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APPLICATION OF THE SCHUR–COHN THEOREM TO THE PRECISE CONVERGENCE DOMAIN FOR A p-CYCLIC SOR ITERATION MATRIX*

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Historical paper.

Abstract. Assume that $A \in C^{n\times n}$ is a block p-cyclic consistently ordered matrix and that its associated Jacobi iteration matrix B, which is weakly cyclic of index p, has eigenvalues μ whose pth powers are all real nonpositive (resp. nonnegative). Usually, one is interested only in the relaxation parameter ω that minimizes the spectral radius of the iteration matrix of the associated SOR iterative method, but here we are interested in all real values for the relaxation parameter ω for which the SOR iteration matrix is convergent. This will be achieved for the values of $p = 2, 3, 4, \ldots$, and for $p \to \infty$.

Key words. block p-cyclic matrix, weakly cyclic of index p matrix, block Jacobi and SOR iteration matrices, Schur–Cohn Algorithm

AMS subject classifications. 65F10

1. Introduction.

Let a square matrix A have the particular block form

$$A = \begin{bmatrix} A_{1,1} & O & O & \cdots & O & A_{1,p} \\ A_{2,1} & A_{2,2} & O & \cdots & O & O \\ \vdots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ O & O & O & \cdots & A_{p,p-1} & A_{p,p} \end{bmatrix}, \tag{1.1}$$

where each diagonal submatrix $A_{i,i}$ $(i = 1, 2, \ldots, p,\ p \geq 2)$ is square and nonsingular. Denoting the block diagonal matrix D by

$$D := \mathrm{diag}(A_{1,1},\ A_{2,2},\ \cdots,\ A_{p,p}),$$

the associated block Jacobi matrix B, defined by

$$B := I - D^{-1}A,$$

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Dedicated to the late Professor Richard S. Varga as a reminiscence of our close collaboration with him during the fall semester of 1984. Although he is gone, his inspiration continues to light up our journey in the academic arena.
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has the form

$$B = \begin{bmatrix} O & O & O & \cdots & O & B_{1,p} \\ B_{2,1} & O & O & \cdots & O & O \\ O & B_{3,2} & O & \cdots & O & O \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ O & O & O & \cdots & B_{p,p-1} & O \end{bmatrix},$$

where $B_{i,i-1} = -A_{i,i}^{-1}A_{i,i-1}$, $i = 1, 2, \ldots, p$, with $B_{1,0}$ and $A_{1,0}$ being interpreted as $B_{1,p}$ and $A_{1,p}$, respectively. Then, B is said to be *weakly cyclic of index* p $(p \geq 2)$ and A is said to be a *p-cyclic matrix* (cf. [20, p. 115]).

As the diagonal submatrices of B are all zero, we write B as the sum of a strictly lower and a strictly upper triangular matrix, i.e.,

$$B = L + U. \tag{1.2}$$

Then, the eigenvalues of the matrix

$$B(\alpha) := \alpha L + \alpha^{(1-p)}U \qquad (\alpha \neq 0)$$

are independent of α (cf. [20, p. 115]), and B and A are said to be *consistently ordered*.

The block successive overrelaxation (SOR) iteration matrix, associated with (1.1) and taking into account (1.2), is defined as

$$\mathcal{L}_\omega := (I - \omega L)^{-1}\{(1-\omega)I + \omega U\},$$

where $\omega \neq 0$ is the real relaxation parameter.

We assume that the eigenvalues of B^p are nonpositive (resp. nonnegative), and we consider the following problem: What is the *exact* convergence domain of the block SOR iteration matrix associated with (1.1).

Several notations are introduced as follows:

Let $p_n(z) := a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad (n \geq 1)$.

- $p_n^*(z) := z^n \overline{p_n\left(\frac{1}{\bar{z}}\right)} = \overline{a_0}z^n + \overline{a_1}z^{n-1} + \cdots + \overline{a_n}$ is called the *reciprocal polynomial* of $p_n(z)$.
- $Tp_n(z) := \overline{a_0}p_n(z) - a_n p_n^*(z) = \sum_{k=0}^{n-1}(\overline{a_0}a_k - a_n\overline{a_{n-k}})z^k$ is called the *Schur transformation* of $p_n(z)$ (cf. [11, p. 493]).
- Define $T^k p_n(z) := T(T^{k-1}p_n(z)) \quad (k = 2, 3, \ldots, n)$.
- Denote $\gamma_1 := Tp_n(0) = |a_0|^2 - |a_n|^2$ and $\gamma_k := T^k p_n(0) \quad (k = 2, 3, \ldots, n)$.

Now, we state the Schur–Cohn Theorem, which will be used to solve the problem raised above.

THEOREM 1.1 (Schur–Cohn [11, p. 493]). *Let $p_n(z)$ be a polynomial of degree n $(n \geq 1)$. Then, all zeros of $p_n(z)$ lie outside the closed unit disk $|z| \leq 1$ if and only if*

$$\gamma_k > 0 \qquad (k = 1, 2, \ldots, n).$$

2. Main results.

In this section, we state two theorems on the exact convergence domain of the block SOR iteration matrix associated with (1.1) and (1.2) in the nonpositive and nonnegative cases. Several examples are also given. The proofs of these theorems will be given in the following section.

THEOREM 2.1 (Nonpositive case). *Let A be a consistently ordered p-cyclic matrix and let B be the associated block Jacobi matrix with $\nu := \rho(B)$. Assume that the eigenvalues of B^p are real and nonpositive, i.e.,*

$$\sigma(B^p) \subseteq [-\nu^p, 0].$$

Let Ω_p^- denote the exact convergence domain of the associated block SOR iteration matrix, with a relaxation parameter ω, in the (ν, ω)-plane. Let the three sequences $\{\alpha_j\}_{j\geq 1}$, $\{\beta_j\}_{j\geq 1}$, $\{\gamma_j\}_{j\geq 1}$, be defined recursively by

$$\alpha_{j+1} := -\alpha_j \beta_j, \qquad \beta_{j+1} := \beta_j \gamma_j, \qquad \gamma_{j+1} := \gamma_j^2 - \alpha_j^2 \qquad (j \geq 1), \tag{2.1}$$

where

$$\alpha_1 := (1-\omega)\nu, \qquad \beta_1 := -\nu, \qquad \gamma_1 := 2-\omega. \tag{2.2}$$

