}$, the operator $B_{m,\ell} : C^0 \to \mathbb{P}_m$ is defined as

(2.3) $$B_{m,\ell} = I - (I - B_m)^\ell,$$

where I is the identity on C^0 and B_m^ℓ the ℓ-th iterate of the Bernstein operator B_m, i.e.,

$$\begin{cases} B_m^0 := I \\ B_m^\ell := B_m(B_m^{\ell-1}), \quad \ell > 0. \end{cases}$$

Obviously, from (2.3) it follows that the polynomial $B_{m,1}(f) = B_m(f)$ and, for each fixed $\ell \geq 1$, one has

$$(2.4) \qquad B_{m,\ell}(f, x) = \sum_{j=0}^m p_{m,j}^{(\ell)}(x) f\left(\frac{j}{m}\right), \quad x \in [0, 1],$$

where

$$p_{m,j}^{(\ell)}(x) = \sum_{i=1}^\ell \binom{\ell}{i} (-1)^{i-1} B_m^{i-1}(p_{m,j}, x)$$

is the *fundamental GB polynomial* of degree m. Such polynomials provide a partition of the unity, i.e.,

$$\sum_{j=0}^m p_{m,j}^{(\ell)}(x) = 1, \quad \forall x \in [0, 1].$$

Moreover, for any $f \in C^0$,

$$B_{m,\ell}(f, 0) = f(0), \quad B_{m,\ell}(f, 1) = f(1).$$

For m fixed and $\ell \to \infty$, the sequence $\{B_{m,\ell}(f)\}_{\ell \geq 0}$ converges uniformly to the Lagrange polynomial $L_m(f)$, interpolating f at the nodes $\{t_k := \frac{k}{m}\}_{k=0}^m$, i.e.,

$$\lim_{\ell \to \infty} \|B_{m,\ell}(f) - L_m(f)\| = 0,$$

where

$$L_m(f, x) = \sum_{k=0}^m l_{m,k}(x) f(t_k), \quad l_{m,k}(t_i) = \delta_{ki},$$

and δ_{ki} is the Kronecker delta.

For an efficient computation of the polynomials $B_{m,\ell}$, it is useful to note that setting

$$\mathbf{p}_m^{(\ell)}(x) := [p_{m,0}^{(\ell)}(x), p_{m,1}^{(\ell)}(x), \dots, p_{m,m}^{(\ell)}(x)]^T,$$

and

$$\mathbf{p}_m(x) := [p_{m,0}(x), p_{m,1}(x), \dots, p_{m,m}(x)]^T,$$

the following vectorial expression holds true [27]

$$(2.5) \qquad \mathbf{p}_m^{(\ell)}(x)^T = \mathbf{p}_m(x)^T C_{m,\ell}.$$

The matrix $C_{m,\ell} \in \mathbb{R}^{(m+1) \times (m+1)}$ is defined as

$$\begin{aligned} C_{m,\ell} &= \mathbf{I} + (\mathbf{I} - \mathbf{A}) + \dots + (\mathbf{I} - \mathbf{A})^{\ell-1} \\ &= \mathbf{A}^{-1}[\mathbf{I} - (\mathbf{I} - \mathbf{A})^\ell] = [\mathbf{I} - (\mathbf{I} - \mathbf{A})^\ell]\mathbf{A}^{-1}, \end{aligned}$$

where $\mathbf{I}$ is the identity matrix of order $m + 1$ and $\mathbf{A} \in \mathbb{R}^{(m+1)\times(m+1)}$ is the matrix with entries

$$a_{i,j} = p_{m,j-1}(t_{i-1}), \quad i, j = 1, \ldots, m + 1.$$

Moreover, for $\ell = 2^p$ with $p \in \mathbb{N}$, the following relation holds true [27]

$$(2.6) \qquad C_{m,2^p} = C_{m,2^{p-1}} + (\mathbf{I} - \mathbf{A})^{2^{p-1}} C_{m,2^{p-1}},$$

