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AN OPTIMAL METHOD FOR RECOVERING THE MIXED DERIVATIVE $f^{(2,2)}$ OF BIVARIATE FUNCTIONS*

Y. V. SEMENOVA† AND S. G. SOLODKY‡

Abstract. The problem of recovering the mixed derivative $f^{(2,2)}$ for bivariate functions is investigated. Based on the truncation method, a numerical differentiation algorithm is constructed that uses perturbed Fourier–Legendre coefficients of the function as input information. Moreover, the idea of a hyperbolic cross is implemented, which makes it possible to significantly reduce computational costs. It is established that this algorithm guarantees order-optimal accuracy (in the power scale) with a minimal amount of Galerkin information involved.

Key words. numerical differentiation, Legendre polynomials, truncation method, information complexity, optimal error estimates

AMS subject classifications. 47A52, 65D25

1. Description of the problem. The problem of numerical differentiation is an actual problem arising in many applied fields such as finance, mathematical physics, image processing, analytical chemistry, viscous elastic mechanics, reliability analysis, pattern recognition, and many others (see, for instance, [4, 10, 12]). The numerical differentiation of real functions is a classical problem that is unstable to small perturbations and therefore requires the application of regularization to ensure the stability of the approximation. It should be noted that intensive and effective research of stable differentiation began in the 60s of the last century due to the development of the theory of ill-posed problems. The first paper on numerical differentiation, which was written in terms of the theory of ill-posed problems, is [5]. Thus far, many researchers have proposed and substantiated different methods of numerical differentiation of univariate functions (see, for example, [1, 7, 9, 10, 11, 16, 19, 21, 31, 33]). As to the functions of several (even two) variables, the problem is still under study (see, in particular, [13, 16, 22, 23, 34]). Within this work, the task of optimally recovering the derivative $f^{(2,2)}$ of bivariate functions is carried out. It should be noted that the mixed derivative $f^{(2,2)}$ appears in many differential equations of fourth order and higher. So, for example, the need to calculate such a derivative arises when solving the mixed differential equations of Boussinesq type [32] as well as some nonclassical higher-order equations in mathematical physics [6].

The article is organized as follows. In Section 1, the problem statement for optimizing numerical differentiation methods in the sense of the minimal Galerkin information radius is given. Sections 2 and 3 describe a modification of the spectral truncation method and establish its accuracy estimates in quadratic and uniform metrics, respectively. Section 4 is devoted to finding order estimates for the minimal radius of Galerkin information, thus establishing the optimality (in the power scale) of the method under consideration.

For the further presentation of the material, we need the following notation and concepts: Let $\{\varphi_k(t)\}_{k=0}^{\infty}$ be the system of Legendre polynomials orthonormal on $[-1,1]$ defined as

$$\varphi_k(t) = \sqrt{k+1/2}(2^k k!)^{-1}\frac{d^k}{dt^k}[(t^2-1)^k], \qquad k = 0, 1, 2, \ldots$$

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†Institute of Mathematics NAS of Ukraine, Kyev, Ukraine; Kyev Academic University (semenovaevgen@gmail.com).

‡Institute of Mathematics NAS of Ukraine, Kyev, Ukraine; University of Giessen, Department of Mathematics, Giessen, Germany (solodky@imath.kiev.ua).

By $L_2 = L_2(Q)$ we mean the space of square-summable functions $f(t,\tau)$ on $Q = [-1,1]^2$ with inner product and norm

$$\langle f, g\rangle = \int_{-1}^{1}\int_{-1}^{1} f(t,\tau)g(t,\tau)d\tau dt \qquad \|f\|_{L_2}^2 = \sum_{k,j=0}^{\infty} |\langle f, \varphi_{k,j}\rangle|^2 < \infty,$$

where

$$\langle f, \varphi_{k,j}\rangle = \int_{-1}^{1}\int_{-1}^{1} f(t,\tau)\varphi_k(t)\varphi_j(\tau)d\tau dt, \qquad k,j = 0,1,2,\ldots,$$

are the Fourier–Legendre coefficients of f. Moreover, let $C = C(Q)$ be the space of continuous bivariate functions on Q equipped with the standard uniform norm, and let ℓ_p, $1 \le p \le \infty$, be the space of numerical sequences $\overline{x} = \{x_{k,j}\}_{k,j\in\mathbb{N}_0}$, $\mathbb{N}_0 = \{0\}\bigcup\mathbb{N}$, such that the corresponding relation

$$\|\overline{x}\|_{\ell_p} := \begin{cases} \left(\sum\limits_{k,j\in\mathbb{N}_0} |x_{k,j}|^p\right)^{\frac{1}{p}} < \infty, & 1 \le p < \infty, \\ \sup\limits_{k,j\in\mathbb{N}_0} |x_{k,j}| < \infty, & p = \infty, \end{cases}$$

is fulfilled.

We introduce the space of functions

$$L_{2,2}^{\mu}(Q) = \left\{ f \in L_2(Q) \,:\, \|f\|_{\mu}^2 = \sum_{k,j=0}^{\infty} (\underline{k\cdot j})^{2\mu} |\langle f, \varphi_{k,j}\rangle|^2 < \infty \right\}, \quad \mu > 0, \tag{1.1}$$

where $\underline{k} = \max\{1, k\}$, $k = 0,1,2,\ldots$ Note that in the sequel we will use the same notations both for the space and for the unit ball in this space:

$$L_{2,2}^{\mu} = L_{2,2}^{\mu}(Q) = \{f \in L_{2,2}^{\mu} \colon \|f\|_{\mu} \le 1\},$$

which is what we call a class of functions. It should be noted that $L_{2,2}^{\mu}$ is a generalization of the class of bivariate functions with dominating mixed derivatives.

We represent a function $f(t,\tau)$ from $L_{2,2}^{\mu}$, $\mu \ge 4$, as

$$f(t,\tau) = \sum_{k,j=0}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi_k(t)\varphi_j(\tau),$$

and by its mixed derivative $f^{(2,2)}$ we mean the following series:

$$f^{(2,2)}(t,\tau) = \sum_{k,j=2}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi_k''(t)\varphi_j''(\tau). \tag{1.2}$$

Assume that instead of the exact values of the Fourier–Legendre coefficients $\langle f, \varphi_{k,j}\rangle$ only some perturbations are known with the error level δ in the metric of ℓ_p, $1 \le p \le \infty$. More precisely, we assume that there is a sequence of numbers $\overline{f^\delta} = \{\langle f^\delta, \varphi_{k,j}\rangle\}_{k,j\in\mathbb{N}_0}$ such that for $\overline{\xi} = \{\xi_{k,j}\}_{k,j\in\mathbb{N}_0}$, where $\xi_{k,j} = \langle f - f^\delta, \varphi_{k,j}\rangle$, and for some $1 \le p \le \infty$, the relation

$$\|\overline{\xi}\|_{\ell_p} \le \delta, \qquad 0 < \delta < 1, \tag{1.3}$$

is true.