Then,

$$\Omega_p^- = \left\{(\nu, \omega) : 0 \leq \nu < \tfrac{p}{p-2},\ 0 < \omega < \tfrac{2}{1+\nu},\ \textit{and}\ \gamma_j > 0,\ j = 1, 2, \ldots, p-1\right\}, \tag{2.3}$$

where for $p = 2$, we define $\frac{p}{p-2} := +\infty$, and we have

$$\Omega_{p+1}^- \subset \Omega_p^-, \quad p = 2, 3, 4, \ldots, \tag{2.4}$$

and

$$\Omega_\infty^- := \bigcap_{p=2}^{\infty} \Omega_p^- = \left\{(\nu, \omega) : 0 \leq \nu \leq 1,\ 0 < \omega < \tfrac{2}{1+\nu}\right\}. \tag{2.5}$$

EXAMPLE 2.2 (Nonpositive case, with $p = 2, 3, 4, 5$). It follows from Theorem 2.1 that

$$\Omega_2^- = \left\{(\nu, \omega) : 0 < \omega < \tfrac{2}{1+\nu}, \qquad \text{when } 0 \leq \nu < +\infty\right\},$$
$$\Omega_3^- = \left\{(\nu, \omega) : \begin{cases} 0 < \omega < \frac{2}{1+\nu}, & \text{when } 0 \leq \nu \leq 2 \\ \frac{\nu-2}{\nu-1} =: \omega_3(\nu) < \omega < \frac{2}{1+\nu}, & \text{when } 2 < \nu < 3 \end{cases}\right\},$$

thus Ω_2^- and Ω_3^- are the same regions as given in [15] and [16], respectively.

It follows in an analogous way that

$$\Omega_4^- = \left\{(\nu, \omega) : \begin{cases} 0 < \omega < \frac{2}{1+\nu}, & \text{when } 0 \leq \nu \leq \sqrt{2} \\ \frac{\nu^2-2}{\nu^2-1} =: \omega_4(\nu) < \omega < \frac{2}{1+\nu}, & \text{when } \sqrt{2} < \nu < 2 \end{cases}\right\},$$
$$\Omega_5^- = \left\{(\nu, \omega) : \begin{cases} 0 < \omega < \frac{2}{1+\nu}, & \text{when } 0 \leq \nu \leq \sqrt{5}-1 \\ \frac{\nu^2-\nu-4+\nu\sqrt{\nu^2+2\nu+5}}{2(\nu^2-1)} =: \omega_5(\nu) < \omega < \frac{2}{1+\nu}, & \text{when } \sqrt{5}-1 < \nu < \frac{5}{3} \end{cases}\right\},$$

thus Ω_4^- and Ω_5^- are the same regions as those given in [21] (see also [6]).

For the nonnegative case, i.e., when the eigenvalues of B^p are real and nonnegative, an analogous theorem can be proved.

THEOREM 2.3. *Under the same assumptions as in Theorem 2.1, except that $\sigma(B^p) \subseteq [0, \nu^p]$, let Ω_p^+ denote the exact convergence domain of the associated block SOR iteration matrix for the nonnegative case. Then,*

$$\Omega_p^+ = \{(\nu, \omega) : 0 \leq \nu < 1 \textit{ and } \gamma_j > 0, \quad j = 1, 2, \ldots, p-1\},$$

where γ_j is defined as in Theorem 2.1 except for $\alpha_1 = \omega - 1$, and we have

$$\Omega_{p+1}^+ \subset \Omega_p^+, \qquad p = 2, 3, 4, \dots,$$

and

$$\Omega_\infty^+ := \bigcap_{p=2}^{\infty} \Omega_p^+ = \left\{(\nu, \omega) : 0 \le \nu < 1,\ 0 < \omega\ \le \tfrac{2}{1+\nu}\right\} \setminus \{(0, 2)\}.$$

EXAMPLE 2.4 (Nonnegative case, with $p = 2, 3, 4$). It follows from Theorem 2.3 that

$$\Omega_2^+ = \{(\nu, \omega) : 0 \le \nu < 1,\ 0 < \omega < 2\},$$
$$\Omega_3^+ = \{(\nu, \omega) : 0 \le \nu < 1,\ 0 < \omega < \tfrac{\nu+2}{\nu+1}\}.$$

Hence, Ω_2^+ and Ω_3^+ are the same regions as given in [19] and [16], respectively. For $p = 4$, we have

$$\Omega_4^+ = \left\{(\nu, \omega) : 0 \le \nu < 1,\ 0 < \omega < \tfrac{8}{(4-\nu^2)+\nu\sqrt{8+\nu^2}}\right\},$$

and Ω_4^+ is the same as that given in [21].

3. The proof of the theorems. It is well-known that the equation which connects the eigenvalues μ of a weakly cyclic consistently ordered of index p block Jacobi matrix B with the eigenvalues λ of the associated block SOR iteration matrix $\mathcal{L}_\omega$ with relaxation parameter ω and $p = 2$ is Young's famous relationship (cf. [22])

$$(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2,$$

which was later extended by Varga to cover all $p \ge 3$ (cf. [19]) as follows:

$$(\lambda + \omega - 1)^p = \lambda^{p-1}\omega^p\mu^p. \tag{3.1}$$

3.1. The nonpositive case. From (3.1), with $\mu^p = -\nu^p$, where $\nu = \rho(B) \ge 0$ and $(-\lambda)^{\frac{1}{p}} = z$, so that $\lambda = -z^p$, we obtain, after taking pth roots (deleting the negative sign of its members), the following polynomial equation:

$$z^p - \omega\nu z^{p-1} + 1 - \omega = 0.$$

We wish to find necessary and sufficient conditions so that

$$g_0^*(z) := z^p - \omega\nu z^{p-1} + (1 - \omega), \qquad \text{where } \nu \ge 0 \text{ and } \omega \text{ is real},$$

has all its zeros less than unity in modulus. If $g_0(z)$ denotes the reciprocal polynomial to $g_0^*(z)$, then

$$g_0(z) := (1 - \omega)z^p - \omega\nu z + 1, \qquad (p \ge 2).$$

We seek necessary and sufficient conditions so that $g_0(z)$ has all its zeros outside the closed unit disk $|z| \le 1$. Write

$$g_j(z) := \alpha_j z^{p-j} + \beta_j z + \gamma_j, \qquad j = 0, 1, \dots, p-1, \tag{3.2}$$

where

$$\alpha_0 := 1 - \omega, \qquad \beta_0 := -\omega\nu, \qquad \gamma_{0:} = 1, \tag{3.3}$$

and let

$$g_j^*(z) := \gamma_j z^{p-j} + \beta_j z^{p-j-1} + \alpha_j, \qquad j = 0, 1, \ldots, p-1,$$

be its reciprocal polynomial. Then, the Schur–Cohn Algorithm is defined as follows:

$$\begin{aligned} g_{j+1}(z) &:= Tg_j(z) = \gamma_j g_j(z) - \alpha_j g_j^*(z) \\ &= -\alpha_j\beta_j z^{p-j-1} + \beta_j\gamma_j z + \gamma_j^2 - \alpha_j^2. \end{aligned}$$