from which one can deduce

$$(2.7) \qquad B_{m,2^p}(f, x) = 2B_{m,2^{p-1}}(f, x) - B_{m,2^{p-1}}^2(f, x).$$

REMARK 2.1. Equation (2.7) is useful for the fast computation of the subsequence $\{B_{m,2^p}\}_p$, for m fixed. Let us also remark that $\mathbf{A} = [a_{ij}]_{i,j=1}^{m+1}$ is a centrosymmetric matrix, i.e., $a_{ij} = a_{m+2-i,m+2-j}$ or, equivalently, $\mathbf{A} = \mathbf{JAJ}$, where $\mathbf{J}$ is the reversal matrix of size $m + 1$; see [18, pp. 33–36]. Hence, its construction requires the computation of $\frac{m^2}{4}$ elements. Moreover, as each entry is the evaluation of the polynomial $p_{m,j-1}(t_{i-1})$, which requires m floating point operations (flops), one need a total of $\frac{m^3}{4}$ flops; see [1, 25, 29] for a deeper description. According to (2.6), the construction of $C_{m,\ell}$, with $\ell = 2^p$, requires $2(\log_2 \ell - 1)$ products of centrosymmetric matrices which again leads to a centrosymmetric matrix. Therefore, for each fixed m, the computational cost of computing $C_{m,\ell}$ is $\mathcal{O}((\log_2 \ell - 1)\frac{m^3}{2})$ flops.

Now, we recall the following result about the error obtained when approximating $f \in C^0$ by $B_{m,\ell}(f)$, as $m \to \infty$ and $\ell \in \mathbb{N}$.

THEOREM 2.2. *[16, Theorem 1 and Corollary] Let $\ell \in \mathbb{N}$ be fixed. Then, for all $m \in \mathbb{N}$ and for any $f \in C^0$, we have*

$$\|f - B_{m,\ell}(f)\| \leq \mathcal{C} \left\{ \omega_\varphi^{2\ell} \left(f, \frac{1}{\sqrt{m}} \right) + \frac{\|f\|}{m^\ell} \right\}, \quad \mathcal{C} \neq \mathcal{C}(m, f).$$

Moreover, for any $0 < \lambda \leq 2\ell$, we obtain for $m \to \infty$

$$\|f - B_{m,\ell}(f)\| = \mathcal{O} \left(\frac{1}{\sqrt{m^\lambda}} \right) \quad \Longleftrightarrow \quad \omega_\varphi^{2\ell}(f, t) = \mathcal{O}(t^\lambda).$$

REMARK 2.3. Note that, differently from the classical Bernstein operator B_m, the Boolean sums $B_{m,\ell}$ may accelerate the speed of convergence of the approximation error, as the smoothness of f increases. In particular, from Theorem 2.2, taking into account (2.1), we deduce that for each $f \in Z_\lambda$, with $\lambda \leq 2\ell$, we have

$$\|f - B_{m,\ell}(f)\| \leq \frac{\mathcal{C}}{\sqrt{m^\lambda}} \|f\|_{Z_\lambda},$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

3. On the approximation of the Volterra integral operator. Let $\mathcal{V} : C^0 \to C^0$ be the linear Volterra integral operator of equation (1.1) defined by

$$(3.1) \qquad (\mathcal{V}f)(s) = \int_0^s k(t, s)f(t)(s - t)^\alpha t^\beta \, dt,$$

where $\alpha, \beta > -1$ and the function $k(t, s)$ is continuous on $\mathcal{D} = \{(t, s) : 0 < t < s \leq 1\}$.

In order for providing an approximation for $\mathcal{V}$, let us express the function $k(t,s)f(t)$ in terms of the fundamental GB polynomials through (2.4), i.e.,

$$B_{m,\ell}(k_s f, t) = \sum_{j=0}^{m} p_{m,j}^{(\ell)}(t)\, k_s\,(t_j)\, f\,(t_j)\,, \quad t_j = \frac{j}{m}.$$

Then, for each fixed $\ell \in \mathbb{N}$, we introduce the sequence $\{\mathcal{V}_m^{(\ell)} f\}_m$ defined as

(3.2)
$$(\mathcal{V}_m^{(\ell)} f)(s) = \sum_{j=0}^{m} Q_j^{(\ell)}(s)\, k_s(t_j) f(t_j),$$

where

$$Q_j^{(\ell)}(s) = \int_0^s p_{m,j}^{(\ell)}(t)(s-t)^{\alpha} t^{\beta}\, dt.$$

Now, introducing the change of variable $t = sz$, by virtue of (2.5) we have

$$
\begin{aligned}
Q_j^{(\ell)}(s) &= s^{\alpha+\beta+1} \int_0^1 p_{m,j}^{(\ell)}\,(zs)\,(1-z)^{\alpha} z^{\beta} dz \\
&= s^{\alpha+\beta+1} \sum_{r=0}^{m} (C_{m,\ell})_{r,j} \left(\int_0^1 p_{m,r}\,(zs)\,(1-z)^{\alpha} z^{\beta} dz \right).
\end{aligned}
$$

