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REUBEN LOUIS ROSENBERG (1909–1986) AND THE STEIN–ROSENBERG THEOREM*

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Historical paper.

Abstract. Rosenberg is well known by numerical analysts, in particular by those working on numerical linear algebra, for an important theorem on the convergence of the methods of Jacobi and Gauss–Seidel for solving systems of linear equations. This paper was published in 1948 with Philip Stein (1890–1974). Although the biography of Stein is well known, that of Rosenberg was almost completely unknown. This paper presents a complete biography of Reuben Louis Rosenberg (1909–1986) with an analysis of all his scientific contributions in numerical analysis and as well as in nuclear physics. Personal information are also included. The Stein–Rosenberg theorem is commented, replaced in its historical context, and a large review of its variants, generalizations, and applications is given.

Key words. Reuben Louis Rosenberg, relaxation methods, Stein–Rosenberg theorem, numerical linear algebra, biography, nuclear physics

AMS subject classifications. 01A60, 01A70, 65F10, 81V35, 82C40

1. Presentation.

Rosenberg is well known by numerical analysts, in particular by those working on numerical linear algebra, for an important theorem he published in 1948 with Philip Stein (1890–1974), another South African scientist (born in Lithuania). This theorem is analyzed in the book [39] of Gérard Meurant and the two authors of this paper. This book contains the biographies of the main researchers who contributed to numerical linear algebra. It was quite easy to write the biography of Stein because obituaries on him were published and also because he wrote his autobiography [115]. The situation was quite different for Rosenberg since, at the beginning, we even did not know the meaning of his initials, R. L. for Reuben Louis. However, we managed to find some more information on him and were able to write his biography in our book. But, since it was quite incomplete, we (the authors of this paper) decided to pursue our search on him, and this paper is the result of our quest. Since the Stein–Rosenberg had, and still has, an important impact on relaxation methods for the iterative solution of systems of linear equations, it is replaced in its historical context with the comments it aroused. We also reviewed its variants, generalizations, and applications since its formulation.

Scientific explanations, not directly connected with the work of Rosenberg but useful for understanding what he did, are included within frames. Additional information describing Rosenberg's environment is also been given. Biographical data of other scientists mentioned in this paper are also framed.

2. Life and work of Reuben Louis Rosenberg.

2.1. First years, 1909–1928.

Reuben Louis Rosenberg was born on November 13, 1909 in Johannesburg, South Africa. His nationality at birth was South African. He was the son of Joseph Rosenberg (about 1880–1963) born in Kaliningrad, Russia, himself son of Morris Rosenberg born in 1855 in Neustadt, Rhineland-Palatinate, Germany, and deceased in Jerusalem, Israel, in 1939, and Itta Blogaslavenski. The family emigrated to South Africa

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Fig. 1. The Rosenberg family. Reuben Louis is the tallest one. ©Rosenberg family

when Joseph was approximately five years old. Joseph became a farmer, and he fought during the Second Boer War around 1900. He had five brothers, one sister, and a half sister. He died in the Orange Free State (OFS) in South Africa. Reuben Louis' mother, Rachel Leah Friedman (c. 1883–c. 1935) also named Ray, was presumably born in Riga, Latvia, and died in the OFS. She was described by her grandson as a fantastic, beautiful, and kind-hearted Lady. He added that she died in her early fifties when she received the news that her son Reuben had married in London in 1937. Reuben Louis had two brothers and three sisters, who are, from left to right, on the photo of the family (see Fig. 1): Ida Freda, born July 13, 1917, and Edwin (?–1986) on the first row, then Reuben Louis, Millicent, Lilly Helena (1906–?), and Barney (?–1984) on the second row behind their parents. They were all born in South Africa.

South Africa has had a very turbulent history. European exploration of the African coast began in the 13th century when Portuguese navigators tried to discover an alternative route to China for the silk road. They continued during the 14th and 15th centuries, and Bartolomeu Dias (c. 1450–1500) discovered the Cape of Good Hope in 1488. The Dutch East India Company established a trading post in Cape Town in 1652, and European workers, named the Boers, began to settle there. The majority of them had Dutch ancestry, but there were also some Germans and French Huguenots. They mixed with the local population. In 1805, Britain inherited the Cape Colony after the Napoleonic Wars. The British government abolishes slavery on December 1, 1838. The discoveries of diamonds and gold in the 19th century introduced a shift between the agrarian-based economy of the Boers and the industrialisation and the development of urban infrastructure.

They led to new conflicts culminating in the war between the Boer settlers and the British Empire, the Boer wars (1880–1881 and 1899–1902).
Following the defeat of the Boers, the Union of South Africa was created as a self-governing dominion of the British Empire on May 31, 1910 in terms of the South Africa Act 1909, which amalgamated the four previously separate British colonies: Cape Colony, Colony of Natal, Transvaal Colony, and Orange River Colony. The first Prime Minister of the Union introduced the policy of formal racial segregation.
The country became a fully sovereign nation state within the British Empire in 1934 following enactment of the Status of the Union Act. It was governed under a form of constitutional monarchy, with the British monarch represented by a Governor-General. Among other segregationist laws, including denial of voting rights to black people, the Union parliament enacted the 1913 Natives' Land Act, which attributed only 8 percent of the land for black occupancy while white people, who constituted 20 percent of the population, received 90 percent of it. The Land Act would form a cornerstone of legalised racial discrimination for the next nine decades.
At the outbreak of World War I, South Africa joined Great Britain and the Allies against the German Empire. In 1924, the Afrikaner-dominated National Party came to power. Afrikaans, previously regarded as a low-level Dutch patois, replaced Dutch as an official language of the Union. English and Dutch became the two official languages in 1925. During World War II, South Africa's ports and harbours, such as Cape Town, Durban, and Simon's Town, were important strategic assets to the British Royal Navy.
Obviously, the political history of South Africa played an important role in its educational system, which is described in [76]. In 1949, South Africa had five universities: Witwatersrand, Cape Town, Pretoria, Stellenbosch, and Natal, and one federal examining university, the University of South Africa, which comprised four constituent colleges. The total of students was roughly 20,000, the double than before WWII. The white population was composed of British and Dutch. With the advent of Afrikaans (evolved from the Dutch of Holland) as one of the two official languages and the fact that it was used as an exclusive medium of instruction in more than half the schools in the Union, university institutions have drifted into two camps differentiated chiefly by the predominant language of instruction, English or Afrikaans. Cape Town, Witwatersrand, and Natal were English, while Stellenbosch was a white Afrikaans university. Until the early 1930s, the University of Pretoria was the only fully bilingual university in South Africa. By 1931, however, 65% of students were Afrikaans speaking, but only 32% of classes were conducted in Afrikaans. To address this imbalance, the University Council decided in 1932 that Afrikaans would be the only medium of instruction. Up until 1959 the Afrikaans-medium universities had traditionally limited admittance to Whites. The University of the Witwatersrand and the University of Cape Town had, however, remained open to all races, colours, or creeds. The University of Natal admitted students of all races but segregated classes.
One of the key law in the establishment of the apartheid system of racial segregation was made in South Africa in 1950, when the country was officially divided into four racial groups, White, Black, Indian, and Coloureds (people of mixed race or non-Whites who did not fit into the other non-White categories). White schools

were far better off than any of the others, and Indian and Coloured schools were better off than those for Africans. Schooling was compulsory for Whites, Indians, and Coloureds but not for Africans. Apartheid ended in the early 1990s in a series of steps that led to the formation of a democratic government in 1994. See [31, 76].

Until December 1925, the young Reuben Louis attended the South African schools. When he was 13 or 14, he entered a secondary school in Frankfort, a small farming town situated on the banks of the Wilge River in the Province of the Orange Free State, where his parents were living. He matriculated at the University of Cape Town (UCT) in 1925 for the year beginning in March 1926 as a student in the first year of Engineering. He wanted to prepare a B.Sc. in Engineering. He followed the courses of English, Afrikaans, History, German, Mathematics, and Chemistry. He was living in the University House, Government Avenue in Cape Town. In 1927, he took the courses of Pure Mathematics, Applied Mathematics, Physics, and Engineering Geometry. The course he chose in 1928 are not mentioned, but he obtained his B.A. at the end of the year. During this year, he was not living with relatives, at City Mansions, Hope street, Cape Town. He continued his studies in mathematics while holding a junior teaching post in the subject. After following the courses of Pure Mathematics I and II, Applied Mathematics I and II, Physics I, Geology I, Economics I, Psychology I, and German I, Rosenberg obtained his M.A. (Magister Artium) in Applied Mathematics, with distinction, on December 19, 1929. The programs of these courses show that they were of a high standard. No details are known on the topic of Rosenberg's thesis, if there was any, or if the degree was awarded by coursework only in those days. For this last year, his address was at the Mens University Residence, Groot Schuur in Cape Town.

UCT was founded in 1829 as the South African College, a high school for boys. Women were admitted in 1887 and the first small group of black students in the 1920s. The College was formally established as a University in 1918 on the basis of the Alfred Beit (1853–1906) bequest and additional substantial gifts from mining magnates Julius Wernher (1850–1912) and Otto Beit (1865–1930).

Let us give some information on the physics and mathematics departments of this university at Rosenberg's time. Sir Basil Ferdinand Jamieson Schonland, OMG, CBE, FRS (1896–1972) matriculated at the age of 14 at St. Andrew's College. After WWI, he was a research student at the Cavendish Laboratory, Cambridge University, where he studied the scattering of beta particles. In 1922, he returned to South Africa and took up a post at UCT as a Lecturer and later Professor of Physics. He left Cape Town in 1937 to become the founding director of the Bernard Price Institute of Geophysics at Witwatersrand University, where he made significant contributions to the study of atmospheric electricity, photographing lightning, and investigating the electric fields generated by thunderclouds.

Alexander Ogg (1870–1948) was Professor of Physics at the University of Cape Town from 1920 to his retirement in July 1936. One of his interests was X-ray crystallography. His work on crystalline structure won general recognition for its originality and scientific value. The high standing of the subject of Physics in South Africa is very largely due to Ogg's own high standards as a teacher and examiner. His students have become members of the staff of practically every university in the country, and his scholarly influence remains imprinted on both schools and universities.

Richard van der Riet Woolley (1906–1986) was an English astronomer who became the eleventh Astronomer Royal. An anecdote is mentioned by the British astronomer and mathematician William McCrea (1904–1999), who joined Imperial College London from 1932 to 1936 as a Lecturer and met Rosenberg there [79]:

Woolley graduated B.Sc. in December 1924 and M.Sc. in December 1925, when he was only 19. He obtained first class honours in both. Later he was given to understand that his performance had been better than those of any other student in the physical sciences for some years both before and after his own. Then for some months of 1926 he served as some sort of junior demonstrator in physics while awaiting the outcome of negotiations for the next step in his scientific progress. Professor R. L. Rosenberg of [an error?] *Halifax, Nova Scotia, was a student to whom he demonstrated. He told me that Woolley would sit opposite him and write out the solution to a problem so that it was the right way up for him, Rosenberg, to read—which was one way to impress students!*

In the Department of Pure Mathematics, teaching loomed far larger than research. The numerous undergraduate classes were not only filled by engineering and surveying students but also by future schoolteachers, applied chemists, actuaries, accountants, and businessmen. In 1927, the staff only consisted in three teachers for 207 students in 11 different courses given in three different faculties. Until 1923 there were only two professors in the department, Lawrence (Lawrie) Crawford (1867–1951), who had assumed his chair in 1899, and Thomas Parkes Kent (1871–1923). Crawford was instrumental for the College to obtain the university status. As a teacher, his vigour was boundless. He regularly gave up to 20 lectures by week, writing everything on the board. The content of his lectures never varied, as most of the other department's courses. They were in the style of the late-19th-century mathematics tripos, with heavy calculations. There was no reference to practical applications. Crawford taught for 39 years what he had himself been taught in the 1880s as a student. Kent was certainly more innovative, and his early death in 1923 threw the department into some disarray. Crawford insisted that he will be replaced with a senior lectureship and a lectureship. Andries Charles Cilliers (1898–1980) was appointed Senior Lecturer in 1925, but he only stayed five months before moving to Stellenbosch University. His successor was Stanley Skewes (1899–1988), who remained at UCT for about forty years. The lecturer's position was filled up in 1925 by Miss G. M. Pagan (later Mrs. Wylie), named "Pancy" because of her penchant for brightly-coloured clothing, a 28-year-old specialist in actuarial mathematics. We found nothing else on her in the internet.