It follows from (3.2) that

$$\alpha_{j+1} = -\alpha_j\beta_j, \qquad \beta_{j+1} = \beta_j\gamma_j, \qquad \gamma_{j+1} = \gamma_j^2 - \alpha_j^2. \tag{3.4}$$

Now,

$$\begin{aligned} g_j(z) &= T^j g_0(z), \quad \text{and} \quad \gamma_j = g_j(0), \qquad j = 0, 1, \ldots, p-1, \\ g_{p-1}(z) &= (\alpha_{p-1} + \beta_{p-1})z + \gamma_{p-1}, \\ g_p(z) &= \gamma_{p-1}^2 - (\alpha_{p-1} + \beta_{p-1})^2. \end{aligned}$$

Write

$$\widetilde{\gamma}_j := \gamma_{j-1}^2 - (\alpha_{j-1} + \beta_{j-1})^2, \qquad j \geq 2. \tag{3.5}$$

We conclude, from the Schur–Cohn Theorem, the following lemma:

LEMMA 3.1.

$$\Omega_p^- = \{(\nu, \omega) : \nu \geq 0,\ \gamma_j > 0,\ j = 1, 2, \ldots, p-1,\ \textit{and}\ \widetilde{\gamma}_p > 0\},$$

where the sequences $\{\alpha_j\}_{j\geq 0}$, $\{\beta_j\}_{j\geq 0}$, $\{\gamma_j\}_{j\geq 0}$ *are given recursively by* (3.3), (3.4), *and* (3.5).

For improving the result, we state without proof the following proposition about α_j, β_j, and γ_j, $(j \geq 0)$:

PROPOSITION 3.2. *If* $\gamma_j > 0$, *(*$j = 1, 2, \ldots, p-1$*), where* $p \geq 2$, *then*

1. $\beta_j < 0, \quad j = 0, 1, \ldots, p-1$
2. $\operatorname{sgn}(\alpha_j) = \operatorname{sgn}(1-\omega), \quad j = 0, 1, \ldots, p-1$
3. $\gamma_{j-1} + \alpha_{j-1} > 0$ *and* $\gamma_{j-1} - \alpha_{j-1} > 0, \quad j = 1, 2, \ldots, p-1.$

From the above proposition, we can show the following result:

LEMMA 3.3. *If* $\gamma_j > 0$ *(*$j = 1, 2, \ldots, p-1$*), where* $p \geq 2$, *then*

$$\widetilde{\gamma}_p > 0 \quad \textit{if and only if} \quad \widetilde{\gamma}_2 > 0, \qquad \textit{i.e.,} \quad 0 < \omega < \tfrac{2}{1+\nu}.$$

Proof. From the definitions of $\widetilde{\gamma}_p$, α_j, β_j, and γ_j in (3.3), (3.4), and (3.5), we have

$$\begin{aligned} \widetilde{\gamma}_p &= \gamma_{p-1}^2 - (\alpha_{p-1} + \beta_{p-1})^2 \\ &= (\gamma_{p-2}^2 - \alpha_{p-2}^2)^2 - \beta_{p-2}^2(\gamma_{p-2} - \alpha_{p-2})^2 \\ &= (\gamma_{p-2} - \alpha_{p-2})^2(\gamma_{p-2} + \alpha_{p-2} - \beta_{p-2})(\gamma_{p-2} + \alpha_{p-2} + \beta_{p-2}) \end{aligned}$$

and

$$\widetilde{\gamma}_{p-1} = \gamma_{p-2}^2 - (\alpha_{p-2} + \beta_{p-2})^2 = (\gamma_{p-2} - \alpha_{p-2} - \beta_{p-2})(\gamma_{p-2} + \alpha_{p-2} + \beta_{p-2}).$$

It follows from Proposition 3.2 that $\widetilde{\gamma}_p > 0$ if and only if $\widetilde{\gamma}_{p-1} > 0$, since $\gamma_{p-2} \pm \alpha_{p-2} > 0$ and $\beta_{p-2} < 0$. By induction, the proof of the lemma follows. $\square$

Since $\omega > 0$ is a necessary condition for the convergence domain, we can replace the sequences in (3.3) and (3.4) for $j \geq 1$ by the sequences in (2.1) and (2.2). Combining Lemmas 3.1 and 3.3, we have shown that

$$\Omega_p^- = \{(\nu, \omega) : \nu \geq 0,\ 0 < \omega < \tfrac{2}{1+\nu},\ \text{and}\ \gamma_j > 0\ \ (j = 1, 2, \ldots, p-1)\}. \tag{3.6}$$

The immediate result from (3.6) is given by

$$\Omega_{p+1}^- = \Omega_p^- \cap \{(\nu, \omega) : \gamma_p(\nu, \omega) > 0\}.$$

It is remarked that the exact convergence domain Ω_p^- can be obtained by a recurrence.

Now, we continue the proof of Theorem 2.1. As an immediate consequence of Lemma 3.3, we have that

$$\Omega_{p+1}^- \subseteq \Omega_p^-, \quad p = 2, 3, 4, \ldots \tag{3.7}$$

The strict inclusion of (2.4) will be proved just after Lemma 3.5.

Next, let the set S be defined as

$$S := \{(\nu, \omega) : 0 \leq \nu \leq 1,\ 0 < \omega < \tfrac{2}{1+\nu}\}. \tag{3.8}$$

First, we wish to show that S is contained in Ω_p^- for each $p \geq 2$. This is equivalent to the statement of the following Lemma 3.4.