We may compute analytically the above integrals by using definition (2.2). However, this would require, for each fixed r, the evaluation of the regularized hypergeometric function and the computation of binomial coefficients, which is expensive. Indeed, one has

$$
\begin{aligned}
\int_0^1 p_{m,r}\,(zs)\,(1-z)^{\alpha} z^{\beta} dz &= \binom{m}{r} s^r \Gamma(1+\alpha)\Gamma(1+\beta+r) \\
&\qquad {}_2\tilde{F}_1(1+\beta+r, r-m, 2+\alpha+\beta+r, s) \\
&= \binom{m}{r} s^r B(1+\alpha, 1+\beta+r) \\
&\qquad {}_2F_1(1+\beta+r, r-m, 2+\alpha+\beta+r, s),
\end{aligned}
$$

where Γ is the Gamma function, ${}_2\tilde{F}_1$ denotes the regularized hypergeometric function, B is the Beta function, and ${}_2F_1$ is the classical hypergeometric function.

To reduce the complexity, we propose to use a Gauss-Jacobi quadrature rule based on the zeros of the Jacobi polynomial $p_n^{\alpha,\beta}$, orthonormal with respect to the weight $(1-z)^{\alpha} z^{\beta}$. Hence, denoting by $\{x_k^{\alpha,\beta}\}_{k=1}^{n}$ the zeros of $p_n^{\alpha,\beta}$ and by $\{\lambda_k^{\alpha,\beta}\}_{k=1}^{n}$ the corresponding Christoffel numbers, choosing $n = \left[\frac{m+2}{2}\right]$ we have

$$\int_0^1 p_{m,r}\,(zs)\,(1-z)^{\alpha} z^{\beta} dz = \sum_{k=1}^{n} \lambda_k^{\alpha,\beta} p_{m,r}\left(x_k^{\alpha,\beta} s\right),$$

since the quadrature rule is exact. Consequently,

(3.3)
$$Q_j^{(\ell)}(s) = s^{\alpha+\beta+1} \sum_{r=0}^{m} (C_{m,\ell})_{r,j} \sum_{k=1}^{n} \lambda_k^{\alpha,\beta} p_{m,r}\left(x_k^{\alpha,\beta} s\right).$$

In the next theorem, we prove that for any $f \in C^0$ the sequence $\mathcal{V}_m^{(\ell)} f$ converges uniformly to $\mathcal{V}f$ as $m \to \infty$, for each fixed ℓ, also providing an estimate of the error.

THEOREM 3.1. *Let* $f \in Z_\lambda$ *and* $\sup\limits_{s \in [0,1]} \|k_s\|_{Z_\lambda} < \infty$. *Assuming* $\alpha + \beta + 1 \geq 0$, *then*

$$(3.4) \qquad \|(\mathcal{V} - \mathcal{V}_m^{(\ell)})f\| \leq \mathcal{C} \sup_{s \in [0,1]} \|k_s\|_{Z_\lambda} \left[\frac{1}{(\sqrt{m})^\lambda} + \frac{1}{m^\ell} \right] \|f\|_{Z_\lambda},$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

We present now two numerical tests which confirm the theoretical estimate (3.4).

EXAMPLE 3.2. Let us consider the integral

$$(\mathcal{V}f)(s) = \int_0^s \sin(st)(s-t)^{\frac{1}{4}} t^{\frac{1}{4}} \, dt,$$

which is of the form (3.1) with $k(t, s) = \sin(st)$, $f \equiv 1$, $\alpha = \beta = 1/4$. In Table 3.1, we report the absolute errors

$$(3.5) \qquad e_m^{(\ell)}(s) = |\mathcal{V}f(s) - \mathcal{V}_m^{(\ell)} f(s)|,$$

for fixed values of the parameter $\ell = 16, 64, 128, 256$. Here, $\mathcal{V}_m^{(\ell)}$ is given by (3.2), with $f \equiv 1$. As we can see, for increasing values of m and at different points s, the proposed approximation (3.2) converges very fast to the exact value of the integral, since the function k is very smooth. Considering the error for $m = 64$, and varying the parameter ℓ, we deduce that the convergence is fast also with respect to ℓ; see Table 3.2.