In the Department of Applied Mathematics, although its four courses contained engineering and surveying students, their number never raised above 100 in the 1920s. Mechanics had a large place in them with a significant amount of practical work in the laboratory. The curriculum was quite modern since the book *The Mathematical Theory of Relativity* by Sir Arthur Stanley Eddington (1882–1944), first published in 1923, was recommended to postgraduate students as soon as 1924. In the 1920's, the staff of the department included Alexander (Sandy) Brown (1878–1947), mostly interested in seismography, and Jan Stephanus van der Lingen (1887–1950), who worked on topics as different as liquid crystals, X-rays, biophysics, and radioactivity. Van der Lingen had an outstanding ability to design his own experimental equipment, and several of his designs came into general use in physics laboratories. He was also equally at home in physics and applied mathematics. He studied under Max von Laue in Zürich for his thesis on X-ray crystallography after graduating in Stellenobosch in 1912 at the age of 25. Rosenberg certainly followed his lectures, and, maybe, it was van der Lingen who supervised his B.A. and his M.A. and suggested him to go to Berlin for his doctorate. We will see below that Max von Laue was a member of the thesis committee of Rosenberg in Berlin in 1933. Maybe Rosenberg also had Crawford, Cilliers, Skewes, and Pagan as his professors [95].

Alexander Brown became Professor of Applied Mathematics at the University of Cape Town in 1903 and kept this position until his death on January 27, 1947. He contributed to the

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study of the ratio of incommensurables in geometry and relations between the distances of a point from three vertices of a regular polygon. He was an excellent lecturer and held the attention of a class of fifty about 1911 to a class of about two hundred later. He was always accessible to ordinary students and ready to give up his time to advise them in their difficulties.

On July 4, 1928, at the Meeting of the South African Association for the Advancement of Science held in Kimberley, June 29–July 4, 1928, Rosenberg (who was 19 years old) read the paper [1], which was published in the December issue of the South African Journal of Science. Its topic was the elastic impact of a sphere on a plane fixed surface. Since this paper is the first one published by Rosenberg, let us quote its introduction:

> *In treating the impact of a sphere on a plane fixed surface the equation of conservation of momentum cannot be used and the subsequent motion of the sphere after impact is solely determined by Newton's kinematical relation: that the velocity of the separation bears a fixed ratio to the velocity of the approach, this ratio depending only on the nature of the materials. Thus, if a body impinges with a velocity V_a on a plane fixed surface and rebounds with a velocity V_s in exactly the opposite direction then $V_s = -eV_a$, e being the "coefficient of restitution", which is a constant. Hence, if a sphere is allowed to drop from a height h_a on to a plane surface, then the height of rebound h_s will be given by $h_s = e^2 h_a$. An investigation into this relation was undertaken and an experiment was arranged with the simple apparatus at one's disposal in any laboratory.*

The results of this experiment are then described and discussed. At the end of the paper Rosenberg wrote *I am greatly indebted to Dr. J. S. van der Lingen under whose direction these investigations have been carried out*.

2.2. The University of the Witwatersrand, 1930. After his M.A. at UCT, on March 1st, 1930, Rosenberg was appointed Junior Lecturer in Applied Mathematics at the University of the Witwatersrand (Wits) in Johannesburg. This is the address given on the paper [2] Rosenberg read on July 9, 1930 in front of the South African Association for the Advancement of Science. The coefficient of restitution (denoted by e) is the ratio of the final to initial relative speed between two objects after they collide. In [1], Rosenberg had shown that this coefficient depends on the velocity of the impact. In [2], he studied the effect of a thin layer of oil on the surface. He gave a curve representing e^2 in term of the height of the drop. It showed that the loss of height of rebound due to the oil (i.e., the loss of energy due to the oil) is not a constant but is a quadratic function of the height and that it appears that the film of oil acts as a sort of cushion, and it is only at a height of 158 cms that the film is completely broken and there is clean metal to metal contact. The method he adopted was the same as in his previous paper.

Rosenberg is mentioned as having a B.Sc. and as one of the secretaries (with John Orr (1870–1954), head of the Department of Mechanical and Electrical Engineering of the University of the Witwatersrand) of the 28th annual meeting of the South African Association for the Advancement of Science, which was held in Caledon on July 7–12, 1930[1]. Caledon is a town in the Western Cape province, located about 100 kilometres east of Cape Town next to mineral-rich hot springs.

The origins of the University of the Witwatersrand lie in mining industry. The South African School of Mines was established in Kimberley in 1896 and transferred to Johannesburg as the Transvaal Technical Institute in 1904, becoming the Transvaal University College in 1906. Then, the Johannesburg campus was renamed the South African School of Mines and Technology four years later. Other departments were added as Johannesburg grew, and in

[1]The South African Journal of Science, Volume XXVIII and Volume XXXIV, 1937, p. viv.

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1920 the name was changed to the University College, Johannesburg. Full university status was granted in 1922, incorporating the College as the University of the Witwatersrand, with effect March 1. The University had 6 faculties (Arts, Science, Medicine, Engineering, Law, and Commerce), 37 departments, 73 members of academic staff, and little more than 1000 students. It was declared that the University "should know no distinctions of class, wealth, race or creed". Between 1928 and 1939, the number of students expanded by about 40%, from 1476 to 2544. The most important growth was in the faculty of Engineering, including Architecture, from 267 to 829. See [84] for a detailed history of Wits.

Rosenberg resigned his appointment at the University of the Witwatersrand on September 30, 1930.

2.3. Dissertation in Berlin, 1930–1933. Having received in 1929 a *Queen Victoria Scholarship* of £150 per annum from the University of Cape Town, Rosenberg registered for a doctorate in nuclear physics in Berlin, Germany, at the Friedrich-Wilhelms-Universität zu Berlin (Humboldt-Universität zu Berlin since 1949) in October 1930. This scholarship was awarded to a graduate obtaining first class Honours degree or Master's degree with distinction, be under 30 years of age, and be prepared to study towards a higher degree [105].

Nuclear physics and quantum mechanics. The end of the 1920s and the beginning of the 1930s were an exciting time for nuclear physics and modern quantum mechanics, which emerged in 1925 when the German physicists Werner Heisenberg (1901–1976), Max Born (1882–1970), and Pascual Jordan (1902–1980) developed matrix mechanics and Schrödinger invented wave mechanics and his non-relativistic equation as an approximation of the generalised case of the wave-particle duality theory of Louis Victor de Broglie (1892–1987) of 1924. Heisenberg formulated an early version of his uncertainty principle in 1927, analyzing a thought experiment where one attempts to measure an electron's position and momentum simultaneously. Beginning in 1927, researchers attempted to apply quantum mechanics to fields instead of single particles, thus resulting in quantum field theories. Starting around 1927, Paul Adrien Dirac (1902–1984) began the process of unifying quantum mechanics with special relativity by proposing the Dirac equation for the electron. His textbook appeared in 1930, and the Hungarian polymath John von Neumann (1903–1957) formulated the rigorous mathematical basis for quantum mechanics as the theory of linear operators in Hilbert spaces described in his likewise famous 1932 book. In 1930, the neutrino was predicted by Wolfgang Ernst Pauli (1900–1958) and antimatter by Dirac. James Chadwick (1891–1974) discovered the neutron in 1932, and Carl David Anderson (1905–1991) the positron. Induced radioactivity was discovered in 1934 by Irène Joliot-Curie (1897–1956), the daughter of Pierre Curie (1859–1906) and Marie Curie (1867–1934), and her husband Jean Frédéric Joliot-Curie (1900-1958), a discovery for which they received the Nobel Prize in Chemistry one year later.

Rosenberg defended his Inaugural Dissertation *Wirkungsquerschnitte von Atomen gegenüber langsamen und schnellen Elektronen* (Effective cross sections of atoms compared with slow and fast electrons[2]) [3] on July 7, 1932 at the Friedrich-Wilhelms-Universität in Berlin. The work was accepted on March 1, 1933 with *magna cum laude*. The committee consisted in Friedrich (Fritz) Möglich (1902–1957), a member of the Institute of Theoretical Physics, and Rosenberg's advisor, Max Theodor Felix von Laue (1879–1960), Nobel Prize laureate in

[2]Translation of the title found in *British Chemical Abstracts*, 1933, part A, p. 203.

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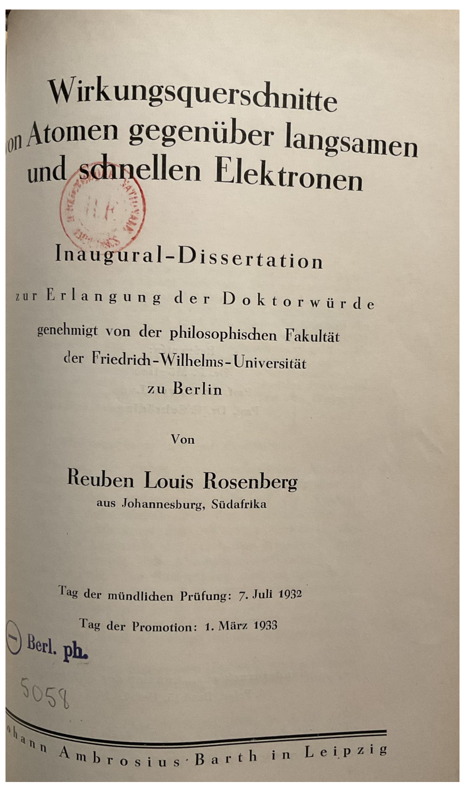
Physics in 1914 for his discovery of the diffraction of X-rays by crystals, and Erwin Rudolf Josef Alexander Schrödinger (1887–1961), who will obtain the Nobel Prize in Physics in 1933. Möglich was professor there and also Von Laue and Schrödinger, who soon became strong opponents to nazism.

Rosenberg's thesis (see Fig. 2) was published as a paper, received August 24, 1932, in the journal *Annalen der Physik* [3]. Its abstract is:[3]

> *The force exerted by a plane matter wave on a spherically symmetrical potential area is calculated. This is applied to the scattering of electrons and atoms to obtain an expression for the cross section in the form of a series of coefficients α_n. As a specific example, the general result for the Allis and Morse atomic model is calculated and the results are plotted for different values of the parameter b^2. The anomalies of the Ramsauer effect are discussed and the positions of the maxima are approximately calculated. Finally, a transition to the limit is made using the asymptotic representations of the occurring functions for large velocities; and a first approximation of the effective cross section of action is calculated.*

Let us remind that scattering refers to particle-particle collisions between molecules, atoms, electrons, photons, and other particles. The effective cross section (or simply, cross section) is a quantity that characterizes the probability of transition of a system of two colliding particles to a certain final state as a result of both elastic and inelastic scattering.


Wirkungsquerschnitte
on Atomen gegenüber langsamen
und schnellen Elektronen

Inaugural-Dissertation

zur Erlangung der Doktorwürde
genehmigt von der philosophischen Fakultät
der Friedrich-Wilhelms-Universität
zu Berlin

Von

Reuben Louis Rosenberg
aus Johannesburg, Südafrika

Tag der mündlichen Prüfung: 7. Juli 1932
Tag der Promotion: 1. März 1933

hann Ambrosius Barth in Leipzig

Fig. 2. Thesis of Reuben Louis Rosenberg.

[3]The German text is given in the Appendix A.