LEMMA 3.4. *Let S be defined as in (3.8). Assume that*

$$g_0(z) := (1-\omega)z^p - \omega\nu z + 1.$$

If $(\nu, \omega) \in S$, then $g_0(z)$ has all its zeros outside the closed unit disk $|z| \leq 1$.

Proof. For the case of $0 < \nu < 1$, let

$$t(z) := -\omega\nu z + 1.$$

The inequality $0 < \omega \leq 1$ implies $1 - \omega\nu > 1 - \omega \geq 0$, and the inequality $1 < \omega < \frac{2}{1+\nu}$ implies $1 - \omega\nu > \omega - 1 > 0$. Then, we have

$$1 - \omega\nu > |1 - \omega| \qquad \text{for } (\nu, \omega) \in S.$$

It follows that

$$|t(z)| > |(1-\omega)z^n| \qquad \text{on } |z| = 1.$$

By applying Rouché's Theorem, we conclude that $g_0(z)$ has no zero in the closed unit disk $|z| \leq 1$.

We next consider the case of $\nu = 1$. Then $0 < \omega < 1$. It is necessary and sufficient to show that if $\nu = 1$, then $g_0^*(z) := z^p - \omega\nu z^{p-1} + (1-\omega)$ has all the zeros in modulus less than unity.

It is clear that $g_0^*(0) \neq 0$. Suppose $|\widetilde{z}| \geq 1$ with $\widetilde{z} \neq 1$. Then,

$$|\widetilde{z}^p - \omega\widetilde{z}^{p-1}| \geq |\widetilde{z} - \omega| > 1 - \omega.$$

Thus, $\widetilde{z}$ is not a zero of $g_0^*(z)$. It is trivial to show the assertion of the lemma in the case of $\nu = 0$. $\square$

It follows from Lemma 3.4 that if the curve $\gamma_p = 0$, in the domain $\Omega_p^- \setminus S$, can be located, then the domain Ω_{p+1}^- can be easily determined by the continuity of the function $\gamma_p(\nu, \omega)$.

For small values of p we obtain, by direct computation, that the intersection point N_p of the curve $\gamma_{p-1} = 0$ and $\omega = 0$ is on the boundary of Ω_p^-. For instance, $N_3 = (2, 0)$, $N_4 = (\sqrt{2}, 0)$, and $N_5 = (\sqrt{5} - 1, 0)$. It is complicated to locate the point N_p for a general value of p. However, it is possible to determine the intersection point M_p of the curve $\gamma_{p-1} = 0$ and the curve $\omega = \frac{2}{1+\nu}$ on the boundary of Ω_p^-, by the following Lemma 3.5.

LEMMA 3.5. *The point $M_p = (\frac{p}{p-2}, \frac{p-2}{p-1})$ is the unique intersection point of the curve $\gamma_{p-1} = 0$ and the curve $\omega = \frac{2}{1+\nu}$ on the boundary of Ω_p^-, where $p \geq 2$.*

Proof. Substituting $\omega = \frac{2}{1+\nu}$ into α_2, β_2, and γ_2 given in Theorem 2.1, we have

$$\alpha_2 = c_2(\nu - 1), \qquad \beta_2 = c_2(-2), \qquad \gamma_2 = c_2(3 - \nu),$$

where $c_2 = \frac{\nu^2}{1+\nu} > 0$. It follows by induction that

$$\alpha_j = c_j(\nu - 1), \qquad \beta_j = c_j((j-2)\nu - j), \qquad \gamma_j = c_j((j+1) - (j-1)\nu),$$

for $j = 2, 3, \ldots, p-1$, where $c_j > 0$. Thus, the point $(\nu, \omega) = (\frac{p}{p-2}, \frac{p-2}{p-1})$ satisfies

$$\begin{aligned} \gamma_j(\nu, \omega) &> 0, & j &= 1, 2, \ldots, p-2, \\ \gamma_{p-1}(\nu, \omega) &= 0, \quad \text{and} & \omega &= \frac{2}{1+\nu}. \end{aligned} \qquad \square$$

We remark that the point M_p converges strictly monotonically to the point $(\nu, \omega) = (1, 1)$ along the curve $\omega = \frac{2}{1+\nu}$, as $p \to \infty$. This remark proves also the strict inclusion in (3.7) or (2.4).

To complete the proof of Theorem 2.1, we need to prove the following lemma.

LEMMA 3.6. *For $p \geq 3$, if $\nu = \frac{p}{p-2}$, then*

$$(\nu, \omega) \notin \Omega_p^- \qquad \textit{for any} \qquad 0 \leq \omega < \tfrac{2}{1+\nu}.$$

Proof. It can be shown (cf. [19]) that, for $0 < \nu < \frac{p}{p-2}$, the optimal relaxation parameter ω_b is the unique positive real root in $(\frac{p-2}{p-1}, 1)$ of the equation

$$(\nu\omega_b)^p = p^p(p-1)^{1-p}(1 - \omega_b)$$

such that

$$\rho(\mathcal{L}_{\omega_b}) = (p-1)(1 - \omega_b),$$

i.e., $\rho(\mathcal{L}_\omega) > \rho(\mathcal{L}_{\omega_b})$ for $\omega \neq \omega_b$. By a simple calculation, if $\nu = \frac{p}{p-2}$, then $\omega_b = \frac{p-2}{p-1}$ satisfies $\rho(\mathcal{L}_{\omega_b}) = 1$. Therefore, letting ν tend to $\frac{p}{p-2}$ from the left, we have

$$\rho(\mathcal{L}_\omega) \geq \rho(\mathcal{L}_{\omega_b}) = 1 \qquad \text{for } \omega \neq \omega_b.$$

The proof of the lemma is complete since $\rho(\mathcal{L}_\omega)$ is a continuous function of ν. $\square$

It should be remarked that, from Lemma 3.6, the exact convergence domain Ω_p^- must be located on the left-hand side of the vertical straight line $\nu = \frac{p}{p-2}$ and that Ω_{p+1}^- is a proper subset of Ω_p^-. So, (2.3) follows from (3.6). Moreover, $\Omega_\infty^- = \bigcap_{p=2}^{\infty} \Omega_p^- = S$, which is the same as (2.5) (see Figure 3.1).