The research of this work is devoted to the optimization of methods for recovering the derivative (1.2) of functions from the class $L^{\mu}_{2,2}$. We now give a precise statement of the problem to be studied. In the subset of a coordinate plane $[2,\infty)\times[2,\infty)$, we take an arbitrary bounded domain Ω. By $\operatorname{card}(\Omega)$ we mean the number of points in Ω, and as information vector $G(\Omega,\overline{f}^{\delta})\in\mathbb{R}^N$, $\operatorname{card}(\Omega)=N$, we take the set of perturbed values of Fourier–Legendre coefficients $\left\{\langle f^{\delta},\varphi_{k,j}\rangle\right\}_{(k,j)\in\Omega}$.

Let $X=L_2(Q)$ or $X=C(Q)$. By a numerical differentiation algorithm, we mean any mapping $\psi^{(2,2)}=\psi^{(2,2)}(\Omega)$ that associates to the information vector $G(\Omega,\overline{f}^{\delta})$ an element $\psi^{(2,2)}(G(\Omega,\overline{f}^{\delta}))\in X$, which is taken as an approximation of the derivative (1.2) of the function f from the class $L^{\mu}_{2,2}$. We denote by $\Psi(\Omega)$ the set of all algorithms $\psi^{(2,2)}(\Omega):\,\mathbb{R}^N\to X$ that use the same information vector $G(\Omega,\overline{f}^{\delta})$.

We do not require, generally speaking, either linearity or even stability for algorithms from $\Psi(\Omega)$. The only condition for these algorithms is to use an input information in the form of perturbed values of the Fourier–Legendre coefficients with indices from the domain Ω of the coordinate plane. Such a general understanding of the algorithm is explained by the desire to consider the widest range of possible methods of numerical differentiation.

The error of the algorithm $\psi^{(2,2)}$ for the class $L^{\mu}_{2,2}$ is determined by the quantity

$$\varepsilon_{\delta}(L^{\mu}_{2,2},\psi^{(2,2)}(\Omega),X,\ell_p)=\sup_{f\in L^{\mu}_{2,2},\,\|f\|_{\mu}\le 1}\ \sup_{\substack{\overline{f}^{\delta}:\,\overline{f}^{\delta}=\overline{f}+\delta\overline{\xi}\\ \|\overline{\xi}\|_{\ell_p}\le 1}}\|f^{(2,2)}-\psi^{(2,2)}(G(\Omega,\overline{f}^{\delta}))\|_X.$$

The minimal radius of the Galerkin information for the problem of numerical differentiation for the class $L^{\mu}_{2,2}$ is given by

$$R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2},X,\ell_p)=\inf_{\Omega:\,\operatorname{card}(\Omega)\le N}\ \inf_{\psi^{(2,2)}\in\Psi(\Omega)}\varepsilon_{\delta}(L^{\mu}_{2,2},\psi^{(2,2)}(\Omega),X,\ell_p).$$

The quantity $R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2},X,\ell_p)$ describes the minimal possible accuracy in the metric of the space X, which can be achieved by numerical differentiation of arbitrary function $f\in L^{\mu}_{2,2}$ while using not more than N values of its Fourier–Legendre coefficients that are δ-perturbed in the ℓ_p-metric. Note that the minimal radius of Galerkin information for the problem of recovering the first partial derivative was studied in [28], and for other types of ill-posed problems, similar studies were previously carried out in [18, 26]. It should be added that the minimal radius characterizes the information complexity of the considered problem and is traditionally studied within the framework of the IBC-Theory (Information Based Complexity Theory), the foundations of which are laid in the monographs [29, 30].

The goal of our research is to find order-optimal estimates (in the power scale) for $R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2},C,\ell_p)$ and $R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2},L_2,\ell_p)$. At the end of this Section 1, we introduce the symbolic notation for inequality and equality in order. For two positive quantities a and b, we write $a\preceq b$ if there exists a constant $c>0$ such that $a\le cb$. We will write $a\asymp b$ if $a\preceq b$ and $b\preceq a$.

2. Truncation method. Error estimates in the L_2-metric. To the best of our knowledge, up to know, a number of approaches were developed for numerical differentiation (see, for example, [2, 3, 20, 22], also [25] and the references therein). All these methods can be divided into three groups (see [22]): difference methods, interpolation methods, and regularization methods. It is well known that the first two types of methods have their advantage in the simplicity of implementation, but they guarantee satisfactory accuracy only in the case of exactly given input data of the differentiable function. At the same time, regularization methods give stable approximations to the desired derivatives in the case of perturbed input data, but most of them (for example, the Tikhonov method and its various variations) are quite complicated for the numerical realization in view of their integral form and require hard-to-implement rules for determining the regularization parameters (see [22]). Recently in [23] a concise numerical method, called the truncation method, has been proposed as a stable and simple approach to the numerical differentiation of multivariate functions. The essence of this method is to replace the Fourier series (1.2) by a finite Fourier sum using perturbed data $\langle f^\delta, \varphi_{k,j}\rangle$. In the truncation method, to ensure the stability of the approximation and achieve the required order accuracy, it is necessary to properly choose the discretization parameter, which here serves as a regularization parameter. So, the process of regularization in the method under consideration consists of matching the discretization parameter with the perturbation level of the input data. The simplicity of implementation is the main advantage of this method.

