Electronic Transactions on Numerical Analysis.
Volume 61, pp. 28–50, 2024.

ISSN 1068–9613.
DOI: 10.1553/etna_vol61s28

ETNA
Kent State University and
Johann Radon Institute (RICAM)

SIMULTANEOUS APPROXIMATION OF HILBERT AND HADAMARD TRANSFORMS ON BOUNDED INTERVALS*

DOMENICO MEZZANOTTE† AND DONATELLA OCCORSIO‡§

Abstract. In this paper, we propose a compound scheme of different product integration rules for the simultaneous approximation of both Hilbert and Hadamard transforms of a given function f. The advantages of such a scheme are multiple: a saving in the number of function evaluations and the avoidance of the derivatives of the density function f when approximating the Hadamard transform. Stability and convergence of the proposed method are proved in the space of locally continuous functions in $(-1, 1)$ with possible algebraic singularities at the endpoints, equipped with weighted uniform norms. The theoretical estimates are confirmed by several numerical tests.

Key words. hypersingular integrals, finite Hilbert transform, Hadamard finite part integrals, polynomial approximation, extended Lagrange interpolation, orthogonal polynomials

AMS subject classifications. 65D32, 65R10, 41A10, 41A28, 44A15

1. Introduction. The present paper deals with the simultaneous approximation of the Hilbert and Hadamard transforms of a function f, namely $\mathcal{H}_0^w(f)$ and $\mathcal{H}_1^w(f)$, defined as

$$\mathcal{H}_0^w(f,t) = \int_{-1}^{1} \frac{f(x)}{x-t} w(x)dx = \lim_{\varepsilon\to 0} \int_{|x-t|>\varepsilon} \frac{f(x)}{x-t} w(x)dx, \quad t \in (-1,1), \tag{1.1}$$

$$\mathcal{H}_1^w(f,t) = \neq\!\!\!\!\!\!\int_{-1}^{1} \frac{f(x)}{(x-t)^2} w(x)dx = \frac{d}{dt}\mathcal{H}_0^w(f,t), \qquad t \in (-1,1), \tag{1.2}$$

where w is a Jacobi weight and the integral in (1.1) is in the Cauchy principal value sense. The topic is of interest since many mathematical models in applied sciences lead to them; see, e.g., [1, 15, 16, 25, 26, 34] and the references therein. Due to its outstanding relevance, many authors introduced and studied many different kinds of numerical approaches; see, for instance, [1, 2, 10, 14, 25, 27, 28].

In particular, global methods to approximate them can be useful in the numerical treatment of singular [7, 22] and hypersingular integral equations [3, 9], which in turn describes many physical and engineering models; see, e.g., [19] and the references therein. Here we approach the case of weighted Hilbert and Hadamard transforms in the finite interval $(-1, 1)$, developing a scheme based on the approximation of the density function f by suitable Lagrange polynomials and their first derivatives. We propose a framework applicable according to different approaches that can be applied independently or together. To be more precise, there are three main approaches (paths):

Path 1 approximate only $\mathcal{H}_0^w(f)$ by using a mixed scheme made of two product integration formulae: the first rule is obtained by approximating f by the Lagrange polynomial $L_{m+1}(\tau, f)$ interpolating f at the zeros of the Jacobi polynomial $p_{m+1}(\tau)$, where $\tau(x) = (1-x)^\rho(1+x)^\sigma$ is a Jacobi weight; the second one, the so-called *extended rule*, is based on the approximation of f by the extended Lagrange polynomial

*Received December 21, 2023. Accepted April 05, 2024. Published online on May 15, 2024. Recommended by L. Fermo.

†Department of Mathematics “Giuseppe Peano”, University of Turin, Via Carlo Alberto 10, 10123 Turin, Italy (domenico.mezzanotte@unito.it).

‡Department of Mathematics, Computer Science and Economics, University of Basilicata, Viale dell’Ateneo Lucano 10, 85100 Potenza, Italy (donatella.occorsio@unibas.it).