TABLE 3.1
Errors for Example 3.2 with $\ell = 16, 64, 128, 256$.

m	$e_m^{(16)}(0.3)$	$e_m^{(16)}(0.6)$	$e_m^{(16)}(0.8)$	$e_m^{(64)}(0.3)$	$e_m^{(64)}(0.6)$	$e_m^{(64)}(0.8)$
4	1.40e-08	1.18e-07	3.57e-07	3.78e-09	9.94e-08	6.43e-07
8	3.40e-11	7.15e-10	9.00e-09	2.44e-14	6.15e-12	5.34e-11
16	1.46e-14	1.10e-12	4.83e-12	1.65e-17	1.87e-16	8.33e-17

m	$e_m^{(128)}(0.3)$	$e_m^{(128)}(0.6)$	$e_m^{(128)}(0.8)$	$e_m^{(256)}(0.3)$	$e_m^{(256)}(0.6)$	$e_m^{(256)}(0.8)$
4	3.86e-09	1.00e-07	6.44e-07	3.86e-09	1.00e-07	6.44e-07
8	9.28e-15	2.70e-12	1.91e-11	9.36e-16	3.08e-13	3.07e-12
16	2.69e-17	1.32e-16	3.61e-16	7.37e-17	1.87e-16	5.27e-16

TABLE 3.2
Errors for Example 3.2 with $m = 64$.

ℓ	$e_{64}^{(\ell)}(0.3)$	$e_{64}^{(\ell)}(0.6)$	$e_{64}^{(\ell)}(0.8)$
4	7.62e-11	1.04e-09	4.37e-10
8	4.03e-16	2.50e-14	6.96e-14
16	0.00e+00	5.55e-17	8.33e-17

EXAMPLE 3.3. Let us consider the integral

$$(\mathcal{V}f)(s) = \int_0^s e^{(s-t)^{3/2}} (s-t)^{\frac{1}{2}} \, dt,$$

which is of the form (3.1) with $k(t, s) = e^{(s-t)^{3/2}}$, $f \equiv 1$, $\alpha = 1/2$, and $\beta = 0$.

In Table 3.3 we report, for increasing values of m and for three different points $s \in [0, 1]$, the errors $e_m^{(\ell)}(s)$ defined in (3.5) with $\ell = 256$. For this choice of ℓ, the first term between the square brackets in (3.4) determines the magnitude of the error which is $m^{-3/4}$, since $k_s \in Z_{3/2}$. In Table 3.4, we display the error $e_m^{(\ell)}(s)$ for $m = 512$ and for increasing value of ℓ. Both tables confirm our theoretical convergence estimate and show that the magnitude of the error does not depend on the point s. In Table 3.4, we see that for $m = 512$ the error does not improve when $\ell \geq 16$.

TABLE 3.3
Errors for Example 3.3 with $\ell = 256$.

m	$e_m^{(256)}(0.2)$	$e_m^{(256)}(0.5)$	$e_m^{(256)}(0.7)$
4	5.63e-04	1.11e-03	2.44e-04
8	2.02e-04	3.71e-04	6.58e-05
16	1.02e-04	2.64e-05	9.43e-06
32	1.08e-05	2.98e-06	1.66e-06
64	1.35e-06	7.78e-07	5.40e-07
128	1.05e-07	2.84e-07	1.90e-07
256	5.18e-08	9.75e-08	7.15e-08
512	1.79e-08	3.41e-08	2.53e-08
1024	6.30e-09	1.20e-08	9.05e-09

TABLE 3.4
Errors for Example 3.3 with $m = 512$.

ℓ	$e_{512}^{(\ell)}(0.2)$	$e_{512}^{(\ell)}(0.5)$	$e_{512}^{(\ell)}(0.7)$
4	1.58e-07	3.14e-07	2.44e-07
8	7.99e-08	1.56e-07	1.19e-07
16	5.06e-08	9.78e-08	7.42e-08
32	3.59e-08	6.91e-08	5.20e-08
64	2.73e-08	5.23e-08	3.92e-08
128	2.18e-08	4.16e-08	3.10e-08
256	1.79e-08	3.41e-08	2.53e-08
512	1.51e-08	2.87e-08	2.12e-08
1024	1.30e-08	2.46e-08	1.81e-08
2048	1.09e-08	2.14e-08	1.56e-08
4096	1.10e-08	1.89e-08	1.37e-08

4. A Nyström-type method. Now we are able to propose a Nyström method based on the quadrature rule (3.2).

Denoting by $\mathcal{V}$ the operator given in (3.1), equation (1.1) can be written as

$$(I + \mu\mathcal{V})f = g.$$

It is well-known that it has a unique solution $f \in C^0$ for each given right-hand $g \in C^0$ and for any $\mu \in \mathbb{R}$; see [6].