In the case of an arbitrary bounded domain Ω of the coordinate plane $[2,\infty)\times[2,\infty)$, the truncation method for differentiating functions of two variables has the form

$$\mathcal{D}_\Omega f^\delta(t,\tau) = \sum_{(k,j)\in\Omega} \langle f^\delta, \varphi_{k,j}\rangle \varphi_k''(t)\varphi_j''(\tau).$$

In order to increase the efficiency of the approach under study, we define the hyperbolic cross as the domain Ω of the following form:

$$\Omega = \Gamma_n := \{(k,j) \,:\, kj \le 2n-1,\ k,j = 2,\dots,n-1\}, \qquad \text{card}(\Gamma_n) \asymp n\ln n.$$

Then, the version of the proposed truncation method can be written as

$$\overline{\mathcal{D}}_n f^\delta(t,\tau) = \sum_{k,j\ge 2,\ kj\le 2n-1} \langle f^\delta, \varphi_{k,j}\rangle \varphi_k''(t)\varphi_j''(\tau). \tag{2.1}$$

We note that the idea of a hyperbolic cross for the problem of numerical differentiation was used earlier in the papers [23, 24, 28]. (For more details about the usage of the hyperbolic cross in solving other ill-posed problems, see [8, 14, 17, 27].) As for [23], the problem of recovering the derivatives $f^{(r,r)}$ of periodic functions was considered when the perturbed values of the Fourier coefficients in the trigonometric system are taken as input information. Unfortunately, it is impossible to automatically transfer the results from the periodic case to the non-periodic one. In particular this is because the best approximation accuracy for derivatives of non-periodic functions (in the case of perturbed input data) has a worse order than the best accuracy for the approximation of derivatives of periodic functions (cf. [23]). Therefore, in the non-periodic case, to construct optimal methods for numerical differentiation, a modification of the previous methodology and the development of new techniques are required. At the moment we have already constructed optimal methods for recovering the derivatives $f^{(1,1)}$ [24] and $f^{(1,0)}$ [28]. The results of [24, 28], together with the results of this work, create the ground and prospects for the development of optimal methods for recovering derivatives of any order for non-periodic functions of any number of variables.

Let us write the error of the method (2.1) as

$$f^{(2,2)}(t,\tau) - \overline{\mathcal{D}}_n f^\delta(t,\tau) = \Big(f^{(2,2)}(t,\tau) - \overline{\mathcal{D}}_n f(t,\tau)\Big) + \big(\overline{\mathcal{D}}_n f(t,\tau) - \overline{\mathcal{D}}_n f^\delta(t,\tau)\big). \tag{2.2}$$

For the first difference on the right-hand side of (2.2), the following representation holds true:

$$f^{(2,2)}(t,\tau) - \overline{\mathcal{D}}_n f(t,\tau) = \triangle_1(t,\tau) + \triangle_2(t,\tau) + \triangle_3(t,\tau), \tag{2.3}$$

$$\triangle_1(t,\tau) = \sum_{k=n+1}^{\infty}\sum_{j=2}^{\infty}\langle f,\varphi_{k,j}\rangle\varphi_k''(t)\varphi_j''(\tau), \tag{2.4}$$

$$\triangle_2(t,\tau) = \sum_{k=2}^{n}\sum_{j=n+1}^{\infty}\langle f,\varphi_{k,j}\rangle\varphi_k''(t)\varphi_j''(\tau), \tag{2.5}$$

$$\triangle_3(t,\tau) = \sum_{k=2}^{n}\sum_{j=\frac{2n}{k}}^{n}\langle f,\varphi_{k,j}\rangle\varphi_k''(t)\varphi_j''(\tau). \tag{2.6}$$

For our calculations, we need the following formula (see [15, Lemma 18]):

$$\varphi_k'(t) = 2\sqrt{k+1/2}\,{\sum_{l=0}^{k-1}}^{*}\sqrt{l+1/2}\,\varphi_l(t), \qquad k\in\mathbb{N}. \tag{2.7}$$

Here and thereafter the notation $\sum^*$ in ${\sum_{l=0}^{k-1}}^{*}\sqrt{l+1/2}\,\varphi_l(t)$ indicates that the summation is taken over only those terms for which $k+l$ is odd.

Let us estimate the error of the method (2.1) in the metric of L_2. An upper bound for the difference (2.3) is given by the following statement:

LEMMA 2.1. *Let $f\in L^{\mu}_{2,2}$, $\mu>4$. Then,*

$$\|f^{(2,2)} - \overline{\mathcal{D}}_n f\|_{L_2} \le c\|f\|_{\mu} n^{-\mu+4}\ln n.$$

Proof. Using the formula (2.7), from (2.4) we get

$$\begin{aligned}
\triangle_1(t,\tau) &= \left(\sum_{k=n+1}^{\infty}\sum_{j=2}^{\infty}\langle f,\varphi_{k,j}\rangle\varphi_k'(t)\varphi_j'(\tau)\right)^{(1,1)} \\
&= 16\sum_{k=n+1}^{\infty}\sum_{j=2}^{\infty}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle \\
&\quad\times {\sum_{l_1=1}^{k-1}}^{*}(l_1+1/2)\,{\sum_{m_1=0}^{l_1-1}}^{*}\sqrt{m_1+1/2}\varphi_{m_1}(t) \\
&\quad\times {\sum_{l_2=1}^{j-1}}^{*}(l_2+1/2)\,{\sum_{m_2=0}^{l_2-1}}^{*}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau).
\end{aligned}$$

We note that in the representation $\triangle_1$, only those terms occur for which all indexes l_1+k, m_1+l_1, l_2+j, m_2+l_2 are odd. Such a rule is valid also for the other terms, namely $\triangle_2$, $\triangle_3$, and $\overline{\mathcal{D}}_n f - \overline{\mathcal{D}}_n f^{\delta}$, appearing in the error representation (see (2.3)–(2.6)). In the following, for simplicity, we will omit the symbol "*" when denoting such summation operations, while taking into account this rule in the calculations.

We change the order of summation and get

$$\triangle_1(t,\tau) = \triangle_{11}(t,\tau) + \triangle_{12}(t,\tau),$$

$$\triangle_{11}(t,\tau) = 16 \sum_{m_1=0}^{n-1} \sqrt{m_1+1/2}\,\varphi_{m_1}(t) \sum_{m_2=0}^{\infty} \sqrt{m_2+1/2}\,\varphi_{m_2}(\tau) \times \sum_{k=n+1}^{\infty} \sum_{j=m_2+2}^{\infty} \sqrt{k+1/2}\sqrt{j+1/2}\,\langle f, \varphi_{k,j}\,\rangle B_{k,j}, \tag{2.8}$$

$$\triangle_{12}(t,\tau) = 16 \sum_{m_1=n}^{\infty} \sqrt{m_1+1/2}\,\varphi_{m_1}(t) \sum_{m_2=0}^{\infty} \sqrt{m_2+1/2}\,\varphi_{m_2}(\tau) \times \sum_{k=m_1+2}^{\infty} \sum_{j=m_2+2}^{\infty} \sqrt{k+1/2}\sqrt{j+1/2}\,\langle f, \varphi_{k,j}\,\rangle B_{k,j}, \tag{2.9}$$

$$B_{k,j} := \sum_{l_1=m_1+1}^{k-1} (l_1+1/2) \sum_{l_2=m_2+1}^{j-1} (l_2+1/2) \le ck^2 j^2. \tag{2.10}$$