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The beginning of the thesis clearly explains the context and the goal of Rosenberg:[4]

> *The methods previously used by different authors to calculate the scattering of electrons from atoms all rely on the analogy between wave mechanics and the wave theory of light. The wave function of the scattered particle is calculated and the amplitude of the spherical scattered wave obtained is used to calculate the absorption coefficient or the cross section. The impact process is understood as a diffraction phenomenon and loses the character of a dynamic process or an impact, as it is known to us in classical mechanics. The reason is that the concept of force has not been discussed at all in wave mechanics.*
>
> *In ordinary classical mechanics, the law of conservation of momentum is used as a fundamental fact for the treatment of a collision problem. Classically, we can imagine a particle colliding with a body described by saying that the particle moving in a certain medium hits the separating surface of another medium. At the collision an impulse is exchanged between the particle and the body.*
>
> *This change of momentum is determined by the law of momentum. The particle transmits a certain momentum to the body. We say that a pressure is exerted on the body. In analogy to the classic treatment, we want to treat the collision of an electron with an atom using wave mechanics. Two media in the wave-mechanical field differ in their potential functions. The impact of an electron on an atom is described by the incidence of a matter wave on a potential field. The collision process is described by the behavior of the wave function in the various media. Associated with a matter wave is a certain momentum density that is, a function of the wave function and its first derivative. The wave function is generally different in the various media, so that the associated momentum is also different. A transfer of a certain momentum to the atom therefore takes place. This transmission, which in the classic case is determined by the law of momentum, is here determined by the requirement of continuity of the wave function and its derivative at the interface. Since we are dealing here with a momentum density, we must ask about the transfer of momentum per volume unit per time unit* [...]
>
> *The replacement of the corpuscular process by a wave process is a reason to set up the concept of the wave-mechanical field. It makes sense to describe the process of absorption using a field theory. As, in the field theory of electricity, Maxwell's stress tensor describes the state of the electric field, so the energy momentum tensor describes the state of the matter field. We may then consider the atoms or the potentials that are in the matter field as sinks of matter, from which the concept of absorption immediately follows. The intensity of the field is then a measure of the absorption. That this representation will lead to the same result as that obtained from classical mechanics by introducing the force lies in the fact that the power and the flux of the field can both be obtained from the energy momentum tensor.*

The problem treated by Rosenberg in his thesis was very topical. In their paper received May 8, 1933, Philip McCord Morse (1903–1985) and William Phelps Allis (1901–1999) gave reasons why the Born approximation was incapable of dealing with the scattering of slow electrons [83]. Since this approximation assumed that the sine of the phase angle δ equals δ, the results computed by the Born method became completely unreliable when δ is greater than $\pi/2$. The exact solutions had not been obtained. They set up the equations, including exchange effects, for the best possible wave function for an electron scattered from hydrogen or helium when the complete wave function is of the separable type usually used in atomic theory. These

[4] The German text is given in the Appendix A.

equations were solved on a differential analyzer to find the best possible curves for the δ's, for the angle distribution of scattering, and for the total cross section for this type of wave function. Obviously Rosenberg knew their work before its publication. Indeed Allis majored in school and received his Bachelor of Science (S.B.) in 1923 and Master of Science (S.M.) in 1924 from the Massachusetts Institute of Technology (MIT). He was granted a *Doctorat ès sciences* (Sc.D.) in physics, in 1925, from the University of Nancy, France. From 1925 to 1929, he was a research associate at MIT. It was there that he met Morse who made arrangements for postdoctoral studies and research with Arnold Johannes Wilhelm Sommerfeld (1868–1951) at the Ludwig Maximilian University of Munich in 1930 and at the University of Cambridge in the spring and summer of 1931. Allis went with Morse to Munich and Cambridge.

The Ramsauer effect, also sometimes called the Ramsauer–Townsend effect, is named after Carl Ramsauer (1879–1955) [97] and also after John Sealy Townsend (1868–1957) and Victor Albert Bailey (1895–1964) [120], who independently studied the collisions between atoms and low-energy electrons in the early 1920s. It is a physical phenomenon involving the scattering of low-energy electrons by atoms of a noble gas. The effect can not be explained by classical mechanics but requires the wave theory of quantum mechanics. If one tries to predict the probability of collision with a classical model that treats the electron and the atom as hard spheres, then one finds that the probability of collision should be independent of the incident electron energy. However, Ramsauer and Townsend observed that for slow-moving electrons in argon, krypton, or xenon, the probability of collision between the electrons and gas atoms attains a minimum value for electrons with a certain amount of kinetic energy. This is the Ramsauer–Townsend effect. No good explanation for the phenomenon existed until the introduction of quantum mechanics, which explains that the effect results from the wave-like properties of the electron. Let us mention that, in his dissertation, Rosenberg made use of Hankel, Bessel, and Laguerre functions.

Friedrich Möglich, Rosenberg's advisor, was a student of Erwin Schrödinger and Max von Laue, from whom he received his doctorate in 1927 with a dissertation *Beugungserscheinungen an Körpern von ellipsoidischer Gestalt* (Diffraction phenomena in bodies of ellipsoidal shape). In 1930 he received his habilitation for conducting students' researches with a work on the quantum effects of oscillating continua. The same year Rosenberg arrived in Berlin. In 1932 Möglich joined the *Nationalsozialistische Deutsche Arbeiterpartei* (NSDAP) and the *Sturmabteilung* (SA). He was one of the first National Socialists to join the teaching staff at Berlin University. In 1934 he was appointed head of the Lectureship at the University of Berlin. From 1935 Möglich came into conflict with the Nazi regime. Since the Gestapo was investigating him for currency offenses and "racial defilement", he temporarily fled to London and Paris. After his return he was arrested in 1937 because of a love affair with Countess Else von Bubna-Littitz-Gerisch, a Jewess. In 1938 Möglich was expelled from the NSDAP for "racial defilement". This also ended his academic career. Through Laue's advocacy, he found a scientific work in industry as a freelance employee at the research company for electrical lighting Osram GmbH. After the end of the war, Möglich was a scientist at the Biomedical Research Center in Berlin. In 1945 he was the spokesperson of the German Central Administration for Public Education (DZVVB). In 1947 he took over the editorship of the *Annalen der Physik*. In 1946 he was appointed full professor of Theoretical Physics. In 1947–1957 he was director of the Institute for Solid State Research of the German Academy of Sciences. He was also the head of the Institute for Theoretical Physics at the Humboldt University. Möglich belonged to the 1st and 2nd German People's Councils, which were the consultative bodies in the Soviet Occupation Zone of Germany that operated in 1948–1949. Their main task was to draw up a constitution on the basis of a draft presented in 1946 by the *Sozialistische Einheitspartei Deutschlands* (SED), the founding and ruling party of the German Democratic Republic. On Möglich, see [63, 64].

At the end of [3], Rosenberg wrote:[5]

> *I would like to thank Dr. F. Möglich for the suggestion of this work and for the encouraging interest in its execution.*
> *I would also like to express my gratitude to the University of Cape Town for the award of a "Queen Victoria Memorial" scholarship.*

As usual, after the thesis, there is a small biography written by the candidate himself:[6]

> *I, Reuben Louis Rosenberg, was born in 1909 in Johannesburg, South Africa. Until December 1925 I attended the South African elementary school. From March 1926 to December 1929 I studied at the University of Cape Town, graduating with an M.A. degree. From the winter semester of 1930 I continued my studies at the University of Berlin.*
> *I owe a great debt of gratitude to all my academic teachers, especially Dr. F. Möglich for his suggestion of this work and his continued interest during its execution. I would also like to take this opportunity to express my gratitude to the University of Cape Town for the granting of a Queen Victoria Memorial Scholarship, which made my studies in Berlin possible.*

An analysis of Rosenberg's thesis was given in *Chemisches Zentralblatt*, Nr. 19, Band I, May 10, 1933, p. 2911.

2.4. Imperial College, 1932–1934. After his stay in Berlin, Rosenberg received a *Beit Fellowship for Scientific Research* of the annual value of £240[7] from the Imperial College of Science, Technology, and Medicine, South Kensington, in London to undertake theoretical investigations in topics connected with quantum mechanics and obtain a *Diploma of Membership of the Imperial College of Science and Technology* (D.I.C.), an academic certificate awarded to postgraduate students upon graduation. He stayed there from October 1932 to July 1934 in the Department of Mathematics.

The College grew out of the ideas of Prince Albert, the consort of Queen Victoria, for an area of culture including other establishments. Until 2007, it was part of the University of London and bestowed the University of London's degrees as well as its own diplomas. To be awarded a D.I.C., the student needs to be registered as a student of the College, which also hosts visiting students from institutions outside the UK who attend classes and complete courses at the College.

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[5] *Hrn. Dr. F. Möglich möchte ich für die Anregung zu dieser Arbeit und für das fördernde Interesse bei ihrer Ausführung herzlich danken.*
Der Universität Kapstadt möchte ich auch an dieser Stelle für die Gewährung eines "Queen Victoria Memorial"-Stipendiums meinen Dank aussprechen.

[6] *Ich, Reuben Louis Rosenberg, bin im Jahre 1909 in Johannesburg, Südafrika, geboren. Bis zum Dezember 1925 besuchte ich die Südafrikanische Volksschule. Von März 1926 bis Dezember 1929 studierte ich an der Universität Kapstadt und erwarb zum Schluß den M.A.-Grad. Seit dem Wintersemester 1930 setzte ich mein Studium an der Universität Berlin fort.*
Allen meinen akademischen Lehrern bin ich zu großem Dank verpflichtet, besonders Herrn Dr. F. Möglich für seine Anregung zu dieser Arbeit und sein förderndes Interesse während der Ausführung. Auch der Universität Kapstadt möchte ich an dieser Stelle meinen Dank aussprechen für die Gewährung eines Queen Victoria Memorial-Stipendium das mein Studium in Berlin ermöglichte.

[7] Beit Fellowships for Scientific Research. Nature 130, 89 (1932). https://doi.org/10.1038/130089b0

Beit Fellowship. Where is the Beit Fellowship coming from? The German brothers Alfred Beit, Otto Beit, British Citizenship in 1896 and FRS in 1924, and German born Julius Wernher, who all became part of the British Establishment, made their fortunes through diamond and gold mining in South Africa. They were financiers and philanthropists and had many other educational, scientific, social, and cultural interests. In 1913 Otto Beit created a Trust Fund of £26,500 to provide Research Fellowships tenable at Imperial College and open to men and women of European descent by both parents, otherwise of any nationality.

Rosenberg's supervisor was Sydney Chapman (1888–1970), a British astronomer and geophysicist, Fellow of the Royal Society, and the title of his diploma was *The Quantum Statistics of Imperfect Gases*. According to the minutes of the committee making the award, the work was split and published into three papers. The first paper is mentioned to have the title *The perturbation of atoms by electrons*, but we were unable to find it (this work is not mentioned in the CV Rosenberg wrote in 1955, but other of his papers are also missing). Since the committee took place in July 1934, presumably after the full assessments of the papers, it would seem unlikely that there was a later change but not impossible perhaps. The two other papers have the titles *The concept of force in wave mechanics*, which corresponds to [5] and *A problem of the Chromosphere*, which is [4].

Before describing the work done by Rosenberg for his D.I.C., let us set up the scene. The brilliant surface of the Sun is called the photosphere. Above the photosphere is the chromosphere. It emits a reddish glow as super-heated hydrogen burns off, but the red rim can only be seen during total solar eclipses or with sophisticated telescopes. At other times, its light is usually too weak to be seen. Above the chromosphere is the corona, which emits the solar wind. In the early 1930s, Chapman and his first graduate student, Vincenzo Consolato Antonio Ferraro (1907–1974), predicted the presence of the terrestrial magnetosphere, a region that meets and blocks the solar wind thus protecting all living organisms from potentially detrimental and dangerous consequences. They also anticipated the characteristics of the magnetosphere that were confirmed 30 years later by the Explorer 12 satellite. A new theory of the chromosphere was described in 1933 by the Norwegian astrophysicist Svein Rosseland (1894–1985) [107]. Its main feature was that an atmosphere of neutral molecules can be supported by the upward drift of positive ions. For solving this hydrodynamical problem, he assumed that the resistance for the ions was proportional to the drift velocity, that is, the average velocity attained by charged particles such as electrons, in a material due to an electric field. It is the range of validity of this assumption that Rosenberg discussed in [4], a paper received March 15, 1934. His investigations led him to propose a general expression of the resistance for all values of the drift velocity as a fairly rapidly converging series. He concluded that Rosseland's result only corresponds to its first term, and hence, that it is only valid for small drift velocities, less than the average thermal velocity. At the end of his paper, Rosenberg thanks the mathematician William Hunter McCrea (1904–1999), who was then a Reader at Imperial College in London, for his many suggestions and discussions.

When he was working in London, Rosenberg met and married Violet Eva Hatcher (London, April 17, 1913–February 20, 1981) on January 30, 1937, in London, England (Fig. 3). She was the daughter of Fred and Emily Hatcher. Reuben and Violet will have two daughters named after the stars Zeta Rachael (b. June 22, 1940) and Gale (b. December 28, 1943), both born in Pietermaritzburg, Kwa-Zulu Natal, South Africa, when Rosenberg was in Natal University College. Because their mother was British, they became British Nationals after leaving South Africa and Reuben Louis too.