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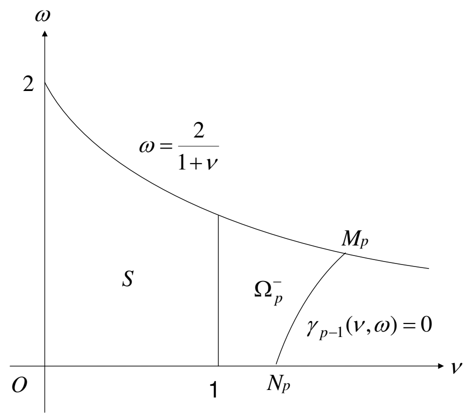

FIG. 3.1. *Exact convergence domain for the nonpositive case.*

3.2. The nonnegative case. As in the *nonpositive case,* we begin with (3.1), where $\mu^p = \nu^p$, $\nu = \rho(B) \geq 0$, and setting $\lambda^{\frac{1}{p}} = z$, we obtain, after taking the pth roots of its members, the following polynomial equation,

$$z^p - \omega\nu z^{p-1} + (\omega - 1) = 0.$$

Denoting

$$\widehat{g}_0^*(z) := z^p - \omega\nu z^{p-1} + (\omega - 1), \qquad \text{where } \nu \geq 0 \text{ and } \omega \text{ is real},$$

we require that $\widehat{g}_0^*(z)$ has all its zeros less than unity in modulus. As in the previous Section 3.1, we denote by $\widehat{g}_0(z)$ the reciprocal polynomial to $\widehat{g}_0^*(z)$, i.e.,

$$\widehat{g}_0(z) := (\omega - 1)z^p - \omega\nu z + 1, \qquad (p \geq 2), \tag{3.9}$$

and we seek necessary and sufficient conditions so that $\widehat{g}_0(z)$ has all its zeros outside the closed unit disk $|z| \leq 1$.

For the nonnegative case, Theorem 2.3 can be proved by showing that similar lemmas as in the nonpositive case hold. Recall that Ω_p^+ is the exact convergence domain of the associated block SOR iteration matrix for the nonnegative case.

LEMMA 3.7.

$$\Omega_p^+ = \{(\nu, \omega) : \nu \geq 0,\ \gamma_j > 0\ (j = 1, 2, \ldots, p-1),\ \textit{and}\ \widehat{\widehat{\gamma}}_p > 0\},$$

where $\widehat{\widehat{\gamma}}_j = \widehat{\gamma}_{j-1}^2 - (\widehat{\alpha}_{j-1} + \widehat{\beta}_{j-1})^2$ *and the sequences* $\{\widehat{\alpha}_j\}_{j\geq}$, $\{\widehat{\beta}_j\}_{j\geq 0}$, $\{\widehat{\gamma}_j\}_{j\geq 0}$ *are the same sequences as in Lemma 3.1 except that* $\widehat{\alpha}_0 = \omega - 1$ *instead of* $1 - \omega$.

Proof. Define $\widehat{\alpha}_j$, $\widehat{\beta}_j$, and $\widehat{\gamma}_j$ $(j = 1, 2, \ldots)$ in the same way as in (3.4), with

$$\widehat{\alpha}_0 := \omega - 1, \qquad \widehat{\beta}_0 = -\omega\nu, \qquad \widehat{\gamma}_0 = 1, \tag{3.10}$$

and

$$\widehat{\widehat{\gamma}}_j := (\widehat{\gamma}_{j-1})^2 - (\widehat{\alpha}_{j-1} + \widehat{\beta}_{j-1})^2.$$

The region of the associated convergence domain is denoted by Ω_p^+.

Note that

$$\widehat{\alpha}_j = -\alpha_j, \qquad \widehat{\beta}_j = \beta_j, \qquad \widehat{\gamma}_j = \gamma_j, \qquad j = 1, 2, \ldots \tag{3.11}$$

Since $\widehat{\gamma}_j = \gamma_j$, $j = 1, 2, \ldots$, the proof of the lemma is completed. ◻

By direct calculation, it follows that

$$\Omega_2^+ = \{(\nu, \omega) : 0 \le \nu < 1 \text{ and } 0 < \omega < 2\}.$$

We now obtain a similar result as in Lemma 3.3.

LEMMA 3.8. *If $\gamma_j > 0$ $(j = 1, 2, \ldots, p-1)$, where $p \ge 2$, then*

$$\widehat{\widehat{\gamma}}_p > 0 \qquad \textit{if and only if} \qquad \widehat{\widehat{\gamma}}_2 > 0, \quad \textit{i.e.,} \quad 0 \le \nu < 1.$$

Combining Lemmas 3.7 and 3.8, we have shown that

$$\Omega_p^+ = \{(\nu, \omega) : 0 \le \nu < 1, \text{ and } \gamma_j > 0 \; (j = 1, 2, \ldots, p-1)\},$$

as a consequence of which it holds that

$$\Omega_{p+1}^+ \subseteq \Omega_p^+, \;\; p = 2, 3, 4, \ldots$$

LEMMA 3.9. *Let S^+ be defined as*

$$S^+ := \{(\nu, \omega) : 0 \le \nu < 1, \, 0 < \omega \le \tfrac{2}{1+\nu}\} \setminus \{(0, 2)\}.$$

For $\widehat{g}_0(z)$ defined in (3.9), the following statement holds: If $(\nu, \omega) \in S^+$, then $\widehat{g}_0(z)$ has all its zeros outside the unit disk $|z| \le 1$.

Proof. For $0 \le \nu < 1$ and $0 < \omega < \frac{2}{1+\nu}$, the result is valid by the same proof as in Lemma 3.4. What remains to be proved is the case of $0 < \nu < 1$ and $\omega = \frac{2}{1+\nu}$. Substituting $\omega = \frac{2}{1+\nu}$ into (3.10) and using (2.1) and (3.11), it can be shown by induction that

$$\widehat{\alpha}_j = \widehat{c}_j(1 - \nu), \qquad \widehat{\beta}_j = \widehat{c}_j((j-2)\nu - j), \qquad \widehat{\gamma}_j = \widehat{c}_j((j+1) - (j-1)\nu),$$

where $\widehat{c}_j$ is a positive number, $j = 2, 3, \ldots$ Thus, $\widehat{\gamma}_j > 0$ for each $j > 2$. In other words,

$$\left\{(\nu, \omega) : 0 < \nu < 1 \text{ and } \omega = \tfrac{2}{1+\nu}\right\} \subseteq \Omega_p^+, \quad \text{for } p \ge 2,$$

and the lemma is proved. ◻

We obtain the following result which is similar to Lemma 3.5.