Using (1.1), (2.10), and the Cauchy-Schwarz inequality we find

$$\begin{aligned} \|\triangle_{11}\|_{L_2}^2 &\le 16^2 \sum_{m_1=0}^{n-1} (m_1+1/2) \sum_{m_2=0}^{\infty} (m_2+1/2) \sum_{k=n+1}^{\infty} \sum_{j=m_2+2}^{\infty} k^{2\mu} j^{2\mu}\, |\langle f, \varphi_{k,j}\,\rangle|^2 \\ &\quad \times \sum_{k=n+1}^{\infty} \sum_{j=m_2+2}^{\infty} \frac{1}{k^{2\mu-1}} \frac{1}{j^{2\mu-1}} B_{k,j}^2 \\ &\le c\|f\|_\mu^2 \sum_{m_1=0}^{n-1} (m_1+1/2) \sum_{m_2=0}^{\infty} (m_2+1/2) \sum_{k=n+1}^{\infty} \frac{1}{k^{2\mu-5}} \sum_{j=m_2+2}^{\infty} \frac{1}{j^{2\mu-5}} \\ &\le c\|f\|_\mu^2 n^{-2\mu+6} \sum_{m_1=0}^{n-1} (m_1+1/2) \sum_{m_2=0}^{\infty} (m_2+1/2)^{-2\mu+7} \le c\|f\|_\mu^2 n^{-2\mu+8}. \end{aligned}$$

Applying the estimating technique above we can bound the norm of $\triangle_{12}(t,\tau)$:

$$\begin{aligned} \|\triangle_{12}\|_{L_2}^2 &\le 16^2 \sum_{m_1=n}^{\infty} (m_1+1/2) \sum_{m_2=0}^{\infty} (m_2+1/2) \sum_{k=m_1+2}^{\infty} \sum_{j=m_2+2}^{\infty} k^{2\mu} j^{2\mu}\, |\langle f, \varphi_{k,j}\,\rangle|^2 \\ &\quad \times \sum_{k=m_1+2}^{\infty} \sum_{j=m_2+2}^{\infty} \frac{1}{k^{2\mu-1}} \frac{1}{j^{2\mu-1}} B_{k,j}^2 \\ &\le c\|f\|_\mu^2 \sum_{m_1=n}^{\infty} m_1^{-2\mu+7} \sum_{m_2=0}^{\infty} (m_2+1/2)^{-2\mu+7} \le c\|f\|_\mu^2 n^{-2\mu+8}. \end{aligned}$$

Summing up the estimates for $\triangle_{11}(t,\tau)$ and $\triangle_{12}(t,\tau)$ we obtain $\|\triangle_1\|_{L_2} \le c\|f\|_\mu n^{-\mu+4}$.

Moreover, using the formula (2.7), from (2.5) we find

$$\begin{aligned}\triangle_2(t,\tau) &= \left(\sum_{k=2}^{n}\sum_{j=n+1}^{\infty}\langle f,\varphi_{k,j}\rangle\varphi_k'(t)\varphi_j'(\tau)\right)^{(1,1)}\\ &= 16\sum_{k=2}^{n}\sum_{j=n+1}^{\infty}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle\\ &\quad\times\sum_{l_1=1}^{k-1}(l_1+1/2)\sum_{m_1=0}^{l_1-1}\sqrt{m_1+1/2}\varphi_{m_1}(t)\\ &\quad\times\sum_{l_2=1}^{j-1}(l_2+1/2)\sum_{m_2=0}^{l_2-1}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau).\end{aligned}$$

Changing the order of summation, we get

$$\begin{aligned}\triangle_2(t,\tau) &= \triangle_{21}(t,\tau)+\triangle_{22}(t,\tau),\\ \triangle_{21}(t,\tau) &= 16\sum_{m_1=0}^{n-2}\sqrt{m_1+1/2}\varphi_{m_1}(t)\sum_{m_2=0}^{n-1}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau)\\ &\quad\times\sum_{k=m_1+2}^{n}\sum_{j=n+1}^{\infty}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle\\ &\quad\times\sum_{l_1=m_1+1}^{k-1}(l_1+1/2)\sum_{l_2=m_2+1}^{j-1}(l_2+1/2),\\ \triangle_{22}(t,\tau) &= 16\sum_{m_1=0}^{n-2}\sqrt{m_1+1/2}\varphi_{m_1}(t)\sum_{m_2=n}^{\infty}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau)\\ &\quad\times\sum_{k=m_1+2}^{n}\sum_{j=m_2+2}^{\infty}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle B_{k,j}.\end{aligned}$$

Taking into account that $\mu>4$, we can bound the term $\triangle_{21}(t,\tau)$ in the L_2-norm:

$$\begin{aligned}\|\triangle_{21}\|_{L_2}^2 \le 16^2\sum_{m_1=0}^{n-2}(m_1+1/2)\sum_{m_2=0}^{n-1}(m_2+1/2)\sum_{k=m_1+2}^{n}\sum_{j=n+1}^{\infty}k^{2\mu}j^{2\mu}\,|\langle f,\varphi_{k,j}\,\rangle|^2\\ \times\sum_{k=m_1+2}^{n}\sum_{j=n+1}^{\infty}\frac{1}{k^{2\mu-1}}\frac{1}{j^{2\mu-1}}B_{k,j}^2.\end{aligned}$$

Using (1.1) and (2.10), we immediately get

$$\begin{aligned}\|\triangle_{21}\|_{L_2}^2 &\le c\|f\|_\mu^2\sum_{m_1=0}^{n-2}(m_1+1/2)\sum_{m_2=0}^{n-1}(m_2+1/2)\sum_{k=m_1+2}^{n}\frac{1}{k^{2\mu-5}}\sum_{j=n+1}^{\infty}\frac{1}{j^{2\mu-5}}\\ &\le c\|f\|_\mu^2 n^{-2\mu+6}\sum_{m_1=0}^{n-2}(m_1+1/2)^{-2\mu+7}\sum_{m_2=0}^{n-1}(m_2+1/2)\le c\|f\|_\mu^2 n^{-2\mu+8}.\end{aligned}$$