The paper [5] of his D.I.C. concerns the concept of forces in wave mechanics. Since the introduction of the paper clearly describes it, let us reproduce it:

> *A collision between two bodies in classical mechanics entails a change of momentum of each body. This change of momentum determines the impulse of the one body on the other. In the scattering of a current of particles by a force centre (this force centre being due to the presence of another body, "the scatterer") we have a redistribution of the current throughout space. There is a change of momentum of the scattered particles, which in this case can be expressed as a rate of change of momentum. To complete the analogy with classical theory one would inquire what is this rate of change of momentum, or the force exerted on the scatterer by the current of particles. We shall therefore attempt to formulate the idea of force in wave mechanics as it arises out of this scattering problem. We shall, however, find general formula which will be applicable also to the case of electrons in discrete negative energy states, i.e., corresponding to the periodic motions of classical mechanics. Further, we shall discuss properties of the force in stationary and non-stationary states in which the effect of a perturbation causing transitions will be found. In finding the force exerted by a current of particles on a scatterer characterized by a field of force, we shall be guided by the correspondence between wave mechanics and the wave theory of light; and the method will be similar to the calculations for the pressure of light on a body. A particular case of this has already been considered by the author* [Rosenberg refers to his thesis [3]]*, and the force expressed in terms of an effective cross-section of the scatterer.*

Rosenberg approached the problem from the point of view of field theory and defined the force-density as the four-dimensional divergence of the energy-momentum tensor of the material field. Möglich is thanked *for the many discussions he has given me on this subject*.

After his D.I.C., his supervisor Sydney Chapman wrote the following recommendation letter for Rosenberg on June 22, 1934:

> *Dr. R. L. Rosenberg, at present Beit Scientific Research Fellow at the College, informs me that he is applying for a university lectureship in mathematics: I have the pleasure in testifying that in my opinion he is well qualified and suitable for such a post.* [A description of his CV follows.] *I wrote to his supervising professor in Berlin, Professor Möglich, enquiring as to his ability in research. Professor Möglich replied in very favourable terms, mentioning, among other things, that Dr. Rosenberg, though not a quick worker in research, was very accurate and showed great understanding of his problems. He was among those elected in 1932 to Beit Fellowships, and during the past two years has continued his researches here, not working under a supervisor, as at Berlin, but independently. He has made good progress in his work, as the list of his publications and manuscript papers will show, and in my opinion he has justified his election to a Beit Fellowship in 1932; in consequence of his work here I am recommending that he shall be awarded the Diploma of the Imperial College of Science and Technology, for Research in Mathematical Physics. Thus, on the score of capacity and actual accomplishment in original research, which is an important part of the work of a mathematical lecturer, Dr. Rosenberg has shown himself to be well qualified, and I have confidence that he will continue to pursue mathematical investigation in any post whose duties afford him the necessary time for such work. As regards his learning, and his ability to teach, I can speak equally favorably. He has a wide knowledge of mathematics both on its pure and applied sides, and has shown himself able to use his pure mathematics in his researches. As regards to his ability to teach, I can testify that he has shown a marked gift for exposition, while reading papers to the mathematical colloquium here: his clarity in thought and expression*

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would, I feel sure, prove equally valuable in the teaching of undergraduates.
Finally, I can recommend him strongly also on personal grounds, as a man of pleasant personality who would be likely to prove persona grata to his colleagues and pupils, and as one who can be relied on to carry out his teaching duties with every care and attention.

Fig. 3. **Reuben Louis Rosenberg and his wife. ©Gale Rosenberg**

2.5. Natal University College, 1935–1949. Thanks to Philip Bernard Stein (1890–1974), Rosenberg received an appointment as a Lecturer in March 1935 and then Senior Lecturer (1938) in Applied Mathematics at the Natal University College (NUC) located in Pietermaritzburg. Notice that Stein was working at the Durban's campus of the NUC. In [60, pp. 151–152], it is mentioned that Rosenberg had taught at the University of the Witwatersrand before arriving at NUC.

Let us give some information about NUC. Around 1903–1904, the Natal Government recognized the need for locally based tertiary education. A commission comprising an even balance of technical/commercial and professional representatives was appointed. There was unanimity that a University College should be established in Pietermaritzburg. In 1908, the Pietermaritzburg City Council indicated its willingness to provide land in Scottsville for the College, and, on 11 December 1909, the Natal University College Act was promulgated. It clearly stated that no religious test would be applied to the appointment of any staff member or admission of any student. NUC was expanded to include a campus in Durban in 1931. In 1936, NUC broke with its policy of accepting only white students into its courses and

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launched an entirely new programme of part-time classes in Durban dedicated to the so-called "non-European students". On 15 March, 1949, NUC became the University of Natal situated both in Pietermaritzburg and Durban. From 2004, it became the University Kwazulu-Natal. See [60, 61] for the establishment and the history of NUC.

From 1910 to 1926 or 1930, the professor of Pure and Applied Mathematics at NUC was William Nicholas Roseveare (1864–1948). In March 1924, in an effort to reverse the declining student numbers in Applied Mathematics (for Roseveare was a pure mathematician), A. Lang Brown (B.A., Cambridge) was appointed to a lectureship in that field and in Physics. Two years later Roseveare retired, and John McKinnell (M.A., B.Sc., Glasgow), the archetypal absent-minded but likeable professor, replaced him as head of Mathematics and Applied Mathematics. He was Stein's best friend at the university. In Pure and Applied Mathematics, there will be a doubling of pre-war student numbers in 1946, with 12 students majoring in Pure Mathematics in 1948, but there were never more than three members of staff to teach the two branches of the discipline, Stein, Rosenberg, and John (Jack) R. H. Coutts (M.Sc., London), who taught Mathematics before assuming the chair of Physics in 1948 [60].

The first work of Rosenberg when at NUC was to present two lectures to the South African Association for the Advancement of Science on July 5, 1938. Only their titles *Note on a monotonic property of a product of Hankel functions* [6] and *The evaluation of certain integrals involving products of confluent hypergeometric functions* [7] are given in the South African Journal of Science, but they are not printed.

Rosenberg was involved in the Dramatic Society that staged, in 1940 (not for the first time), an outstanding performance of Oscar Wilde's "The Importance of Being Earnest". At this occasion he was described "superb" for his "stage presence" and "diction". He himself financed the award of an annual "Oscar" for the best stage performance. Since, for some years, he was warden of the Men's Residence, or "Lord Warden" as he called himself with ironic reference to the Lady Warden on campus. When he donated a cup to be presented annually to the best student actor in a University Dramatic Society production, it was appropriately engraved "The Lord Warden's Trophy" [60].

2.5.1. War time, 1940–1943. In 1940, Rosenberg joined the armed forces as the Council of the College was specifically requested to release certain staff members from their academic responsibilities so that they could assist in military training and in other areas of expertise required for the war effort. He volunteered for active service with the South Africa Forces in the Middle East from July 1940 to 1943.

In a letter to Larry Goldberg (husband of Reuben's great niece, Jillian Goldberg, née Zausmer), Milton Hirschfield (Reuben's nephew) wrote (in March 2009):

> *Rueben* [sic] *who joined the army and told them he was a Plumber in order to avoid being an Officer a long story. In SA (South Africa) he was a Professor of Applied Mathematics and he was brilliant, he joined the Army at the outbreak of the war. He told them he was a Plumber and managed to persuade them to enlist him in to the Natal Field Artillery. Before leaving for North Africa the unit was given intensive training on Range Finding, Reuben was bored and starting shooting little pebbles about unfortunately one hit his Sergeant. He was sent to Detention Barracks for ten days. After the exams he received 98% and the unit average was 60% they investigated him and discovered his true occupation. He was immediately transferred to the Gas Laboratories as Office Commander and was sent North but stayed well behind the front line. The Natal Field Artillery was wiped out to the last man at the battle of Tobruk. Reuben returned to SA at the end of the war and called the family together to advise that South Africa was not for white people as it belonged to the*

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Blacks. When he was working in London he met and married Violet they had two Daughters named after the Stars Gale and Zeta. In 1944 I lived with Reuben, Violet and Gale and Zeta in Pietermaritzburg, Violet passed away and Rueben [sic] *visited us in Australia.*

2.5.2. Natal University College, 1944–1947. In 1944, Rosenberg is mentioned as belonging to the Department of Mathematics of the NUC at the occasion of a meeting of the Durban Library Committee regarding the safety of the collections. Duplication of library materials between the three Durban libraries was not encouraged, but practical difficulties were encountered in the prevention of duplication between the two centres. Via the Pietermaritzburg Library Committee, Rosenberg requested, at a meeting held on June 26, 1944, that "a complete catalogue of the books and journals of the N.U.C. Library in Durban be made available in the Pietermaritzburg Library, pointing out that this will avoid excessive duplication and increase our library facilities and also that current numbers of certain journals, which by arrangement are obtained in Durban and not in Pietermaritzburg, be circulated in the Pietermaritzburg Library one month after their arrival in Durban" [43, p. 106].

2.5.3. The Stein–Rosenberg Theorem, 1947–1948. It is during this period that the paper Rosenberg coauthored with Philip Stein on the solution of linear equations by an iterative relaxation method, which contains the famous Stein–Rosenberg theorem, was written. It was submitted on June 16, 1947, read on October 16, 1947, and appeared in 1948 [9]. This paper is analyzed in detail in [39]. Let us explain it and its context.

Consider the system of linear equations $Ax = b$ and decompose the matrix A into $A = M - N$. If M is nonsingular, then this expression is named a *splitting* of the matrix A, and associated with it is the iterative method

$$Mx_{k+1} = Nx_k + b, \qquad k = 0, 1, \ldots,$$

where x_0 is an arbitrary vector. Equivalently, this method can be written as

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, \qquad k = 0, 1, \ldots$$

This iterative method (often named a *relaxation method*, see [109]) converges to the solution of the system if and only if $\varrho(M^{-1}N) < 1$. The smaller this spectral radius is, the faster the method converges. Thus it is of interest to compare the spectral radii of two methods, and this is precisely what the Stein–Rosenberg theorem is doing for the methods of Gauss–Seidel and Jacobi.

Usually, the following splitting $A = D - E - F$ is considered where D is the diagonal of A, E its strictly lower part, and F its strictly upper part. The method of Jacobi consists in the iterations

$$x_{k+1} = D^{-1}(E + F)x_k + D^{-1}b,$$

and the method of Gauss–Seidel in the iterations

$$x_{k+1} = (D - E)^{-1}Fx_k + (D - E)^{-1}b.$$

The iteration matrices of these methods are respectively denoted as $B = D^{-1}(E + F)$ for the method of Jacobi and $\mathcal{L}_1 = (D - E)^{-1}F$ for that of Gauss–Seidel. The Stein–Rosenberg theorem is as follows:

THEOREM 2.1.
If B is nonnegative, then one and only one of the following mutually exclusive relations is valid:
1. $\rho(B) = \rho(\mathcal{L}_1) = 0$.
2. $0 < \rho(\mathcal{L}_1) < \rho(B) < 1$.
3. $\rho(B) = \rho(\mathcal{L}_1) = 1$.
4. $1 < \rho(B) < \rho(\mathcal{L}_1)$.

This theorem shows that both methods are simultaneously convergent or divergent. When convergence occurs, the method of Gauss–Seidel is faster than that of Jacobi, and, in the case of divergence, it diverges faster. Let us remark that the names of Jacobi, Gauss, and Seidel are not mentioned in the paper. The proof of the theorem is based on the Perron–Frobenius theorem on the dominant eigenvalue of a nonnegative matrix. Another proof of it, also based on the Perron–Frobenius theorem, was given in 1958 by William "Velvel" Morton Kahan (born 1933) [68].

The Stein–Rosenberg theorem is sometimes written under the following form:

THEOREM 2.2.
Let A be irreducible and $A = I - L - U$, where $L, U > 0$ are the strictly lower and upper triangular parts of A, respectively. Then, one and only one of the following mutually exclusive relations holds:
1. $0 < \varrho((I - L)^{-1}U) < \varrho(L + U) < 1$.
2. $\varrho(L + U) = \varrho((I - L)^{-1}U) = 1$.
3. $1 < \varrho(L + U) < \varrho((I - L)^{-1}U)$.