To approximate a solution, let us consider the finite dimensional equation

(4.1) $$(I + \mu\mathcal{V}_m^{(\ell)})f_m^{(\ell)} = g,$$

where $f_m^{(\ell)}$ is the unknown and $\mathcal{V}_m^{(\ell)}$ is defined in (3.2). By collocating (4.1) at the points $s_i = \frac{i}{m}$, for $i = 0, \ldots, m$, we obtain the linear system

(4.2) $$\sum_{j=0}^{m} \left[\delta_{ij} + \mu Q_j^{(\ell)}(s_i) k(t_j, s_i) \right] a_j = g(s_i), \quad i = 0, \ldots, m,$$

where $a_j = f_m^{(\ell)}(t_j)$ are the unknowns and the coefficients $Q_j^{(\ell)}$ are given by (3.3). System (4.2) is equivalent to (4.1). Indeed, the solution $\mathbf{a} = (a_0^*, \ldots, a_m^*)^T$ of (4.2) allow us to write the unique solution of (4.1), i.e., the so-called Nyström interpolant

(4.3) $$f_m^{(\ell)}(s) = g(s) - \mu \sum_{j=0}^{m} Q_j^{(\ell)}(s) k(t_j, s) a_j^*.$$

Conversely, the latter provides a solution for system (4.2). We just have to evaluate (4.3) at the nodes $t_j = \frac{j}{m}$.

Next theorem states the stability and convergence of the proposed method.

THEOREM 4.1. *Consider the functional equation (4.1) for a fixed parameter ℓ. Then, for m sufficiently large (say $m \geq m_0$), the operators $(I + \mu \mathcal{V}_m^{(\ell)})$ are invertible and their inverse are uniformly bounded w.r.t. m on C^0. Moreover, denoted by f^* the unique solution of (3.1). If $g \in Z_\lambda$, $\sup_{s \in [0,1]} \|k_s\|_{Z_\lambda} < \infty$, and $\alpha + \beta + 1 \geq 0$, then one has*

(4.4) $$\|f^* - f_m^{(\ell)}\| \leq \mathcal{C} \left[\frac{1}{(\sqrt{m})^\lambda} + \frac{1}{m^\ell} \right] \|f^*\|_{Z_\lambda},$$

where $\mathcal{C} \neq \mathcal{C}(m, f^*)$.

5. Numerical Tests. The aim of this section is to present some numerical examples to assess the accuracy of the Nyström method, as well as the well conditioning of system (4.2). The accuracy is measured by the errors

$$\epsilon_m^{(\ell)}(s) = |f^*(s) - f_m^{(\ell)}(s)|, \quad s \in (0, 1],$$

where ℓ is fixed, f^* is the exact solution of the given equation, and $f_m^{(\ell)}$ is defined in (4.3). When the exact solution is not known, we consider as exact the approximated solution $f_m^{(\ell)}$ with $m = 1024$ and $\ell = 256$. For each example, the errors are computed at three different points s of $(0, 1]$.

The well-conditioning of system (4.2) is illustrated by showing that, for increasing value of the size of the system, the condition number in infinity norm of the coefficient matrix A of (4.2)

$$\text{cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty, \quad \text{where} \quad A_{i,j} = \delta_{ij} + \mu Q_j^{(\ell)}(s_i) k(t_j, s_i),$$

does not increase.

All the computations were carried out in Matlab R2021b in double precision on an Intel Core i7-2600 system (8 cores), under the Debian GNU/Linux operating system.

EXAMPLE 5.1. In the first equation we consider, the kernel and right-hand side are smooth functions

$$f(s) + \frac{1}{2} \int_0^s \log(t + s + 2) f(t) \sqrt{t}\, dt = \frac{\cos(s)}{s^2 + 2}.$$

In Table 5.1, we report the error as well as the condition number of system (4.2) for increasing values of m . As we can see, the convergence is fast, due to the high regularity of the known functions. Moreover, the condition number of system (4.2) does not increase with m. Figure 5.1 displays the maximum absolute errors attained over 512 equally spaces points of the interval $(0, 1)$ when m is fixed and the parameter ℓ varies. It aims to underline that the error improves for increasing values of ℓ.