LEMMA 3.10. *The point $M_p^+ = (1, \frac{p}{p-1})$ is the unique intersection point of the part of the curve $\widehat{\gamma}_{p-1}(\nu, \omega) = 0$, located in the closed set $\overline{(\Omega_p^+ \setminus S)}$ and the straight line $\nu = 1$ on the boundary of Ω_p^+.*

Proof. Let $\nu = 1$. Substituting the value of ν into (3.11), we have

$$\widehat{\alpha}_0 = \omega - 1, \qquad \widehat{\beta}_0 = -\omega, \qquad \widehat{\gamma}_0 = 1,$$

and

$$\widehat{\alpha}_1 = d_1(\omega - 1), \qquad \widehat{\beta}_1 = d_1(-1), \qquad \widehat{\gamma}_1 = d_1(2 - \omega),$$

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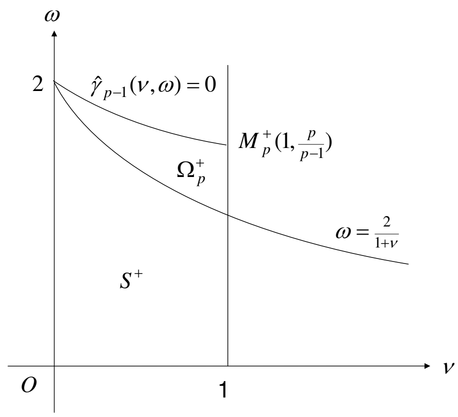

FIG. 3.2. *Exact convergence domain for the nonnegative case.*

where $d_1 = \omega > 0$. Then $\widehat{\gamma}_1 > 0$ if and only if $0 < \omega < 2$. It follows by induction that for $j = 1, 2, \ldots, p-1$,

$$\widehat{\alpha}_j = d_j(\omega - 1), \qquad \widehat{\beta}_j = d_j((j-1)\omega - j), \qquad \widehat{\gamma}_j = d_j(j + 1 - j\omega),$$

where d_j is a positive number since $\widehat{\gamma}_j(1,\omega) > 0$ $(j = 1, 2, \ldots, p-2)$. It is clear that $\widehat{\gamma}_{p-1}(1,\omega) = 0$ if and only if $\omega = \frac{p}{p-1}$, which completes the proof of the lemma. □

It is noted that, as $p \to \infty$, M_p^+ converges strictly monotonically to the point $(1,1)$ along the straight line $\nu = 1$. Based on this observation and the consequence of Lemma 3.8, we have the validity of the following statement.

COROLLARY 3.11. *For the convergence domains Ω_p^+ there holds*

$$\Omega_{p+1}^+ \subset \Omega_p^+, \qquad p = 2, 3, 4, \ldots$$

The regions Ω_p^+ and S^+ are depicted in Figure 3.2.

Finally, to complete the proof of Theorem 2.3, we need to prove the following lemma:

LEMMA 3.12. *Let $0 < \nu < 1$. Then for a positive integer $p > \max\{4, \left(\frac{1+\nu}{2\nu}\right)^2\}$, the point $(\nu, \omega_p) \notin \Omega_p^+$, where $\omega_p = \frac{2}{1+\nu} + \frac{1}{\sqrt{p}} \in (\frac{2}{1+\nu}, 2)$.*

Proof. Fix $\nu \in (0,1)$. From Lemma 3.9, it is known that for $\omega = \frac{2}{1+\nu}$, the zeros of the polynomial $\widehat{g}_0(z)$ in (3.9) lie outside the closed unit disk. Hence, all zeros of its reciprocal polynomial

$$\widehat{g}_0^*(z) := z^p - \omega\nu z^{p-1} + \omega - 1$$

are less than unity in modulus. For an integer $p > \max\{4, \left(\frac{1+\nu}{2\nu}\right)^2\}$, we denote

$$\omega_p = \frac{2}{1+\nu} + \frac{1}{\sqrt{p}}\,.$$

It follows that

$$\frac{p}{p-1} < 1 + \frac{1}{\sqrt{p}} < \omega_p < \frac{2}{1+\nu} + \frac{2\nu}{1+\nu} = 2.$$

Now, by Theorem 3, part (2), of Wild and Niethammer [21] and our Lemma 3.7, in order to show that the assertion of the lemma holds, namely that $(\nu,\, \omega_p) \notin \Omega_p^+$, it suffices to show that

$$\nu\omega_p \geq \cos\theta + (\omega_p - 1)\cos((p-1)\theta), \tag{3.12}$$

where $\cos\theta \in (\cos\frac{\pi}{p}, 1)$, and $\theta \in (0, \frac{\pi}{p})$ is the unique solution of

$$\omega_p = \frac{\sin((p-1)\theta) + \sin\theta}{\sin((p-1)\theta)}\,. \tag{3.13}$$

Substituting (3.13) into (3.12), following Wild and Niethammer [21], we need to show that

$$\nu \geq \frac{\sin(p\theta)}{\sin((p-1)\theta) + \sin\theta}\,. \tag{3.14}$$

Forming $\omega_p - \dfrac{2}{1+\nu}$ and substituting ω_p and ν from (3.13) and (3.14), respectively, we have

$$\begin{aligned} \omega_p - \frac{2}{1+\nu} &\geq \frac{\sin((p-1)\theta) + \sin\theta}{\sin((p-1)\theta)} - \frac{2\,(\sin((p-1)\theta) + \sin\theta)}{\sin((p-1)\theta) + \sin\theta + \sin(p\theta)} \\ &= \frac{(\sin((p-1)\theta) + \sin\theta)\,(\sin\theta + \sin(p\theta) - \sin((p-1)\theta))}{\sin\left((p-1)\theta\right)\left(\sin\left((p-1)\theta\right) + \sin\theta + \sin(p\theta)\right)}\,. \end{aligned} \tag{3.15}$$

The numerator and denominator of the fraction above can be respectively factorized as

$$8\sin\left(\frac{p\theta}{2}\right)\cos\left((\frac{p}{2}-1)\theta\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{p\theta}{2}\right)\cos\left(\frac{(p-1)\theta}{2}\right)$$

and

$$8\sin\left(\frac{(p-1)\theta}{2}\right)\cos\left(\frac{(p-1)\theta}{2}\right)\cos\left(\frac{(p-1)\theta}{2}\right)\sin\left(\frac{p\theta}{2}\right)\cos\left(\frac{\theta}{2}\right).$$