Let us remind that a matrix is said to be irreducible if it is not similar via a permutation to a block upper triangular matrix. A simple proof and a proof of another theorem by Stein [117] are given in [98]. In this second paper, Stein proved that $\varrho(G) < 1$, where G is a given matrix, if and only if there exists a positive definite matrix P such that the matrix M given by $M = P - GPG^H$ is positive definite. If G is real, then P can be taken to be real [135, p. 80]. On numerical methods for linear equations, a major reference is the book of Richard Steven Varga (1928–2022) [124].

Let us now comment on the publication of this paper. In their paper, Stein and Rosenberg thanked Olga Taussky (1906–1995) for her advice on some points. In [119], Olga Taussky wrote (the references added are those of our bibliography):

> *Mordell asked me to look at a manuscript of Stein and Rosenberg, before it went out to a referee. I liked it, and it has since become a classic. It has been particularly studied by Francois Robert [100] in France, and is discussed in detail in Varga's 1962 book [124]. The Perron–Frobenius theorem, which concerns the eigenvalues of matrices with nonnegative entries plays a big part in this paper.*

Louis Joel Mordell (1888–1972), a number theorist, had been President of the London Mathematical Society from 1943 to 1945. He was an old friend of Stein [115]. François Robert is a French numerical analyst whose work will be described below.

An analysis, by Ewald Konrad Bodewig (born July 22, 1901 in Duisburg–?), of the paper can be found in *Math. Reviews*, vol. 10, no. 7, July–August 1949, p. 485:

> *The authors study the convergence of the Seidelian iteration O and the usual iteration S. In both cases the errors of the nth approximation vector are proportional to the nth power of a matrix. In discussing the convergence of the powers of a matrix the authors start from some fundamental theorems of Frobenius and give some other known theorems [the first of which is not due to Berry, but to E. Schmidt] and some new ones. Theorem III states that A^n converges to zero if $a - bq < (1 - c)(q - 1)$,*

where $q = (a/b)^{1/m}$, m *is the order of* A, *and* a, b, c, *respectively, are the moduli of the dominant elements of* A *below, above, or in the diagonal. Theorem VI states that iteration* O *converges better than* S *if in the system* $(E - C)x = d$ *of equations the elements of* C *are nonnegative. This theorem does not hold if* C *contains positive and negative elements, so that in this case the convergence of* S *is independent of that of* O*. Furthermore, the order in which the equations are taken is important for the convergence of* O*. Theorem VII says that if the elements of* C *are nonnegative the convergence of* O *will be better if the equations are so ordered that the smaller elements of* C *lie above the diagonal.*
E. Bodewig.

It is not known exactly why Stein and Rosenberg decided to work on this problem. However, Stein wrote in his autobiography [115, p. 26]:

In 1936 I again went over to Cambridge on my own. Littlewood [his Ph.D. advisor, see below] *found the money, partly from Caius and partly from the Royal Society. It was during the time of the German-Jewish refugees. I spent most of my time in social visits to the various colleges and to the refugees. I did no work at all. It was a waste of public money, Littlewood said. I was finished as a creative mathematician. I agreed. But we were both wrong as to the latter for since then I have developed my interest in matrix theory and iteration and have solved quite a few problems.*

It must be noticed that, in 1936, Richard Vynne Southwell (1888–1970), the father of relaxation methods, was in Cambridge too, and he could have suggested Stein to study iterative methods for solving systems of linear equations. At that time, Southwell was publishing many papers on this method, he was interested by its convergence (see, for example, [35, 37]), and he could have discussed it with Stein. Southwell was also at Trinity College in Cambridge at the time Stein did his thesis with Littlewood, and they could have met then (see below for a short biography of Stein). Anyway, the study of convergence of relaxation methods was in the air; see, for example, the paper of Jacob Lionel Bakst Cooper (1915–1979) [51], a South African mathematician who studied mathematics and physics at the University of Cape Town. Cooper's paper was submitted on 29 August 1947, and published in 1948. Back to the Durban's campus of Natal University College (NUC), Stein wrote:

The college had been growing steadily in staff and students. After the war there was a big influx of students, ex-servicemen and others. I suddenly found myself with a staff of seven or eight and two Senior Lecturers. By a happy chance, both Senior Lecturers were good administrators and I had leisure. Quite accidentally I had some years before got interested in matrix theory and iteration processes. I had published some years before a paper in association with Rosenberg on the subject of matrices and iteration. This paper is still alive and is regarded as an important pioneer effort in this subject. It had been refereed by Olga Tausky, a Hungarian-Jewish refugee, and recommended for publication. As a very young mathematician, she had been unsure of herself and was worried how this paper would be received. When it received a good review she was relieved. I sent other articles to Olga Tausky who liked them.

In their paper, Stein and Rosenberg quote five papers (in the order of their quotation): a paper of 1940 by the American mathematician and mechanical engineer Rufus Oldenburger (1908–1969) [90], the 1908 paper of Ferdinand Georg Frobenius (1849–1917) [58], a paper of 1945 by Clifford Edward Berry (1918–1963) [32], one of the creators of the first digital electronic computer in 1939, a paper of 1946 by Alfred Theodor Brauer (1894–1985) [38], a German-American mathematician who did work in number theory, and a paper of 1941 by

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R. J. Schmidt [110] on whom nothing is known except his affiliation to Imperial College in September 1939. The paper [90] was on powers of matrices and characteristic roots and [38] on characteristic roots only. In his paper [58], Frobenius proved that if the elements of A are all real and positive, it has a characteristic root which is real, positive, simple, and greater in absolute value than any other characteristic root. As Berry explained in [32], he gave *the first development of a necessary and sufficient criterion for convergence in the general case* of relation methods. Let us quote Schmidt [110]:

> *In the present paper theorems are given which enable the solution of any system of linear simultaneous equations to be found by the method of successive approximations, without the necessity of first putting the equations into their normal form.*

He showed that the components of each iterate of a relaxation method are the solution of a linear difference equation and can be expressed as a ratio of two determinants. These determinants are exactly those that will be used in 1949 [112] and 1955 [113] by Daniel Shanks (1917–1996) for defining the sequence transformation named after him. This transformation can be recursively implemented by the ε-algorithm due to Peter Wynn (1931–2017) in 1956 [133]. On the history and the developments of these topics, see [40, 41].

It can easily be deduced that Stein was inspired by the papers [32] and [110], where the relaxation methods of Jacobi and Gauss–Seidel are described, and that, with Rosenberg who was a good mathematician, they used the theoretical results on matrix norms and characteristic roots given in the papers [58, 90] and [38] for proving their theorem.

The Stein–Rosenberg theorem became rapidly and widely known. Let us quote a report by Magnus Rudolph Hestenes (1906–1991) and John (Jack) Todd (1911–2007), the husband of Olga Taussky, [62, p. 48]:[8]

> *There was a notable contribution from S. Africa. In the early post war years Philip Stein, together with a pupil, R. L. Rosenberg, submitted a paper to the London Mathematical Society. This paper was concerned with comparison of the two classical iterative methods for the solution of linear systems associated with the names of Jacobi and Gauss–Seidel. Olga Todd, as referee, noted its importance and novelty and gave them detailed advice and made sure that it was published. This paper has become a classic and the Stein–Rosenberg Theory is a standard chapter in courses on iterative matrix analysis. During his visit to NBS* [National Bureau of Standards] *and later, Stein worked on other problems suggested by Olga Todd, e.g., in Gerschgorin Theory and in Lyapunov Theory, with which his characterization of matrices C such that $C^n \to 0$, is closely connected.*
> *R. S. Varga and his many students built up a large body of work in this area. There were many generalizations, e.g., to infinite dimensional problems, and developments from all over the world from the early 1950's to the present day.*

The Stein–Rosenberg's paper is mentioned (with a wrong year) in the thesis of David Monagham Young Jr. (1923–2008) [134, p. 35] dating 1950, in the 1951 bibliography of Olga Taussky [118], in the 1953 volume of the National Bureau of Standards [93], in the 1953 paper by George Elmer Forsythe (1917–1972) [56], claiming that *Solving linear algebraic equations can be interesting*, and, under this name and with explanations, in the 1956 book of Bodewig [36, p. 166–168], whose dedication is *To my wife who believes more in me than in Mathematics*, in the 1956 bibliography on numerical analysis by Alston Scott Householder (1904–1993) [65], and in the proceedings of the Sixth Symposium in Applied Mathematics,

[8] https://archive.org/details/nbsinatheinstitu730hest is an url for downloading the pdf file.

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which took place in Santa Monica in 1953 [54]. It seems that other publications of the same period do not mention the paper.

After this paper, the next papers of Stein do not mention the theorem [116, 117], and Rosenberg never returned to numerical linear algebra.

Historically, the theory of regular splitting was introduced in the classical book of Varga [124] in 1962. It gives a large place to the Stein–Rosenberg theorem. Around the fifties, the methods of choice for solving linear systems were those of Jacobi and Gauss–Seidel. Thus most of the papers were quoting this result, and it is quite impossible to refer to all of them. Let us, for example, mention the report [91] by James McDonough Ortega on the application of relaxation methods for approximating solutions of systems of equations arising from a mildly non-linear elliptic boundary value problems, and the paper by Walter Volodymy Petryshyn (1929–2020) on the extrapolated Jacobi method published the same year [94]. Later the method of Successive Over Relaxation (SOR) of Young and the method of Cornelius Lanczos (1893–1974) became the favorite ones, but the latter, which was used as a direct method at its beginning, was rapidly abandoned when the dimension of the systems began to grow and then reconsidered as an iterative method.

In his Doctoral Dissertation in 1973, Zbigniew Ignacy Woźnicki (1937–2008) extended the theory of regular splittings and gave several comparison theorems for regular splittings of monotone matrices [127]. His results, recalled in [129], were based on the Perron–Frobenius theory of nonnegative matrices. Further extensions of regular splittings were obtained by George Csordas and Varga [53] in 1984. They generalize the original results of Varga [123] of 1960 and those of Woźnicki's thesis [127]. Let us mention that, in 1993, Woźnicki introduced a double splitting of the form $A = P - R - S$ and the corresponding iterative method $x_{k+1} = P^{-1}Rx_k + P^{-1}Sx_{k-1} + P^{-1}b$ [128], which was later studied by Cui-Xia Li and Su-Hua Li [71]. As noted by Woźnicki in [131]:

> *from this time a renewed interest in comparison theorems, proven under progressively weaker hypotheses for different splittings, has been permanently observed in the literature. These new results lead to successive generalizations and were accompanied with an increased complexity in the verification of the hypotheses.*

Therefore, in his paper, the author provided a systematic analysis of the convergence conditions derived from their implications for the regular splitting case.

Let us mention that Woźnicki published a book related to splitting methods [132]. In his analysis of this book (Scitech Book News, vol. 34, no. 2, June 2010, p. 28), the reviewer wrote:

> *Polish mathematician Woznicki (1937–2008) was one of the founders of incomplete factorization algorithms and the associated iterative methods, and this account of his later work was in nearly finished form when he died unexpectedly. It is devoted to the description and convergence analysis of iterative methods based on matrix splittings and their implementation in mesh structures. Matrix splitting iterative methods are especially attractive for solving linear equations with non-symmetric matrices, he says, because they yield more accurate solutions with significantly less computations work than Krylov subspace methods. He pays special attention to developing efficient techniques for a priori estimations of optimal acceleration parameters, which can be useful tools for solving many practical problems.*

Let us now give an account of the extensions and generalizations of the Stein–Rosenberg theorem (simply denoted below as "the theorem").

An extension of the theorem was proposed by Ivo Marek in 1967 [77]. It concerns the so-called u_0-positive operators in a real Banach space, whose definition is too long to be given here.

At the University of Grenoble, in France, Noël Gastinel (1925–1984) was very much interested in numerical linear algebra, and in 1966 he published his book [59] in French, later translated into English, a part of which is devoted to the theorem. He also launched the doctoral thesis of several students on this domain. These students published papers for their thesis and even after, and some of them later also had their own students in Grenoble and in the universities where they obtained a position. Thus, for many years, numerical linear algebra was quite well represented in France. Let us give some details about these contributions.