TABLE 5.1
Numerical results for Example 5.1.

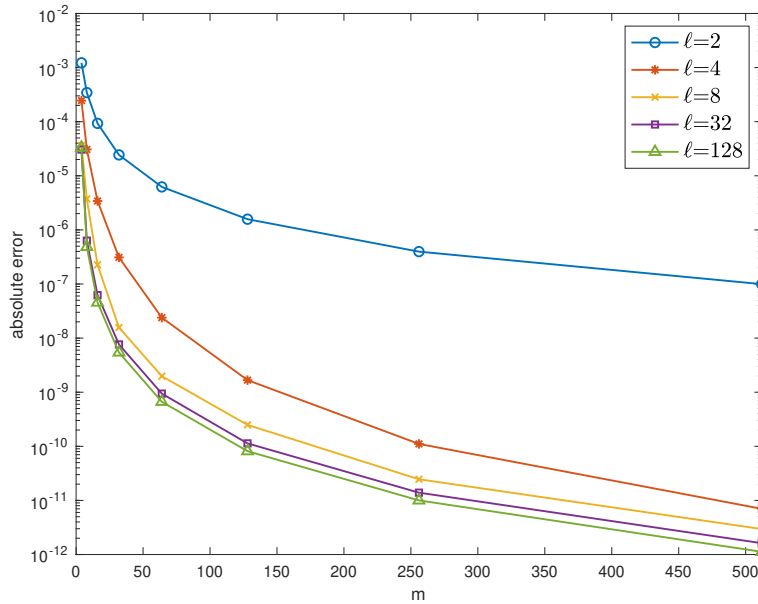| m	$\epsilon_m^{(256)}(0.1)$	$\epsilon_m^{(256)}(0.3)$	$\epsilon_m^{(256)}(0.8)$	$\mathrm{cond}(A)$
4	3.89e-06	1.12e-05	1.33e-05	1.78e+00
8	3.44e-07	3.12e-07	3.38e-07	1.86e+00
16	3.57e-08	3.71e-08	3.87e-08	1.89e+00
32	4.53e-09	4.59e-09	4.75e-09	1.91e+00
64	5.27e-10	5.79e-10	5.85e-10	1.92e+00
128	6.66e-11	7.16e-11	7.25e-11	1.92e+00
256	8.18e-12	8.82e-12	8.91e-12	1.92e+00
512	9.06e-13	9.78e-13	9.89e-13	1.93e+00

FIG. 5.1. *Example 5.1: plot of the maximum errors attained for different values of ℓ.*

EXAMPLE 5.2. Let us approximate the solution of the following equation

$$f(s) + \int_0^s (t \sin s)^{\frac{3}{2}} f(t)\, dt = s^2 + 3 \tanh(2s).$$

In this case the kernel belongs to $Z_{3/2}$ and the right-hand side is smooth. Then, according to Theorem 4.1, we expect a theoretical error of order $\mathcal{O}(m^{-3/4})$. However, the convergence of the method is faster than expected, as also confirmed by the estimated order of convergence

$$EOC_m^{(\ell)}(s) = \frac{\log\left(\epsilon_m^{(\ell)}(s)/\epsilon_{2m}^{(\ell)}(s)\right)}{\log 2},$$

that we report in Table 5.2 next to each error. The condition numbers in the last column of the table confirm the well-conditioning of system (4.2).

TABLE 5.2
Numerical results for Example 5.2.

m	$\epsilon_m^{(256)}(0.4)$	$EOC_m^{(\ell)}$	$\epsilon_m^{(256)}(0.7)$	$EOC_m^{(\ell)}$	$\epsilon_m^{(256)}(0.99)$	$EOC_m^{(\ell)}$	$\text{cond}(A)$
4	2.73e-04	8.07	6.50e-04	8.26	2.24e-04	5.61	1.52e+00
8	1.02e-06	1.69	2.12e-06	1.73	4.58e-06	2.48	1.59e+00
16	3.15e-07	3.65	6.38e-07	3.66	8.20e-07	3.67	1.63e+00
32	2.51e-08	3.57	5.05e-08	3.56	6.44e-08	3.56	1.65e+00
64	2.11e-09	3.53	4.28e-09	3.53	5.45e-09	3.53	1.66e+00
128	1.82e-10	3.53	3.70e-10	3.53	4.71e-10	3.53	1.66e+00
256	1.58e-11	3.63	3.21e-11	3.63	4.09e-11	3.62	1.66e+00
512	1.28e-12		2.60e-12		3.31e-12		1.66e+00