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Furthermore, we can bound the norm of $\triangle_{22}(t,\tau)$:

$$\begin{aligned}\|\triangle_{22}\|_{L_2}^2 &\le 16^2 \sum_{m_1=0}^{n-2}(m_1+1/2)\sum_{m_2=n}^{\infty}(m_2+1/2)\sum_{k=m_1+2}^{n}\sum_{j=m_2+2}^{\infty}k^{2\mu}j^{2\mu}\,|\langle f,\varphi_{k,j}\,\rangle|^2\\ &\quad\times\sum_{k=m_1+2}^{n}\sum_{j=m_2+2}^{\infty}\frac{1}{k^{2\mu-1}}\frac{1}{j^{2\mu-1}}B_{k,j}^2\\ &\le c\|f\|_\mu^2\sum_{m_1=0}^{n-2}(m_1+1/2)^{-2\mu+7}\sum_{m_2=n}^{\infty}(m_2+1/2)^{-2\mu+7}\le c\|f\|_\mu^2 n^{-2\mu+8}.\end{aligned}$$

Summing up the estimates for $\triangle_{21}(t,\tau)$ and $\triangle_{22}(t,\tau)$, we obtain $\|\triangle_2\|_{L_2}\le c\|f\|_\mu n^{-\mu+4}$.

Using formula (2.7), from (2.6) we arrive at

$$\begin{aligned}\triangle_3(t,\tau) &= \left(\sum_{k=2}^{n}\sum_{j=\frac{2n}{k}}^{n}\langle f,\varphi_{k,j}\rangle\varphi_k'(t)\varphi_j'(\tau)\right)^{(1,1)}\\ &= 16\sum_{k=2}^{n}\sum_{j=\frac{2n}{k}}^{n}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle\\ &\quad\times\sum_{l_1=1}^{k-1}(l_1+1/2)\sum_{m_1=0}^{l_1-1}\sqrt{m_1+1/2}\varphi_{m_1}(t)\\ &\quad\times\sum_{l_2=1}^{j-1}(l_2+1/2)\sum_{m_2=0}^{l_2-1}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau).\end{aligned}$$

Changing the order of summation we get

$$\triangle_3(t,\tau)=\triangle_{31}(t,\tau)+\triangle_{32}(t,\tau)+\triangle_{33}(t,\tau), \tag{2.11}$$

$$\begin{aligned}\triangle_{31}(t,\tau) &= 16\sum_{m_1=0}^{n-2}\sqrt{m_1+1/2}\varphi_{m_1}(t)\sum_{m_2=0}^{\frac{2n}{m_1+2}-2}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau)\\ &\quad\times\sum_{k=m_1+2}^{\frac{2n}{m_2+2}}\sum_{j=\frac{2n}{k}}^{n}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle B_{k,j},\\ \triangle_{32}(t,\tau) &= 16\sum_{m_1=0}^{n-2}\sqrt{m_1+1/2}\varphi_{m_1}(t)\sum_{m_2=1}^{\frac{2n}{m_1+2}-1}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau)\\ &\quad\times\sum_{k=\frac{2n}{m_2+1}}^{n}\sum_{j=m_2+2}^{n}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle B_{k,j},\\ \triangle_{33}(t,\tau) &= 16\sum_{m_1=0}^{n-2}\sqrt{m_1+1/2}\varphi_{m_1}(t)\sum_{m_2=\frac{2n}{m_1+2}}^{n-2}\sqrt{m_2+1/2}\,\varphi_{m_2}(\tau)\\ &\quad\times\sum_{k=m_1+2}^{n}\sum_{j=m_2+2}^{n}\sqrt{k+1/2}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\,\rangle B_{k,j}.\end{aligned}$$

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Using (1.1) and (2.10) we obtain

$$
\begin{aligned}
\|\triangle_{31}\|_{L_2}^2 \le 16^2 \sum_{m_1=0}^{n-2} (m_1+1/2) \sum_{m_2=0}^{\frac{2n}{m_1+2}-2} (m_2+1/2) \\
\times \sum_{k=m_1+2}^{\frac{2n}{m_2+2}} \sum_{j=\frac{2n}{k}}^{n} k^{2\mu} j^{2\mu} \, |\langle f, \varphi_{k,j}\rangle|^2 \sum_{k=m_1+2}^{\frac{2n}{m_2+2}} \sum_{j=\frac{2n}{k}}^{n} \frac{1}{k^{2\mu-1}} \frac{1}{j^{2\mu-1}} B_{k,j}^2 \\
\le c\|f\|_\mu^2 n^{-2\mu+6} \sum_{m_1=0}^{n-2} (m_1+1/2) \sum_{m_2=0}^{\frac{2n}{m_1+2}-2} (m_2+1/2) \sum_{k=m_1+2}^{\frac{2n}{m_2+2}} \frac{1}{k} \\
\le c\|f\|_\mu^2 n^{-2\mu+8} \ln^2 n.
\end{aligned}
$$

Similarly for $\|\triangle_{31}\|_{L_2}$, we find

$$
\begin{aligned}
\|\triangle_{32}\|_{L_2}^2 \le 16^2 \sum_{m_1=0}^{n-2} (m_1+1/2) \sum_{m_2=1}^{\frac{2n}{m_1+2}-1} (m_2+1/2) \\
\times \sum_{k=\frac{2n}{m_2+1}}^{n} \sum_{j=m_2+2}^{n} k^{2\mu} j^{2\mu} \, |\langle f, \varphi_{k,j}\rangle|^2 \sum_{k=\frac{2n}{m_2+1}}^{n} \sum_{j=m_2+2}^{n} \frac{1}{k^{2\mu-1}} \frac{1}{j^{2\mu-1}} B_{k,j}^2 \\
\le c\|f\|_\mu^2 n^{-2\mu+6} \sum_{m_1=0}^{n-2} (m_1+1/2) \sum_{m_2=1}^{\frac{2n}{m_1+2}-1} (m_2+1/2) \\
\le c\|f\|_\mu^2 n^{-2\mu+8} \sum_{m_1=0}^{n-2} \frac{1}{m_1+1/2} \asymp \|f\|_\mu^2 n^{-2\mu+8} \ln n,
\end{aligned}
$$

$$
\begin{aligned}
\|\triangle_{33}\|_{L_2}^2 \le c\|f\|_\mu^2 \sum_{m_1=0}^{n-2} (m_1+1/2) \sum_{m_2=\frac{2n}{m_1+2}}^{n-2} (m_2+1/2) \\
\times \sum_{k=m_1+2}^{n} \sum_{j=m_2+2}^{n} \frac{1}{k^{2\mu-1}} \frac{1}{j^{2\mu-1}} B_{k,j}^2 \\
\le c\|f\|_\mu^2 \sum_{m_1=0}^{n-2} (m_1+1/2)^{-2\mu+7} \sum_{m_2=\frac{2n}{m_1+2}}^{n-2} (m_2+1/2)^{-2\mu+7} \\
\le c\|f\|_\mu^2 n^{-2\mu+8} \sum_{m_1=0}^{n-2} \frac{1}{m_1+1/2} \asymp \|f\|_\mu^2 n^{-2\mu+8} \ln n.
\end{aligned}
$$