In 1969, François Robert, a doctoral student of Gastinel, extended the theorem to the convergence of the methods of Jacobi and Gauss–Seidel for block H-matrices [99], a notion introduced by Alexander Markowich Ostrowski (1893–1986) in 1955 [92]. Let us remind that $A = (a_{ij})$ is an M-matrix if $a_{ij} \leq 0$, for all $i \neq j$, and if it is nonsingular and $A^{-1} \geq 0$. Let $A = (a_{ij})$ be a $n \times n$ complex matrix, and let $N(A)$ be the matrix whose elements are $|a_{ii}|$ and $-|a_{ij}|$, $j \neq i$. If $N(A)$ is a M-matrix, then A is an H-matrix.

Robert gathered a group of students around him. In 1971, Christiane Odiard was able to show that the assumption of irreducibility, which was present in some statements of the theorem of Perron–Frobenius for ensuring the existence of a nonnegative eigenvector associated to the spectral radius, was not necessary [89]. Nonlinear fixed point problems can be solved by parallel iterations of the type of Jacobi or by serial ones of the type of Gauss–Seidel. A Stein–Rosenberg theorem for the nonlinear operators represented by contraction matrices was given by Michèle Chambat and Michel Charnay, two students of Robert, in 1972 [48]. In their paper, they also treated the case of block Gauss–Seidel iterations. In 1974, François Musy, another student of Robert, and Charnay established an extension of the theorem for chaotic iterations when delays in the transmission of data are not taken into consideration [85]. The same year, this work was extended, with delays, to nonlinear problems by Claude Jacquemard [67]. Then, in a paper with two other of his students, Michel Charnay and François Musy, submitted in 1973 but only published in 1975, Robert studied the case of serial parallel chaotic iterations using a nonlinear Gauss–Seidel method for fixed point equations [104]. In a paper received February 9, 1976, Robert presented a synthesis of the recent applications and extensions of the theorem for nonnegative matrices: convergence of nonlinear iterative process, truncated, or chaotic, or Boolean Stein–Rosenberg-type theorems [100] (see also [103] and [102]). His conclusion was that, as it works through various mathematical contexts, the theorem expresses a rather deep idea of an essentially algorithmic nature. The result of Robert was extended by Juan Pedro Milaszewicz in 1980 to real Banach spaces with a normal reproducing cone and where the operators involved are positive and completely continuous [81, 82]. In [101], Robert came back to the Boolean case and proved a Boolean Perron–Frobenius theorem, a "truncated" Boolean Stein–Rosenberg theorem, and a Boolean Stein–Rosenberg theorem for matrices with Boolean elements. They are the exact Boolean analogues of the usual corresponding theorems concerning real nonnegative matrices.

The term *chaotic relaxation*, suggested by Jack L. Rosenfeld [106], a computer scientist from IBM Thomes J. Watson Research Center, was introduced in the paper by Daniel Chazan and Willard Miranker (1932–2011) in 1969 [50] and then was replaced by the term *asynchronous relaxation* due to Gérard M. Baudet [30]. It was a way for exploiting parallel processing, which was then emerging. Chaotic relaxation, in which the next iterate to be treated is chosen to optimize performance, was taken up by the group of Grenoble in 1975 with Robert [104] and Jean-Claude Miellou who incorporated delays into the algorithm to reduce the time to wait for results from other processors [80]. Later Miellou published a series

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of papers in the *Comptes rendus de l'Académie des sciences*, which generated much research in France on convergence and parallel implementations. In [108], Yousef Saad wrote:

> *Some of the work done in France in those days was truly visionary. Discussions that I attended as a student in Grenoble could be tense sometimes, with one camp claiming that the methods were utopian. They were not necessarily utopian but certainly ahead of their time by a few decades. In fact this work has recently staged a strong come back with the advent of very large high-performance computers where communication is expensive* [...]

In 1993, Jacques Bahi and Miellou gave a result that can be considered as an extension of the theorem in the case of parallel asynchronous algorithms [28]. It allowed these authors to build asynchronous convergent algorithms and to compare their speed of convergence [29, pp. 119ff]. See the paper by Pierre Spiteri for a review [114].

In 1978, Michael Neumann (1946–2011) and Robert James Plemmons extended the theorem to singular and consistent systems [86]. In 1979, John J. Buoni and Varga generalized the theorem and compared the spectral radii of the Jacobi Over Relaxation (JOR) and Successive Over Relaxation (SOR) methods [46, 47]. In 1982, Buoni, Neumann, and Varga covered the singular case [44], and, in 1983, Buoni and Balakrishnan Subramanian (1928–2022) extended it to rectangular systems [45]. The Symmetric Successive Over Relaxation (SSOR) was considered bu Götz Alefeld in 1982 [26]. Günter Mayer extended the theorem when A is an interval matrix in 1986 [78]. In 1993, Xinmin Wang generalized the theorem and compared the spectral radii of the Accelerated Over Relaxation (AOR) and the Jacobi iterative methods [125, 126]. The Unsymmetric Successive Overrelaxation method (USSOR) was treated by Da-Wei Chang in 1995 [49]. In 2002, Wen Li, Ludwig Elsner, and Linzhang Lu generalized it again and compared the spectral radii of the iterative methods stemming from two different M-splittings of the matrix A [74].

In their paper, Stein and Rosenberg remarked that the condition that U is strictly upper triangular was not necessary. Several papers dealt with generalizing the above result by relaxing the conditions on L and U [72, 73, 125, 126].

Research on the comparison between different splittings has been and is still quite active. It is quite impossible to quote all results. In 1991, Paul J. Lanzkron, Donald J. Rose, and Daniel B. Szyld presented a two-stage iterative method, named a *nested method* and corresponding to inner-outer iterations, and gave a Stein–Rosenberg theorem for it [70]. The case of iteration matrices associated with block partitions of the matrix A were proven by Woźnicki in 1997 by means of nonnegative splitting theory [130]. In 1999, the theorem was extended to stair matrices, which are block tridiagonal matrices with special properties, by Hao Lu [75] and, quite recently, to bilinear games, which are two-player, non-cooperative, single shot games represented by two payoff matrices and two polytopal compact strategy sets, by Guojun Zhang and Yaoliang Yu [136, 137]. In 2002, Chi-Kwong Li and Hans Schneider applied the theorem to the problem of population dynamics [72]. In 2008, Dimitrios Noutsos studied Stein–Rosenberg-type theorems for nonnegative splittings, namely splittings in which $M^{-1}N$ is a nonnegative matrix [88], as well as for Perron–Frobenius splittings he introduced two years earlier, where $M^{-1}N$ satisfies the conditions of the Perron–Frobenius theorem.

The classical Stein–Rosenberg theorem states that, in a serial setting where one Jacobi iteration takes as much as one Gauss–Seidel iteration, the rate of convergence of the Gauss–Seidel iteration is always faster. But, in 1991, the authors of the paper [34] mentioned that, surprisingly, in a parallel setting, this conclusion is reversed since one Gauss–Seidel iteration takes as much parallel time as several Jacobi iterations.

For an history of relaxation methods for solving systems of linear equations and the corresponding convergence results, consult [39]. Old references can be found also in [122, 124], and more recent developments are mentioned in [42]. Applications and extension are still under consideration and can be easily found on the internet.

Philip Bernard Stein was presumably born in Švėkšna, Šilutė District Municipality, Klaipėda County, Lithuania, in a Jewish family. His official birthday was the January 25th, 1890, but, as he himself wrote, he did not know his actual birthday within six months [52, 115].

In 1897, the family emigrated to Cape Colony in South Africa. Stein attended the Normal College Boy's High School in Cape Town. He was awarded a minor bursary for one year in the School Higher Examination of 1905. After graduating in 1906, he studied at the South African College in Cape Town. In 1909, Stein was awarded the B.A. degree with honours in Applied Mathematics by the University of the Cape of Good Hope, and he also won the Ebden Scholarship that allowed him to study abroad for three years. He entered Caius College in Cambridge and spent three years, studying for the Mathematical Tripos.

After his return to South Africa, Stein worked at the South African Railway Offices in Johannesburg, but, in about 1917, he joined the staff of the South African School of Mines and Technology, Johannesburg (from 1923 the University of the Witwatersrand). In about 1918, he moved to the Natal University College in Pietermaritzburg (from 1950 the University of Natal) and was appointed there as a Lecturer in 1920. The same year, he married Lily Rollnick born in South Africa on February 15, 1920, with whom he had three children. His daughter, Zena Stein (1922–2021) became professor of epidemiology at the University of the Witwatersrand[a].

Up to this time, Stein was not a researcher, and he had never studied for a doctorate. William Nicholas Roseveare (1864–1948), Professor of Pure and Applied Mathematics at this College, encouraged him to undertake research, and, in 1926, Stein went to the University of Cambridge, where John Edensor Littlewood (1885–1977) was his thesis advisor. He was awarded a doctorate in 1931 for two dissertations: *On Equalities for Certain Integrals in the Theory of Picard Functions* and *On the Asymptotic Distribution of the Values of an Integral Function*.

Back home, Stein was appointed Professor of Mathematics and Applied Mathematics at the college's newly established Durban campus of the NUC in 1931, a position he held until 1955. During these years he produced several papers. In his paper *On a theorem of M. Riesz* published in 1933, he gave a proof of the fundamental result of Marcel Riesz (1886–1969) on which the theory of conjugate functions in L_p-spaces depends.

As a result of his paper with Rosenberg in 1948, Stein received an invitation from Olga Taussky to visit the National Bureau of Standards, and he wrote three papers there. Philip Bernard Stein died on 7 January 1974 in London.

More detail on Stein's life and work are given in [39], and the list of his publications can be found in [52].

[a]see https://www.wits.ac.za/alumni/obituaries/obituary-content-by-year/ for her biography.

2.5.4. Bristol University, 1947–1948. In 1947–1948, on leave from the Department of Mathematics of the NUC, Rosenberg spent time continuing his research in Atomic Physics at the H. H. Wills Physics Laboratory, Physics Department, Bristol University, England. This

Laboratory was named after Henry Herbert (Harry) Wills (1856–1922), a businessman and philanthropist from Bristol and a member of the Wills tobacco family. He was responsible for meeting the funding needed to build several residences for students and laboratories. Bristol's school of physics had a leading position in science for over a hundred years. In 1918, Paul Dirac entered Bristol University at the age of 16, studying for a degree in electrical engineering. The largest research group and the most well-known was devoted to the study of cosmic rays. In the 1930s Bristol welcomed many physicists fleeing from the Nazis and hosted several Nobel laureates. From 1919 to 1948, the chair of physics was occupied by Arthur Mannering Tyndall (1881–1961), a specialist of the discharge of electricity in gases, who was also the director of the Physics Laboratory until his retirement. He was followed in this position by Nevill Francis Mott (1905–1996), who won the Nobel Prize in Physics in 1977 for his work on the electronic structure of magnetic and disordered systems, especially amorphous semiconductors [96].

When in Bristol, Rosenberg coauthored in the journal *Nature* a small paper on *Decay and capture of slow mesons in dielectrics* with Herbert Fröhlich (1905–1991) a German-born British physicist (a former student of Arnold Johannes Wilhelm Sommerfeld (1868–1951)), obliged to escape from Germany because of his Jewish origin, Ronald Huby, and R. Kolodziejski [8]. In this paper, comparing the radiative energy loss with that due to the Auger effect, it was shown that the latter is very much weaker in a dielectric than in a metal, so much so that it is here likely to play only a negligible role [66, p. 73]. Let us mention that, in 1950, the British physicist from Bristol, Cecil Frank Powell (1903–1969), was awarded the Nobel Prize in Physics for heading the team that developed the photographic method of studying nuclear processes and for the resulting discovery of the pion (π-meson) in 1947 in collaboration with the Italian physicist Giuseppe (Beppo) Paolo Stanislao Occhialini (1907–1993) and the Brazilian experimental physicist Cesare Mansueto Giulio Lattes (1924–2005) [33, 69]. Thus, again, Rosenberg worked on a quite topical research theme.