EXAMPLE 5.3. In this example, we consider an equation in which the kernel is smooth, but the right-hand side is not.

$$f(s) + 2\int_0^s (t+s+2)f(t)\sqrt[3]{(s-t)t}\,dt = s^{5/2}, \quad s \in (0,1).$$

Table 5.3 displays the errors. They are smaller than the expected theoretical results, according to which errors should behave like $\mathcal{O}(m^{-5/4})$. In Figure 5.2, we illustrate the approximate solution for different values of m.

TABLE 5.3
Numerical results for Example 5.3.

m	$\epsilon_m^{(256)}(0.01)$	$\epsilon_m^{(256)}(0.5)$	$\epsilon_m^{(256)}(0.99)$	$\text{cond}(A)$
4	2.14e-07	7.83e-05	7.97e-04	8.62e+00
8	2.04e-08	2.45e-06	1.20e-06	9.87e+00
16	4.12e-09	1.10e-07	1.11e-08	1.01e+01
32	9.08e-10	6.93e-09	5.02e-10	1.03e+01
64	1.47e-10	4.70e-10	3.23e-11	1.04e+01
128	1.39e-11	3.22e-11	2.15e-12	1.04e+01
256	8.94e-13	2.23e-12	1.37e-13	1.04e+01
512	6.14e-14	1.45e-13	1.04e-14	1.04e+01

EXAMPLE 5.4. As a last example, we consider an equation that arises in the direct scattering problem for the initial value problem associated to the Korteweg-de Vries (KdV)
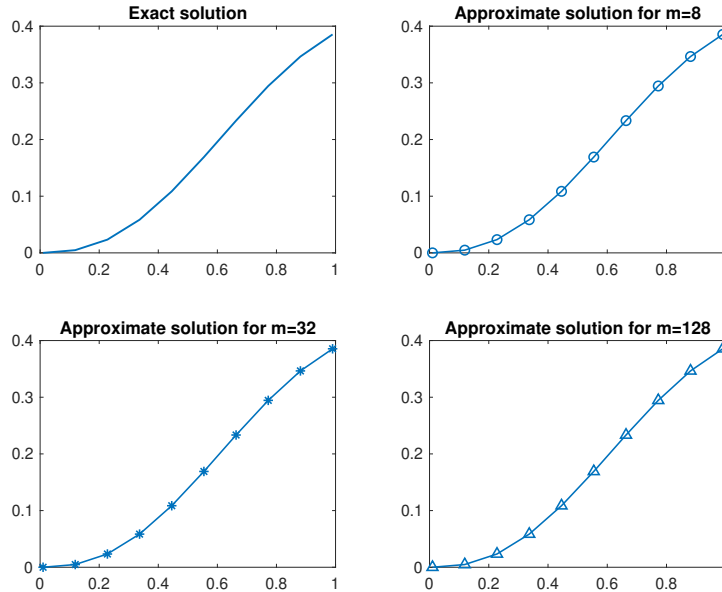
FIG. 5.2. *Example 5.3: plot of the solutions $f_{1024}^{(256)}$, $f_8^{(256)}$, $f_{32}^{(256)}$, and $f_{128}^{(256)}$.*

equation [15, Section 6]

$$\begin{cases} q_t - 6qq_x + q_{xxx} = 0, & x \in \mathbb{R}, \quad t > 0, \\ q(x,0) = q_0(x). \end{cases}$$

The equation is the following

$$f(s) - \frac{1}{2i\omega^2} \int_0^s q_0(s)(e^{2i\omega(s-t)} - 1)f(t)\,dt = q_0(s), \quad s \in (0,1],$$

where $\omega \in \mathbb{R}$ and $i = \sqrt{-1}$.

In Table 5.4, we report the absolute errors in the case $\omega = 10$ and $q_0 = 1$ is the well-known square-well potential. Since all the involved functions are analytic, machine precision is easily achieved. Furthermore, system (4.2) is well-conditioned, as the condition number appears to be always the same for increasing values of m.