From (2.11) and the estimates above we obtain $\|\triangle_3\|_{L_2} \le c\|f\|_\mu n^{-\mu+4} \ln n$. The combination of (2.3) and the bounds for the norms of $\triangle_1$, $\triangle_2$, $\triangle_3$ conclude the proof of the lemma. ∎

The following statement contains an estimate for the second term from the right-hand side of (2.2) in the metric of L_2.

LEMMA 2.2. *Let condition (1.3) be satisfied. Then, for an arbitrary function $f \in L_2(Q)$, it holds that*

$$
\|\overline{\mathcal{D}}_n f - \overline{\mathcal{D}}_n f^\delta\|_{L_2} \le c\delta n^{\frac{9}{2}-\frac{1}{p}} \ln^{\frac{3}{2}-\frac{1}{p}} n.
$$

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Proof. Let us write down the representation

$$\overline{\mathcal{D}}_n f(t,\tau) - \overline{\mathcal{D}}_n f^\delta(t,\tau) = \left(\sum_{k,j\geq 2,\ kj\leq 2n-1} \langle f - f^\delta, \varphi_{k,j}\rangle \varphi'_k(t)\varphi'_j(\tau) \right)^{(1,1)}.$$

Using formula (2.7) we get

$$\begin{aligned}\overline{\mathcal{D}}_n f(t,\tau) - \overline{\mathcal{D}}_n f^\delta(t,\tau) = 16 \sum_{k=2}^{n-1} \sum_{j=2}^{\frac{2n-1}{k}} \sqrt{k+1/2}\sqrt{j+1/2}\,\langle f - f^\delta, \varphi_{k,j}\,\rangle \\ \times \sum_{l_1=1}^{k-1} (l_1+1/2) \sum_{m_1=0}^{l_1-1} \sqrt{m_1+1/2}\varphi_{m_1}(t) \\ \times \sum_{l_2=1}^{j-1} (l_2+1/2) \sum_{m_2=0}^{l_2-1} \sqrt{m_2+1/2}\,\varphi_{m_2}(\tau).\end{aligned}$$

We change the order of summation in the above formula to get

$$\begin{aligned}&\overline{\mathcal{D}}_n f(t,\tau) - \overline{\mathcal{D}}_n f^\delta(t,\tau) \\ &\quad = 16 \sum_{m_1=0}^{n-3} \sqrt{m_1+1/2}\varphi_{m_1}(t) \sum_{m_2=0}^{\frac{2n-1}{m_1+2}-2} \sqrt{m_2+1/2}\,\varphi_{m_2}(\tau) \\ &\quad\quad \times \sum_{k=m_1+2}^{\frac{2n-1}{m_2+2}} \sum_{j=m_2+2}^{\frac{2n-1}{k}} \sqrt{k+1/2}\sqrt{j+1/2}\,\langle f - f^\delta, \varphi_{k,j}\,\rangle B_{k,j}.\end{aligned}$$

First, let $1 < p < \infty$. Then, using the Hölder inequality and the estimate (2.10), we find that

$$\begin{aligned}&\|\overline{\mathcal{D}}_n f - \overline{\mathcal{D}}_n f^\delta\|_{L_2}^2 \\ &\leq c \sum_{m_1=0}^{n-3} (m_1+1/2) \sum_{m_2=0}^{\frac{2n-1}{m_1+2}-2} (m_2+1/2) \\ &\quad \times \left(\sum_{k=m_1+2}^{\frac{2n-1}{m_2+2}} \sum_{j=m_2+2}^{\frac{2n-1}{k}} |\langle f - f^\delta, \varphi_{k,j}\,\rangle|^p \right)^{2/p} \left(\sum_{k=m_1+2}^{\frac{2n-1}{m_2+2}} \sum_{j=m_2+2}^{\frac{2n-1}{k}} (kj)^{\frac{5p}{2(p-1)}} \right)^{2(p-1)/p} \\ &\leq c\delta^2 n^{5+\frac{2(p-1)}{p}} \ln^{\frac{2(p-1)}{p}} n \sum_{m_1=0}^{n-3} (m_1+1/2) \sum_{m_2=0}^{\frac{2n-1}{m_1+2}-2} (m_2+1/2) \asymp \delta^2 n^{9-2/p} \ln^{3-2/p} n,\end{aligned}$$

which gives us the wanted estimate. In the case of $p = 1$ and $p = \infty$, the assertion of the lemma is proved similarly. ◻

The combination of Lemmas 2.1 and 2.2 leads to:

THEOREM 2.3. *Let $f \in L^\mu_{2,2}$, $\mu > 4$, and let condition (1.3) be satisfied. Then, for $n \asymp \left(\delta^{-1} \ln^{\frac{1}{p}-\frac{1}{2}} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$, the following bound is valid:*

$$\|f^{(2,2)} - \overline{\mathcal{D}}_n f^\delta\|_{L_2} \leq c \left(\delta \ln^{\frac{1}{2}-\frac{1}{p}} \frac{1}{\delta}\right)^{\frac{\mu-4}{\mu-1/p+1/2}} \ln\frac{1}{\delta}.$$

3. Truncation method. Error estimate in the metric of C. Now we would like to deal with the error of (2.1) using the metric of C.