The Auger effect, or Auger–Meitner effect, is a physical phenomenon in which the filling of an inner-shell vacancy of an atom is accompanied by the emission of an electron from the same atom. This effect was first discovered by the Austrian-Swedish physicist Lise Meitner (1878–1868) in 1922. The French physicist Pierre Victor Auger (1899–1993) independently discovered the effect shortly after (1923) and is credited with the discovery in most of the scientific community. Meitner received many awards and honours late in her life, but she was dispossessed of her discoveries a second time. Indeed, she was never recognized for her important role in the discovery of nuclear fission, and she did not share the 1944 Nobel Prize in Chemistry, exclusively awarded to her long-time collaborator, the German chemist Otto Hahn (1879–1968).

2.5.5. Natal University College, 1949. In 1949, while at NUC, Rosenberg published the paper [10] (received May 2, 1949) as a continuation of [8]. The rate of loss of energy of negative mesons in matter had been calculated by the Italian physicist Enrico Fermi (1901–1954) and the Hungarian-American theoretical physicist Edward Teller (1908–2003) in 1947 on the assumption that the electrons can be regarded as free. As this problem was of some importance with regard to the interpretation of the experiments already discussed in [8], Rosenberg felt advisable, in the case of non-metals, to make a more detailed calculation of this rate of loss for velocities of the meson less than the orbital velocity of the electron. In this paper he was only concerned with the loss of energy due to ionization. The numerical values he predicted were used in [27], and his paper is also mentioned in [57] due to three researchers of the H. H. Wills Physics Laboratory.

In his paper, Rosenberg thanks the head of the H. H. Wills Physical Laboratory in Bristol, Sir Nevill Francis Mott, for allowing him to spend a session in his department and Herbert Fröhlich for many discussions and help in the preparation of this paper. Let us mention that a Summer School in Theoretical Physics was held in the department during September 16–24, 1949.

Rosenberg resigned from NUC in 1949. The family left South Africa in 1949, and they became British Nationals some time after that.

2.6. Chelsea Polytechnic, 1949–1955. After leaving NUC, Rosenberg became a Senior Lecturer in Applied Mathematics at the Chelsea Polytechnic in London in October 1949. He was also a "Recognized Teacher" and staff examiner in the Faculty of Science of the University of London since March 1950. His name is listed in the University of London Calendars for the academic years 1950–1951 until 1956–1957 (inclusive). It does not appear in the 1957–1958 Calendar. Chelsea Polytechnic was one of a number of educational institutions in London which were not formally part of the University of London but did have staff who were recognised by the University. Rosenberg was a Recognised Teacher in his capacity as Teacher of Mathematics at Chelsea, so University of London Recognised Teacher status was not in addition to but through his employment at Chelsea Polytechnic.

Since 1928, the Principal of Chelsea Polytechnic was Frederick James Harlow, head of the Chemistry Department, who throughout the 1930s and 1940s steered the College through many changes. In 1949, Harlow was appointed by the government of Nigeria to asses the need of establishing a College or Colleges of Higher Technical Education. On January 1, 1950, he was replaced by the mathematician Nicholas Morpeth Hutchinson Lightfoot (1902–1962), who kept this position until 1962 and played an important role in the post-war development of the College, which saw its designation as a college of advanced technology. It was then renamed Chelsea College of Science and Technology after it was granted its Royal Charter in 1971.

Beginning January 11, 1950, on Wednesdays at 6 p.m., Rosenberg gave an advanced course of about 12 lectures on *Quantum Mechanics of the Atomic Nucleus*. The fee for the course was 20s., and membership 1s.[9] He resigned from Chelsea Polytechnic on October 31, 1955.

2.7. University of New Brunswick, 1955–1966. As his cousin, Lillian Goldberg, wrote us:

> *After the war, he* [Rosenberg] *realized that South Africa was facing an uncertain future as the white population was a tiny minority of the country and he felt that the white community did not have a future in a black country. He immigrated to Canada.*

In 1955, Rosenberg took a professorship at the Mathematics Department of the University of New Brunswick in Fredericton, Canada, and was appointed as his head the same year. In his obituary in the Carleton University paper, it is written that *he also was interested in the mathematics curriculum at the provincial junior high school level. He co-authored several junior high school mathematics textbooks*. In a document, we found that he complained privately and publicly that students coming to the university were not well prepared. He was followed by other educators, and his complaint led to a reform of secondary education.

In the *Curriculum Vitaæ* he will later give to Carleton University, Rosenberg described his teaching experience at the University of New Brunswick:

[9] Journal and Proceedings of the Royal Institute of Chemistry, Part VI, 1949, p. 564, and also Part I, 1950, p. 80.

My teaching has been equally divided between Pure and Applied Mathematics. My main interest is in Applied Mathematics, but since coming to the University of New Brunswick I have been more concerned with the teaching of Pure Mathematics. My involvement with the current reorganization of the school curriculum has led me to devote a great deal of time to elementary teaching. In this connection I have been a co-author of three school texts for Grades 7, 8, 9 respectively, published by W. J. Gage for the new Brunswick schools.

2.7.1. National Research Council, 1959–1963. From 1960, Rosenberg was affiliated to the Division of Pure Physics, National Research Council in Ottawa, on sabbatical leave from University of New Brunswick. During this period he began a research collaboration in plasma physics with Ta-You Wu (1907–2000), a Chinese physicist who has been called the Father of Chinese Physics. They co-authored five papers.

Ta-You Wu was born on 27 September 1907 in Panyu, Guangzhou (Canton) in the last years of the Qing dynasty. In 1929 he took his undergraduate degree at Nankai University in Tianjin (Tientsin). He moved to the United States in 1931 for graduate schooling and obtained a Ph.D. from the University of Michigan in 1933 under the guidance of Samuel Abraham Goudsmit (1902–1978). Wu returned to China (then Republic of China) in 1934, and, up to 1949, he taught at various institutions there, including Peking University in Beijing and National Southwestern Associated University in Kunming. In 1949, the year of the defeat of the Nationalists by the Communists in the Chinese Civil War, Wu moved to Canada. There he headed the Theoretical Physics Division of the National Research Council until 1963. In the 1960s, he was Chair of the Department of Physics and Astronomy at the University at Buffalo. After 1962, he held various positions in Taiwan (Republic of China) and was President of the Academia Sinica (1983–1994). He continued lecturing into his 90s and died on March 4, 2000.
Wu's Ph.D. thesis dealt with theoretical predictions of the chemical properties of the yet undiscovered transuranic elements of the actinide series, which includes such well-known elements as plutonium and americium. Later in his career, he worked on solid-state physics, molecular physics, statistical physics, and other areas of theoretical physics. He was known as a teacher as much as a theoretician. His many students include Chen Ning (Franklin) Yang (born September 22, 1922) and Tsung-Dao Lee (born November 24, 1926), co-winners of the Nobel Prize in Physics in 1957 for their work on parity non-conservation of weak interaction.

The first of the five papers [11, 12, 13, 14, 15] written by Wu and Rosenberg presents a formulation of the theory of irreversible processes in ionized gases on the basis of the theory of Bogoliubov for neutral gases.

The paper [11] (received November 13, 1959) was written also with H. Sandstrom. The discovery of the nuclear fusion $p + d + \mu \rightarrow He^3 + \mu$ by μ-catalysis in 1956 was the occasion for a number of studies of the cross sections of some collisions involving a μ-mesonic hydrogen atom and a proton, a deuteron or a hydrogen atom. While, in principle, such collision processes present no essential new problems, in the actual calculations they do present certain features arising from the considerable value of the ratio μ/M of the meson and the proton mass, which is negligible when the processes involve an electron instead of the μ. In this paper, the authors compute several quantities related to this ratio. However, they were limited in accuracy by their use of the Born–Oppenheimer approximation, while later calculations have retained the full (non-relativistic) Hamiltonian [55]. The results are too technical to be given here.

Let us now describe the content of [12] (received October 31, 1961). As explained in its introduction, up to about 1946, the Boltzmann equation was the basis of the theory of the irreversible processes in a gas towards thermodynamic equilibrium. This equation describes the statistical behaviour of a thermodynamic system not in a state of equilibrium. It was obtained by Ludwig Eduard Boltzmann (1844–1906) in 1872. It was known very early that the theory involves the so-called Stosszahlansatz, which is not derivable from the dynamical equations of motion but must be understood on statistical considerations in the macroscopic description of the gas. The Stosszahlansatz is a terminology due to Paul Ehrenfest (1880–1933), that is, the molecular chaos hypothesis, which assumes that the velocities of colliding particles are uncorrelated and independent of position. It is the Stosszahlansatz that gives the Boltzmann equation the "irreversible character" in time. However, it was soon realized by many physicists that the theory involving the explicit assumption of binary collisions between molecules cannot be easily generalized to gases at higher densities in which higher-order collisions become significant. About 1946, several attempts were made to obtain a theory of irreversible processes starting from the Liouville equation (due to the French mathematician Joseph Liouville (1809–1882)) and introducing assumptions other than that of Boltzmann. Of these many theories, perhaps that of Nikolaï Nikolaïevitch Bogoliubov (1909–1992) in 1946 was the most satisfactory in providing a formal scheme for treating the irreversible processes in neutral gases of any densities and in making clear the fundamental assumption, the "initial condition", that introduces the irreversibility into the theory.

A theory of the kinetic equation of an ionized gas along the lines of the Bogoliubov theory for neutral gases had been given by Ralph Lewis Guernsey (1930–1982) and his advisor, the Dutch-American theoretical physicist George Eugene Uhlenbeck (1900–1988), in 1960. These authors obtained a kinetic equation, to the first order in the (Coulomb) interaction, for the case of spatially homogeneous systems.

In [12], received October 31, 1961, Wu and Rosenberg extended the work of Guernsey and Uhlenbeck to the case of spatially inhomogeneous systems to the same (first) order in the (Coulomb) interaction. They studied such questions as the divergence difficulty at large distances arising from the Coulomb interaction, the Landau damping, the general theory of irreversible processes in gases of Ilya Romanovich Prigogine (1917–2003), who received the Nobel Prize in Chemistry in 1977 for his definition of dissipative structures and their role in thermodynamic systems far from equilibrium, and Radu Bălescu (1932–2006) in 1959–1960, the theory of ionized gases of Balescu (1960), and even the question of the "meaning" of a theory of irreversible processes. In a footnote on page 474, it is written that *The study of this problem, namely the basic equation of magnetohydrodynamics, on the basis of the work of Guernsey and Uhlenbeck and of the present paper is being undertaken by Dr. Sundaresan and the present writers and will be reported in the near future*. This paper seemed to have never been published. Papers by Sundaresan alone, and Sundaresan and Wu have been found but nothing else.

A footnote on page 504, mentions that the following theorem was established by Rosenberg. Let the Hilbert transform be defined by

$$H_u[f(w)] = \frac{1}{\pi} P \int \frac{f(w)}{w-u}\, dw,$$

where P designates the Cauchy principal value, and the integral runs on the entire real axis. Then

$$H_u[f(v)H_v[\phi(w)]] + H_u[\phi(v)H_v[f(w)]] = H_u[f] \cdot H_u[\phi] - f(u)\phi(u),$$

and also that

$$\begin{aligned} H_{\alpha u}[f(w)] &= H_u[f(\alpha w)], \qquad \alpha = \text{Cst.}, \\ H_{\alpha+u}[f(w)] &= H_u[f(w+\alpha)], \\ H_{u+\Delta u}[f] &= H_u[f] + \Delta u H_u[f'] + \cdots \end{aligned}$$

It is added in the same footnote *We are, however, indebted to Professor Erdelyi for calling our attention to another proof given by F. G. Tricomi, Quart. J. Math. (Oxford), Ser. (2), 199 (1951)*; see [121], received March 24, 1950.

The paper [13] (received May 22, 1962) presents corrections in some formulæ of [12], which do not affect the results. A better and consistent approximation of a quantity is also given by adding a second term to it.