Thus, (3.15) is equivalent to

$$\omega_p - \frac{2}{1+\nu} \geq \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{(p-1)\theta}{2}\right)}\,\frac{\cos\left((\frac{p}{2}-1)\theta\right)}{\cos(\frac{\theta}{2})}\,\frac{\cos\left(\frac{p\theta}{2}\right)}{\cos\left(\frac{(p-1)\theta}{2}\right)}\,. \tag{3.16}$$

Since $\theta \in (0, \frac{\pi}{p})$ and $p > 4$, the following inequalities hold:

$$0 < \frac{\theta}{2} < (\frac{p}{2}-1)\theta < \frac{(p-1)\theta}{2} < \frac{p\theta}{2} < \frac{\pi}{2},$$

which means that all the angles in (3.16) are located in the open first quadrant. So the last two factors on the right-hand side in (3.16) are both positive and less than 1. This suggests that it suffices to prove that

$$\frac{1}{\sqrt{p}} = \omega_p - \frac{2}{1+\nu} \geq \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{(p-1)\theta}{2}\right)} = \frac{1}{p-1}\,\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}}\,\frac{\frac{(p-1)\theta}{2}}{\sin\left(\frac{(p-1)\theta}{2}\right)}\,.$$

It can be easily verified that $\frac{\sin x}{x}$ is monotonically decreasing and $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ on $(0, \frac{\pi}{2})$. Thus, it suffices to show that for $p \geq 5$ there holds

$$\frac{1}{\sqrt{p}} \geq \frac{1}{p-1}\frac{\pi}{2} \qquad \text{or} \qquad 4p^2 - (8+\pi^2)p + 4 \geq 0.$$

The last inequality holds since the larger root of the quadratic polynomial $4t^2 - (8+\pi^2)t + 4$ is 4.23, less than 5. This completes the proof of the lemma. ∎

4. Addendum—Applications. This section is prepared only by the first two authors to provide some information about the manuscript, relevant research works, and applications to scientific computing.

There is an unpublished paper under the title "Application of the Schur–Cohn Theorem to the precise convergence domain for a p-cyclic SOR iteration matrix" written by the three authors of this paper.

In the fall 1984 the first two authors visited the Institute for Computational Mathematics at the Mathematics and Computer Science Department at Kent State University.

The first author (A. Hadjidimos) found a way to extend the precise convergence domain for a p-cyclic SOR iteration matrix in the cases of $p = 2$ and 3 in [16, 22] to the cases of $p = 4, 5$, and 6. It was Professor Varga who suggested that the *Schur polynomials* may work and generate an algorithm for the solution for all p's of the problem, and the second author, X. Li, joined the team.

After a few months, the problem was solved using the *Schur–Cohn Theorem* except for the proof of Lemma 3.12. Then, a draft of the manuscript was available under the title mentioned above [6].

Professor Varga presented the findings in a 1985 ILAS or SIAM Conference in Boston. He also sent a message to Professor Niethammer who worked with his student P. Wild, and a nice paper using hypocycloids appeared in [21] citing the message they received as "Unpublished Notes". The same message was also cited in [3] as "(in preparation)".

The only other work using SOR and Schur polynomials was done by A. Hadjidimos with his colleagues Dimitrios Noutsos and Michael Tzoumas, which appeared in [8].

After the NUMAN 2008 Kalamata Conference in Greece, celebrating earlier Professor Varga's 80th birthday, the three of us continued working on Lemma 3.12 to complete the paper and submit it for publication. Eventually, the first two authors came up with the very first complete proof, which satisfied Professor Varga, who took the manuscript to have a final look, and the current manuscript, except for the last section Addendum–Applications and the additional references cited in it, was completed.

Next, applications of block p-cyclic matrix iterations and relevant research works are discussed: Suppose that we want to solve a linear system where the coefficient matrix is block p-cyclic consistently ordered, $p \geq 3$. Then, the block SOR iterative method is most suitable for its solution. Under certain conditions, a block repartitioning into a block weakly cyclic of index q $(2 \leq q \leq p)$ block Jacobi iteration matrix, and therefore the associated preconditioners, play a significant role due to better convergence rates.

For the first time the aforementioned idea appeared in 1985 [13], where a block $q = 2$ cyclic repartitioning is always asymptotically better than the original $p = 3$ cyclic one.

For any $\sigma^p \in [0, \beta^p]$, $0 \leq \beta < 1$, or $\sigma^p \in [-\alpha^p, 0]$, $\alpha \in [0, \infty)$, where σ represents the eigenvalues of the Jacobi iteration matrix, Pierce, Hadjidimos, and Plemmons showed that the best block cyclic repartitioning was that for $q = 2$ in 1990 [17]. Soon Eiermann, Niethammer, and Ruttan proved that if $\sigma^p \in [-\alpha^p, \beta^p]$, $\alpha \in [0, \infty)$, $\beta \in [0, 1)$, there were cases where the block 2-cyclic repartitioning was not always the best [3]. Based on one of the results in [3], A. Hadjidimos and his colleague S. Galanis showed [4] that the best block cyclic repartitioning

q for any p was dependent on the ratio $\frac{\beta}{\alpha} \in [0, \infty]$ (with ∞ corresponding to $\alpha = 0$), and they presented an associated table for the best repartitioning q from the ratio $\frac{\beta}{\alpha}$ in 1992.

A. Hadjidimos with M. Neumann found optimal results for the case $p = 2$, using l_2-norms, and $\sigma^2 \in [0, \beta^2]$, $\beta \in [0, 1)$, for the Modified SOR iterative method in 1998 [7], which extended the results given by Gene Golub with John de Pillis and David Young, respectively, [5, 23]. The corresponding optimal results for $\sigma^2 \in [-\alpha^2, 0]$, $\alpha \in [0, \infty)$ was proved by Milléo, Yin, and Yuan in 2006 [14]. The last relevant result was obtained by A. Hadjidimos with his MSc. student P. Stratis in 2007 [10]. It generalizes the two previous cases for $\sigma^2 \in [-\alpha^2, \beta^2]$, $\alpha \in [0, \infty)$, $\beta \in [0, 1)$.