LEMMA 3.1. *Let $f \in L^{\mu}_{2,2}$, $\mu > 5$. Then,*

$$\|f^{(2,2)} - \overline{\mathcal{D}}_n f\|_C \le c\|f\|_\mu n^{-\mu+5} \ln^{\frac{3}{2}} n.$$

Proof. Using (2.8), (1.1), and (2.10) we get

$$\begin{aligned}\|\triangle_{11}\|_C &\le c \sum_{m_1=0}^{n-1} (m_1+1/2) \sum_{m_2=0}^{\infty} (m_2+1/2) \\ &\quad \times \sum_{k=n+1}^{\infty} \sum_{j=m_2+2}^{\infty} \frac{\sqrt{k+1/2}}{k^{\mu-2}} \frac{\sqrt{j+1/2}}{j^{\mu-2}} |\langle f, \varphi_{k,j} \rangle| k^\mu j^\mu \\ &\le c\|f\|_\mu n^{-\mu+3} \sum_{m_1=0}^{n-1} (m_1+1/2) \sum_{m_2=0}^{\infty} (m_2+1/2)^{-\mu+4} = c\|f\|_\mu n^{-\mu+5}.\end{aligned}$$

Moreover, from (2.9) it follows that

$$\|\triangle_{12}\|_C \le c\|f\|_\mu \sum_{m_1=n}^{\infty} (m_1+1/2)^{-\mu+4} \sum_{m_2=0}^{\infty} (m_2+1/2)^{-\mu+4} = c\|f\|_\mu n^{-\mu+5}.$$

Thus, we have $\|\triangle_1\|_C \le c\|f\|_\mu n^{-\mu+5}$. Similarly, we find that $\|\triangle_2\|_C \le c\|f\|_\mu n^{-\mu+5}$, $\|\triangle_3\|_C \le c\|f\|_\mu n^{-\mu+5} \ln^{3/2} n$. Substituting the estimates for the norms of $\triangle_1, \triangle_2, \triangle_3$ into the identity (2.3) allows us to prove the lemma. ∎

The following statement contains an estimate for the second difference from the right-hand side of (2.2) in the metric of C.

LEMMA 3.2. *Assume that condition (1.3) is satisfied. Then, for an arbitrary function $f \in C(Q)$, the following bound is valid:*

$$\|\overline{\mathcal{D}}_n f - \overline{\mathcal{D}}_n f^\delta\|_C \le c\delta n^{\frac{11}{2}-\frac{1}{p}} \ln^{2-\frac{1}{p}} n.$$

Lemma 3.2 is proved in a similar way to Lemma 2.2.

The combination of Lemmas 3.1 and 3.2 leads to

THEOREM 3.3. *Let $f \in L^{\mu}_{2,2}$, $\mu > 5$, and condition (1.3) be satisfied. Then for $n \asymp \left(\delta^{-1} \ln^{\frac{1}{p}-\frac{1}{2}} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$ the following bound is valid:*

$$\|f^{(2,2)} - \overline{\mathcal{D}}_n f^\delta\|_C \le c \left(\delta \ln^{\frac{1}{2}-\frac{1}{p}} \frac{1}{\delta}\right)^{\frac{\mu-5}{\mu-1/p+1/2}} \ln^{\frac{3}{2}} \frac{1}{\delta}.$$

4. Minimal radius of Galerkin information. Now, we are in the position to find order estimates for the minimal radius. First, we would like to establish a lower estimate for the quantity $R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, C, \ell_p)$. To this end we fix an arbitrarily chosen domain $\hat{\Omega}$ of the coordinate plane $[2,\infty) \times [2,\infty)$, card$(\hat{\Omega}) \le N$, and construct an auxiliary function

$$f_1(t,\tau) = \widetilde{c}\left(\varphi_0(t)\varphi_0(\tau) \,+\, N^{-\mu-1/2} 2^{-\mu} \varphi_2(\tau) \sum_{k=N+1}^{3N}{}' \varphi_k(t)\right),$$

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where the sum $\sum_{k=N+1}^{3N}{}'$ is taken over N of any pairwise distinct functions $\varphi_k(t)$ such that $N+1 \le k \le 3N$ and $(k,2) \notin \hat{\Omega}$. There is always at least one set of such functions. To satisfy the condition $\|f_1\|_\mu \le 1$, it is enough to take $\widetilde{c} = \left(1+3^{2\mu}\right)^{-1/2}$.

We consider another function $f_2(t,\tau) = \widetilde{c}\,\varphi_0(t)\varphi_0(\tau)$ from the class $L^\mu_{2,2}$ and find a lower bound for $\|f_1^{(2,2)} - f_2^{(2,2)}\|_C$. For this we need the following representation:

$$
\begin{aligned}
f_1^{(2,2)}(t,\tau) = 4\widetilde{c}\,N^{-\mu-1/2}\,\frac{3\sqrt{5}}{2^\mu}\varphi_0(\tau) \sum_{k=N+1}^{3N}{}' \sqrt{k+1/2} \\
\times \sum_{l=1}^{k-1}{}^{*}(l+1/2) \sum_{j=0}^{l-1}{}^{*} \sqrt{j+1/2}\varphi_j(t).
\end{aligned} \tag{4.1}
$$

We note that on the right-hand side of (4.1) only terms with odd indexes $l+k, l+j$ appear.

Furthermore, we have

$$
\begin{aligned}
\|f_1^{(2,2)} - f_2^{(2,2)}\|_C &\ge \frac{12}{\sqrt{2}}\frac{\sqrt{5}}{2^\mu}\widetilde{c}\,N^{-\mu-1/2} \sum_{k=N+1}^{3N}{}' \sqrt{k+1/2} \sum_{l=1}^{k-1}{}^{*}(l+1/2) \sum_{j=0}^{l-1}{}^{*}(j+1/2) \\
&\ge \frac{3\sqrt{5}}{2^{\mu+7/2}}\,\widetilde{c}\,N^{-\mu-1/2} \sum_{k=N+1}^{3N}{}' (k-1)^{9/2} \ge \frac{3\sqrt{5}}{2^{\mu+7/2}}\,\widetilde{c}\,N^{-\mu+5}.
\end{aligned}
$$

Since for any $1 \le p \le \infty$ it holds true that $\|\overline{f}_1 - \overline{f}_2\|_{\ell_p} = \frac{\widetilde{c}}{2^\mu}\,N^{-\mu-1/2+1/p}$, it follows that in the case of $N^{-\mu-1/2+1/p} \le 2^\mu\delta/\widetilde{c}$, the functions

$$
f_1^\delta(t,\tau) = f_2(t,\tau), \qquad f_2^\delta(t,\tau) = f_1(t,\tau)
$$

can be considered as δ-perturbations of f_1 and f_2, respectively.