TABLE 5.4
Numerical results for Example 5.4.

m	$\epsilon_m^{(256)}(0.01)$	$\epsilon_m^{(256)}(0.5)$	$\epsilon_m^{(256)}(0.99)$	$\text{cond}(A)$
4	6.57e-06	2.38e-03	1.31e-03	1.01e+00
8	4.27e-05	9.48e-04	1.09e-03	1.01e+00
16	5.51e-06	3.07e-05	2.75e-05	1.01e+00
32	2.91e-08	3.75e-08	3.46e-08	1.01e+00
64	1.65e-12	3.59e-12	1.60e-12	1.01e+00
128	1.06e-19	1.12e-16	5.98e-17	1.01e+00

6. Proofs. In this section we report the proofs of the main results.

Proof of Theorem 3.1. By definitions (3.1) and (3.2), we can write

$$|(\mathcal{V}f)(s) - (\mathcal{V}_m^{(\ell)}f)(s)| = \int_0^s [(fk_s)(t) - B_{m,\ell}(fk_s,t)](s-t)^\alpha t^\beta\, dt$$

$$= \int_0^1 [(fk_s)(t) - B_{m,\ell}(fk_s,t)]\,\widetilde{v}(t)\, dt,$$

where

$$\widetilde{v}(t) := \begin{cases} (s-t)^\alpha t^\beta, & s > t, \\ 0, & s < t. \end{cases}$$

Then, by virtue of Theorem 2.2, we can claim that

$$|(\mathcal{V}f)(s) - (\mathcal{V}_m^{(\ell)}f)(s)| \leq \mathcal{C}\left[\omega_\varphi^{2\ell}\left(fk_s; \frac{1}{\sqrt{m}}\right) + \frac{\|fk_s\|}{m^\ell}\right]\int_0^1 \widetilde{v}(t)\, dt$$

$$= \mathcal{C}s^{\alpha+\beta+1}\left[\omega_\varphi^{2\ell}\left(fk_s; \frac{1}{\sqrt{m}}\right) + \frac{\|fk_s\|}{m^\ell}\right]\int_0^1 (1-x)^\alpha x^\beta\, dx.$$

By the assumption, we deduce that the product fk_s is a function of the space Z_λ, so that, by (2.1) and $\alpha + \beta + 1 \geq 0$, we obtain

$$|(\mathcal{V}f)(s) - (\mathcal{V}_m^{(\ell)}f)(s)| \leq \mathcal{C}s^{\alpha+\beta+1}\left[\frac{\|fk_s\|_{Z_\lambda}}{(\sqrt{m})^\lambda} + \frac{\|fk_s\|}{m^\ell}\right]$$

$$\leq \mathcal{C}s^{\alpha+\beta+1}\|k_s\|_{Z_\lambda}\|f\|_{Z_\lambda}\left[\frac{1}{(\sqrt{m})^\lambda} + \frac{1}{m^\ell}\right],$$

from which the assertion follows after taking the supremum of s at both sides.   ☐

Proof of Theorem 4.1. By virtue of Theorem 3.1, the quadrature rule we use in the proposed method is convergent, then the sequence $\{\mathcal{V}_m^{(\ell)}\}_m$ is collectively compact and pointwise convergent [19, Theorem 12.8]; see also [28]. Then, for each fixed ℓ [2, p. 114]

$$\sup_m \|\mathcal{V}_m^{(\ell)}\|_{C^0 \to C^0} < +\infty \qquad \text{and} \qquad \lim_{m\to\infty} \|(\mathcal{V} - \mathcal{V}_m^{(\ell)})\mathcal{V}_m^{(\ell)}\| = 0.$$

Consequently, from [2, Theorem 4.1.1] we deduce that the operators

$$(I + \mu\mathcal{V}_m^{(\ell)})^{-1} : C^0 \to C^0$$

exist and are uniformly bounded with respect to m. To prove the error estimate, we first note that, by the assumption on g and k_s, we deduce that the solution f^* of equation (3.1) is at least in Z_λ. Then, by virtue of [2, Theorem 4.1.1] and Theorem 3.1, one has

$$\|f^* - f_m^{(\ell)}\| \leq \mathcal{C}\|(\mathcal{V} - \mathcal{V}_m^{(\ell)})f^*\| \leq \mathcal{C}\sup_{s\in[0,1]}\|k_s\|_{Z_\lambda}\left[\frac{1}{(\sqrt{m})^\lambda} + \frac{1}{m^\ell}\right]\|f^*\|_{Z_\lambda},$$

that is, inequality (4.4).   ☐

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