In the following we continue with some applications where p-cyclic matrices appear in practical problems: It seems that it was Tee [18], who first considered p-cyclic matrices, with p arbitrarily large, to solve the problem of finite-difference equations for steady-state parabolic equations with periodic boundary conditions.

Next, Chen [2] considered iterative methods for the solution of the linear least-squares problem where a 3-cyclic matrix was involved.

Then, in 1985, Markham, Neumann, and Plemmons [13] used a direct-iterative method for the solution of large-scale least-squares problems by repartitioning the 3-cyclic coefficient matrix of the system yielding a 2-cyclic one. They observed and proved that the results obtained with the latter cyclicity were much better than those with the former one.

Later, there was an excellent work by Kontovasilis, Plemmons, and Stewart [12], who studied both theoretically and experimentally their new idea of introducing an extension of the classical SOR method to analyze the block p-cyclic SOR method for Markov Chains with a p-cyclic infinitesimal generator. The fact that they found real optimal relaxation parameters outside the classical interval $(0, 2)$ was very surprising.

After the previous work appeared, Hadjidimos and Plemmons [9] analyzed the p-cyclic iterations for Markov Chains and Queuing Theory with interesting results.

We conclude this section by referring to the latest known work in this area by Chaysri, Hadjidimos, Noutsos, and Tachyridis [1]. There in Section 4, the authors study the Schur complement of M_{v}- and GM-matrices associated with a p-cyclic matrix. Under certain conditions the powers of these matrices become eventually nonnegative, a property that makes them suitable for applications to dynamical systems that appear in biology, economics, etc.

According to all of the above, the role of preconditioners of the block p-cyclic (M)SOR method has become clearly of vital importance.

Acknowledgments. The basic ideas of this work were developed when the first two authors visited the Institute for Computational Mathematics at Kent State University in the fall 1984. They sincerely thank Professors Lothar Reichel and Richard Brualdi for the valuable suggestions of the insertion of Section 4, Addendum—Applications.

REFERENCES

[1] T. CHAYSRI, A. HADJIDIMOS, D. NOUTSOS, AND G. TACHYRIDIS, *Extensions of M-matrices and properties inherited by their Schur compliments*, Linear Multilinear Algebra, (2022), revised.

[2] Y. T. CHEN, *Iterative methods for least square problems*, Research Report CS-75-04, Dept. of Computer Science, Univ. of Waterloo, Ontario, 1975.

[3] M. EIERMANN, W. NIETHAMMER, AND A. RUTTAN, *Optimal successive overrelaxation iterative methods for p-cyclic matrices*, Numer. Math., 57 (1990), pp. 593–606.

[4] S. GALANIS AND A. HADJIDIMOS, *Best cyclic repartitioning for optimal successive overrelaxation convergence*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 102–120.

[5] G. H. GOLUB AND J. E. DE PILLIS, *Toward an effective two-parameter SOR method*, in Iterative Methods for Large Linear Systems, D. R. Kincaid and L. Hayes, eds., Academic Press, New York, 1990, pp. 107–119.

[6] A. Hadjidimos, X. Li, and R. S. Varga, *The Schur–Cohn algorithm applied to the p-cyclic SOR matrix*, unpublished notes, 1985.
[7] A. Hadjidimos and M. Neumann, *Euclidean norm minimization of the SOR operators*, SIAM J. Matrix Anal. Appl., 19 (1998), pp. 191–204.
[8] A. Hadjidimos, D. Noutsos, and M. Tzoumas, *On the convergence domains of the p-cyclic SOR*, J. Comp. Appl. Math., 72 (1996), pp. 63–83.
[9] A. Hadjidimos and R. J. Plemmons, *Analysis of p-cyclic iterations for Markov chains*, in Linear Algebra, Markov Chains and Queeing Models, C. D. Meyers and R. J. Plemmons, eds., IMA Vol Math. Appl. 48, Springer, New York, 1993, pp. 111–124.
[10] A. Hadjidimos, and P. Stratis, *Minimization of the spectral norm of the SOR operator in a mixed case*, Electron. Trans. Numer. Anal., 28 (2007/08/2008), pp. 78–94.
https://etna.ricam.oeaw.ac.at/vol.28.2007-2008/pp78-94.dir/pp78-94.pdf
[11] P. Henrici, *Applied and Computational Complex Analysis. Vol. 1*, Wiley, New York, 1974.
[12] K. P. Kontovasilis, R. J. Plemmons, and W. J. Stewart, *Block cyclic SOR for Markov chains with p-cyclic infinitesimal generator*, Linear Algebra Appl., 154/156 (1991), pp. 145–223.
[13] T. L. Markham, M. Neumann, and R. J. Plemmons, *Convergence of a direct-iterative method for large-scale least-squares problems*, Linear Algebra Appl., 69 (1985), pp. 155–167.
[14] I. Milléo, J. Yin, and J. Y. Yuan, *Minimization of the l_2-norms of the SOR and MSOR operators*, J. Comput. Appl. Math., 192 (2006), pp. 431–444.
[15] W. Niethammer, *Relaxation bei Matrizen mit der Eigenschaft “A”*, Z. Angew, Math. Mech., 44 (1964), pp. T49–T52.
[16] W. Niethammer, J. de Pillis, and R. S. Varga, *Convergence of block iterative methods applied to sparse least-squares problems*, Linear Algebra Appl., 58 (1984), pp. 327–341.
[17] D. J. Pierce, A. Hadjidimos, and R. J. Plemmons, *Optimality relationships for p-cyclic SOR*, Numer. Math., 56 (1990), pp. 635–643.
[18] G. J. Tee, *An application of p-cyclic matrices, for solving periodic parabolic problems*, Numer. Math., 6 (1964), pp. 142–159.
[19] R. S. Varga, *p-cyclic matrices: A generalization of the Young-Frankel successive overrelaxation scheme*, Pacific J. Math., 9 (1959), pp. 617–628.
[20] ———, *Matrix Iterative Analysis*, 2nd ed., Springer, Berlin, 2000.
[21] P. Wild, and W. Niethammer, *Over- and underrelaxation for linear systems with weakly cyclic Jacobi matrices of index p*, Linear Algebra Appl., 91 (1987), pp. 29–52.
[22] D. M. Young, *Iterative methods for solving partial difference equations of elliptic type*, Trans. Amer. Math. Soc., 76 (1954), pp. 92–111.
[23] ———, *Iterative Solution of Large Linear Systems*, Academic Press, New York, 1971.