Let us find an upper bound for the difference $\|f_1^{(2,2)} - f_2^{(2,2)}\|_C$. Taking into account the relation $G(\hat{\Omega}, \overline{f}_1^\delta) = G(\hat{\Omega}, \overline{f}_2^\delta)$, for any $\psi^{(2,2)}(\hat{\Omega}) \in \Psi(\hat{\Omega})$, we find

$$
\begin{aligned}
\|f_1^{(2,2)} - f_2^{(2,2)}\|_C &\le \|f_1^{(2,2)} - \psi^{(2,2)}(G(\hat{\Omega}, \overline{f}_1^\delta))\|_C + \|f_2^{(2,2)} - \psi^{(2,2)}(G(\hat{\Omega}, \overline{f}_2^\delta))\|_C \\
&\le 2 \sup_{f \in L^\mu_{2,2}} \sup_{\substack{\overline{f}^\delta:\, \overline{f}^\delta = \overline{f} + \delta\overline{\xi} \\ \|\overline{\xi}\|_{\ell_p} \le 1}} \|f^{(2,2)} - \psi^{(2,2)}(G(\hat{\Omega}, \overline{f}^\delta))\|_C \\
&=: 2\,\varepsilon_\delta(L^\mu_{2,2}, \psi^{(2,2)}(\hat{\Omega}), C, \ell_p).
\end{aligned}
$$

That is $\varepsilon_\delta(L^\mu_{2,2}, \psi^{(2,2)}(\hat{\Omega}), C, \ell_p) \ge \overline{c}\,N^{-\mu+5}$, where $\overline{c} = \frac{3\sqrt{5}}{2^{\mu+9/2}}\,\widetilde{c}$. From the fact that the domain $\hat{\Omega}$ and the algorithm $\psi^{(2,2)}(\hat{\Omega}) \in \Psi(\hat{\Omega})$ are arbitrary, it follows that

$$
R^{(2,2)}_{N,\delta}(L^\mu_{2,2}, C, \ell_p) \ge \overline{c}\,N^{-\mu+5}.
$$

Thus, the following statement is proved:

LEMMA 4.1. *Let $\mu > 5$, $1 \le p \le \infty$, $N \ge (2^\mu\delta/\widetilde{c})^{-1/(\mu+1/2-1/p)}$. Then,*

$$
R^{(2,2)}_{N,\delta}(L^\mu_{2,2}, C, \ell_p) \ge \overline{c}\,N^{-\mu+5}.
$$

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THEOREM 4.2. *Let $\mu > 5$, $1 \leq p \leq \infty$. Then, for $N \asymp \left(\delta^{-1} \ln^{\mu} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$ it holds that*

$$N^{-\mu+5} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, C, \ell_p) \preceq N^{-\mu+5} \ln^{\mu-7/2} N,$$

$$\left(\delta \ln^{-\mu} \frac{1}{\delta}\right)^{\frac{\mu-5}{\mu-1/p+1/2}} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, C, \ell_p) \preceq \left(\delta \ln^{\frac{1}{2}-\frac{1}{p}} \frac{1}{\delta}\right)^{\frac{\mu-5}{\mu-1/p+1/2}} \ln^{\frac{3}{2}} \frac{1}{\delta}.$$

The upper bound is attained by (2.1) for $n \asymp \left(\delta^{-1} \ln^{\frac{1}{p}-\frac{1}{2}} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$.

Proof. The upper bound for $R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, C, \ell_p)$ follows from Theorem 3.3. The lower bound is found in Lemma 4.1. ∎

Let us turn to the estimation of the minimal radius in the L_2-metric.

LEMMA 4.3. *Let $\mu > 4$, $1 \leq p \leq \infty$. Then for $N \geq (2^{\mu}\delta/\widetilde{c})^{-1/(\mu+1/2-1/p)}$ it holds that*

$$R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, L_2, \ell_p) \geq \overline{\overline{c}}\, N^{-\mu+4}, \qquad \overline{\overline{c}} = \frac{3\sqrt{15}}{2^{\mu+3}} \widetilde{c}.$$

Proof. Lemma 4.3 is proved in a similar way as Lemma 4.1. ∎

THEOREM 4.4. *Let $\mu > 4$, $1 \leq p \leq \infty$. Then, for $N \asymp \left(\delta^{-1} \ln^{\mu} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$, it holds that*

$$N^{-\mu+4} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, L_2, \ell_p) \preceq N^{-\mu+4} \ln^{\mu-3} N,$$

$$\left(\delta \ln^{-\mu} \frac{1}{\delta}\right)^{\frac{\mu-4}{\mu-1/p+1/2}} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, L_2, \ell_p) \preceq \left(\delta \ln^{\frac{1}{2}-\frac{1}{p}} \frac{1}{\delta}\right)^{\frac{\mu-4}{\mu-1/p+1/2}} \ln \frac{1}{\delta}.$$

The upper bound is attained (2.1) for $n \asymp \left(\delta^{-1} \ln^{\frac{1}{p}-\frac{1}{2}} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$.

Proof. The upper bound for $R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, L_2, \ell_p)$ follows from Theorem 2.3. The lower bound is found in Lemma 4.3. ∎

REMARK 4.5. As can be seen from Theorems 4.1 and 4.4, the upper and lower estimates of the minimal radius coincide in power order and differ by a logarithmic factor. Such an accuracy is usually called order-optimal in the power scale. Moreover, the method (2.1) that realizes the upper bounds is also called order-optimal in the power scale. As for the question of which logarithmic factor, from the upper or lower estimate, represents the exact order, in our opinion, the lower estimate can be "pulled up" to the upper one. This seems to be achievable by modifying the proof of Lemma 4.1. However, a detailed consideration of this issue is beyond the scope of this work.

5. Conclusions. The main statements of the paper (see Theorems 4.2 and 4.4) contain estimates that are order-optimal (in the power scale) for the minimal radius of Galerkin information for the problem of recovering the mixed derivative $f^{(2,2)}$. This allows us to find the orders of best accuracy (up to a logarithmic factor) of recovering the derivative $f^{(2,2)}$, as well as the minimal number of perturbed Fourier–Legendre coefficients necessary to achieve this accuracy. In addition, in Theorems 2.3 and 3.3 we consider the spectral truncation method (2.1), which realizes the optimal orders of the quantity under study. Thus, the proposed method (2.1) not only has a simple implementation but also achieves the best order of accuracy, while using the smallest (in order) amount of discrete information in the form of Fourier–Legendre coefficients. The authors plan to continue these studies in the following areas:

- optimal recovery of mixed derivatives $f^{(r,r)}$, for any $r = 1, 2, 3, \ldots$;
- numerical differentiation of functions of any number of variables;
- numerical differentiation of functions from wider classes than $L^{\mu}_{2,2}$.

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