The paper [14] (received February 8, 1963) is a continuation of [12]. Since the introduction clearly describes the aim and the contents of the paper, let us partially reproduce it:

> *In a previous paper [12], the present authors extended the work of Guernsey and Uhlenbeck to the case of a spatially inhomogeneous ionized gas and obtained the kinetic equation for the case when the system deviates only slightly from spatial homogeneity. Furthermore, in order to cope with the complexities of the calculation, the spatial inhomogeneities were assumed to be spatially slowly varying perturbations and finally only a very coarse approximation to infinitely slowly varying perturbations was made. Nevertheless, the authors attempted to indicate the general line of approach and how a better approximation could be arrived at. It is the object of this paper to carry out this approximation somewhat more carefully and also to embody certain corrections to the original calculations which were already indicated in a previous note [13]. Thus the present work is concerned mainly with the development of the calculations indicated in the first paper without any change in the basic ideas underlying the previous work.*

In the last paper of Rosenberg and Wu [15] (received August 8, 1963), the kinetic equation of Guernsey–Balescu for spatially homogeneous plasmas was solved as an initial value problem in the linearized approximation. The distribution functions for the electrons and the positive ions were expanded in a series of associated Laguerre polynomials in the momentum, with coefficients that are functions of the time. The solution of the (infinite) systems of linear equations for these coefficients leads to the "spectrum of relaxation times", but the method used is not explained. The abstract of this paper is as follows:

> *The kinetic equation of Guernsey–Balescu for spatially homogeneous plasmas is solved as an initial value problem in the linearized approximation. The distribution functions F_1 for the electrons and the positive ions are expanded in series of associated Laguerre polynomials in the momentum, with coefficients which are functions of the time. The solution of the (infinite) systems of linear equations for these coefficients leads to the "spectrum of relaxation times". The damping constants vary from 10^{-4} to $O(1)$ in units of $(V/r_D^3)\omega$, where V is the "volume per particle", r_D is the Debye length, and ω is the plasma frequency of the electrons $\omega^2 = 4\pi e^2/Vm$. Numerical results are given for a few finite numbers of terms in the expansions of F_1. For a homogeneous plasma with a given initial velocity distribution, the asymptotic approach, after an initial rapid damping, to the Maxwellian distribution has a relaxation time of the order $10^4(r_D^3/V\omega)$. The results of the present calculation and the question of the appropriateness, in the case of ionized gases, of the use of the Bogoliubov "initial condition" and "functional Ansatz" in the formulation of a theory of irreversible processes are discussed.*

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2.7.2. University of New Brunswick, 1964–1966. Then, Rosenberg went back to the University of New Brunswick, and, from 1964 to 1966, he was the President of the Association of University of New Brunswick Teachers. He co-authored several books for students in English, some of them translated or written in French [16, 17, 18, 19, 20, 21, 22, 23, 24]. The thesis *Correlation Functions in Ionized Gases* on the kinetic theory of plasma using singular integral equation techniques was defended under him at New Brunswick in 1965 by Mizan Rahman (1932–2015), a Bangladeshi Canadian mathematician who became a well-known specialist in hypergeometric series and orthogonal polynomials.

Let us notice that, from 1961 to 1965, Rosenberg was the Vice-President of the Canadian Mathematical Society and the Vice-President of the Canadian Mathematical Congress in 1962–1963.

2.8. Carleton University, 1966–1975. From 1966, Rosenberg held a position at Carleton University in Ottawa as a Professor in the Department of Mathematics and became its chairman from 1967 to 1969.

He belonged to the Senate of the University (1966–1968), and, in 1971–1972, he was chairman for Mathematics and Physics of the Interdepartmental Committees and a member of General Science Committee for Mathematics (1972–1973). Rosenberg was chairman of his department from 1967 to 1969, and he was replaced by Derek William Sida (1926–2019), a physicist. As a chairman, his salary was $20,000, reduced to $19,200 after.

In 1970–1976, he taught, three hours a week during the fall and winter terms, the course of Advanced Classical Mechanics with Sida and then with Rahman, his former Ph.D. student at the University of New Brunswick. The program was Hamiltonian mechanics, integral invariants, non-holonomic systems, and rigid body systems. He also taught, in 1970–1972, a course on the Kinetic Theory of Gases and Plasmas with Rahman, again three hours a week during the fall and winter terms. The program consisted in irreversible processes in gases, Boltzmann and Fokker–Planck equations, theories of Bogoliubov and of Frieman and Sandri, inhomogeneous plasmas, initial and boundary value problems of gases and plasmas, the hydrodynamical stage. Rosenberg retired in 1975, but, from 1976 to 1986, he continued to teach as a sessional lecturer in mathematics of the Department of Mathematics, and he also published lecture notes [25].

In the President's Report for 1966–1967, it is indicated that Rosenberg was working on a boundary value problem with a moving boundary[10], but no trace of it could be found. Moving boundary and boundary value problems occur in many physical and engineering processes involving heat transfer and phase changes.

In his curriculum, written round that time and transmitted to us by his daughter Gale, he wrote:

> *Whilst living in South Africa I was equally fluent in English and Afrikaans. At present I would have difficulties in speaking Afrikaans although I understand Afrikaans quite well when spoken or written.*

His wife Violet died on February 20, 1981. Rosenberg dealt with cancer for the year before his death in August 25, 1986 in Ottawa. He was 76. He is buried in the Jewish Memorial Gardens, Bank Street Cemetery, Section 4, Row 16, Plot 4, 2692 Rue Bank, Gloucester, Ontario, K1T 1K9, in the suburbs of Ottawa (Fig. 5).

Endowed in 1986 by the daughters, friends, and academic colleagues of the late Reuben L. Rosenberg, who served his University and his Department with distinction, the R. L. Rosenberg Memorial Scholarship in Mathematics is still awarded to an outstanding student entering

[10] https://archive.org/details/cp401

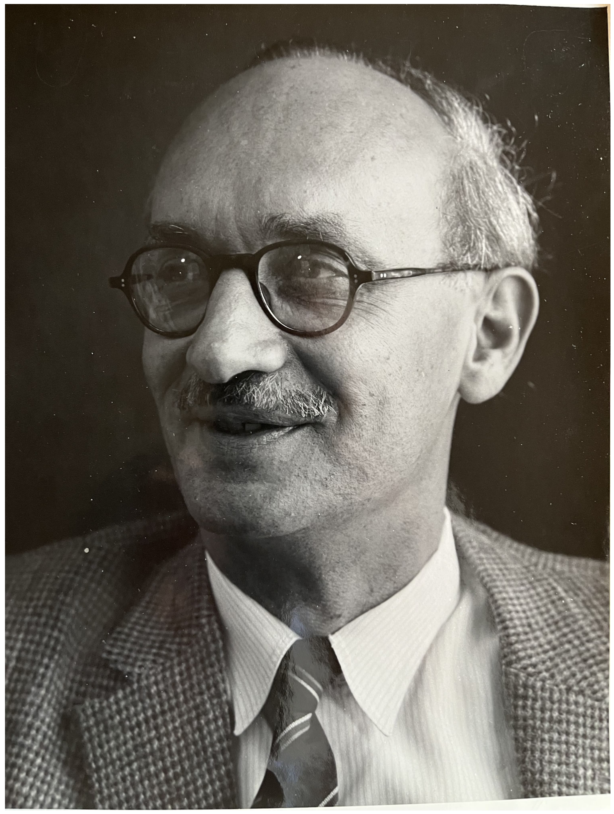

Fig. 4. **Reuben Louis Rosenberg a few years before he died. ©Gale Rosenberg**

a first-year honours programme in the Department of Mathematics and Statistics at Carleton University.

According to the *Alumni Weekend*, Volume 7, Number 20, dated September 26–29, 1986, Reuben Louis Rosenberg, "Rosy" to his many colleagues and friends at the University, devoted a great deal of time to educational theory and teaching methods. He was the co-author of junior school textbooks as well as of a book on elementary calculus. He was an avid art collector and was also known as a gourmet cook. His daughter Gale wrote us that he was a very private and unassuming person, and that he had a wonderful sense of humour.

Appendix A.

Abstract (in German) of Rosenberg's thesis [3]:

Es wird die Kraft, die eine ebene Materiewelle auf einen kugelsymmetrischen Potentialbereich ausübt, berechnet. Dies wird angewendet für die Streuung von Elektronen and Atomen, um einen Ausdruck für den Wirkungsquerschnitt in der Form einer Reihe der Koeffizienten a_n zu erhalten. Als spezielles Beispiel wird das allgemeine Ergebnis für das Atommodell von Allis und Morse berechnet und die Resultate für verschiedene Werte des Parameters b^2 graphisch dargestellt. Die Anomalien des Ramsauereffektes werden diskutiert und die Lagen der Maxima näherungsweise berechnet. Zum Schluß wird der Grenzübergang mit Hilfe der asymptotischen Darstellungen der auftretenden Funktionen für große Geschwindigkeiten gemacht; und ein Wert des Wirkungsquerschnittes in erster Näherung berechnet.

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Fig. 5. The grave of Reuben Louis Rosenberg.

Beginning of the first section of Rosenberg's thesis [3]:

I. Wirkungsquerschnitt und Kraftdichte

Die Methoden, die bisher von verschiedenen Autoren benutzt worden sind, um die Streuung von Elektronen an Atomen zu berechnen, stützen sich alle auf die Analogie zwischen der Wellenmechanik und der Wellentheorie des Lichtes. Die Wellenfunktion des gestreuten Teilchens wird berechnet und die Amplitude der gestreuten Kugelwelle wird benutzt zur Berechnung des Absorptionskoeffizienten oder Wirkungsquerschnittes. Der Stoßvorgang wird im ganzen als Beugungserscheinung aufgefasst und verliert den Charakter eines dynamischen Vorganges oder eines Stoßes, wie er uns in der klassischen Mechanik bekannt ist. Die Ursache liegt darin, dass der Begriff der Kraft überhaupt in der Wellenmechanik nicht erörtert worden ist.
In der gewöhnlichen klassischen Mechanik wird als fundamentale Tatsache für die Behandlung eines Stoßproblems der Satz von der Erhaltung des Impulses verwendet. Wir können klassisch einen Zusammenstoß eines Teilchens mit einem Körper dadurch beschreiben, dass wir sagen, das Teilchen, das sich in einem gewissen Medium bewegt, trifft auf die Trennungsoberfläche eines anderen Mediums auf. Bei dem

Zusammenstoß wird ein Impuls zwischen Teilchen und Körper ausgewechselt. Dieser Wechsel des Impulses wird durch den Impulssatz bestimmt. Das Teilchen überträgt einen gewissen Impuls auf den Körper. Wir sagen, dass ein Druck auf den Körper ausgeübt wird. Ganz analog der klassischen Behandlung wollen wir den Zusammenstoß eines Elektrons mit einem Atom wellenmechanisch behandeln. Zwei Medien im wellenmechanischen Felde unterscheiden sich durch ihre Potentialfunktionen. Das Auffallen eines Elektrons auf ein Atom wird beschrieben durch das Auffallen einer Materiewelle auf einen Potentialbereich. Der Vorgang des Stoßes wird durch das Verhalten der Wellenfunktion in den verschiedenen Medien beschrieben. Verbunden mit einer Materiewelle ist eine gewisse Impulsdichte, die eine Funktion der Wellenfunktion und ihrer ersten Ableitung ist. Die Wellenfunktion ist in den verschiedenen Medien im allgemeinen verschieden, so daß der zugehörige Impuls auch verschieden ist. Eine Übertragung eines gewissen Impulses auf das Atom findet deshalb statt. Diese Übertragung, die im klassischen Fall durch den Impulssatz bestimmt wird, wird hier durch die Stetigkeitsforderung der Wellenfunktion und ihrer Ableitung an der Trennungsfläche bestimmt. Da wir es hier mit einer Impulsdichte zu tun haben, so müssen wir nach der Übertragung des Impulses pro Volumeneinheit pro Zeiteinheit fragen.

Die Ersetzung des korpuskularen Vorganges durch einen Wellenvorgang gibt Anlass, den Begriff des wellenmechanischen Feldes zu bilden. Es liegt nahe, den Vorgang der Absorption mittels einer Feldtheorie zu beschreiben. Wie in der Feldtheorie der Elektrizität der Maxwellsche Spannungstensor den Zustand des elektrischen Feldes beschreibt, so beschreibt in der Wellenmechanik der Energieimpulstensor der Materie den Zustand des Materiefeldes. Die Atome oder Potentialbereiche, die sich im Materiefelde befinden, dürfen wir dann als Senken der Materie betrachten, woraus der Begriff der Absorption unmittelbar folgt. Die Ergiebigkeit des Feldes ist dann ein Maß der Absorption. Daß diese Darstellung zu demselben Resultat führen wird wie die aus der klassischen Mechanik durch Einführung der Kraft gewonnene liegt darin, daß die Kraft und die Ergiebigkeit des Feldes beide aus dem Energieimpulstensor gebildet werden können.

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