§C.N.R. National Research Council of Italy, Institute for Applied Computation “Mauro Picone”, Via P. Castellino, 111, 80131 Naples, Italy.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

$\mathcal{L}_{2m+1}(\tau, \bar{\tau}, f)$, being $\bar{\tau}(x) = (1 - x^2)\tau(x)$, and interpolating f at the zeros of $p_{m+1}(\tau)p_m(\bar{\tau})$. This approach, described in Section 3, allows us to double in some sense the degree of the corresponding ordinary formulas by reusing the computations employed in the ordinary ones.

Path 2 approximate only $\mathcal{H}_1^w(f)$ according to a mixed scheme similar to that used for $\mathcal{H}_0^w(f)$, i.e., made up of a suitable composition of ordinary and extended rules, both of them avoiding the derivatives of the density function f. This approach will be discussed in Section 4.

Path 3 Both the previous procedures can be fruitfully employed to simultaneously approximate $\mathcal{H}_0^w(f)$ and $\mathcal{H}_1^w(f)$, without further samples of f than those required in Path 1. This scheme will be developed in Section 5.

Extended interpolation processes have been considered by many authors in both finite and infinite intervals, and their behavior has been studied in different weighted normed spaces [5, 6, 12, 17, 29, 30]; see also [20] and the references therein. Mixed schemes of ordinary and extended rules in quadrature have been developed in [23, 31, 32], with the purpose of a fast computation of weakly singular integrals, both in $\mathbb{R}^+$ and the interval $[-1, 1]$. The mixed quadrature scheme has been fruitfully employed in the fast solution of Fredholm integral equations by Nyström-type methods [24, 33], for which the reduction of samples of f corresponds to reducing the sizes of the final linear systems. The simultaneous approximation of Hilbert and Hadamard transform through the Lagrange approximation tool has been considered in [8] when such transforms are defined on $\mathbb{R}^+$.

As we have previously announced, all the schemes we propose here combine the advantages of the mixed quadrature scheme (Path 1 and 2) and of the simultaneous approximation of $\mathcal{H}_0^w(f)$ and $\mathcal{H}_1^w(f)$ (Path 3), without computing any derivative of the density function f. Moreover, from the theoretical point of view, we have proved that for any function f' satisfying a Dini-type condition, the weighted Hadamard transform $\mathcal{H}_1^w(f, t)$ is bounded for any $t \in (-1, 1)$ and algebraically diverges at the endpoints ± 1. About the introduced numerical scheme, we have proved the stability of the formulae and studied the convergence in uniform spaces of functions equipped with weighted norms.

The outline of the paper is as follows. In Section 2, some preliminary results are collected. In Section 3 the main results on the approximation of the Hilbert transform and the scheme obtained by mixing ordinary and extended rules are presented. The successive Section 4 contains the main results and the mixed scheme for the Hadamard transforms, while in Section 5 we are finally able to propose the compound scheme for the simultaneous approximation of the Hilbert and Hadamard transforms. Section 6 provides some details about the coefficients of all the rules we use. Finally, in Section 7, a selection of numerical tests is given.

2. Preliminaries. Throughout the paper, $\mathcal{C}$ will denote a generic positive constant having different meanings at different occurrences. We write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to indicate that $\mathcal{C}$ is independent of $a, b, \ldots$ and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. If $A, B > 0$ are quantities depending on some parameters, then we write $A \sim B$ if there exists a constant $\mathcal{C} \neq \mathcal{C}(A, B)$ such that $\mathcal{C}^{-1}B \leq A \leq \mathcal{C}B$.

For $m \in \mathbb{N}$, we denote by $\mathbb{P}_m$ the space of the algebraic polynomials of degree at most m. Finally, in all the paper, we will use the short notation

$$v^{a,b}(x) := (1 - x)^a(1 + x)^b, \quad a, b \in \mathbb{R}, \quad x \in (-1, 1).$$

2.1. Function spaces. Let u be the Jacobi weight

$$u(x) = v^{\gamma,\delta}(x) := (1 - x)^\gamma(1 + x)^\delta, \qquad x \in [-1, 1], \quad \gamma, \delta \geq 0.$$

We denote by C_u the space of locally continuous functions f on $(-1,1)$ such that the following limit conditions are satisfied:

$$\lim_{x\to 1^-} f(x)u(x)=0, \quad \text{if } \gamma>0, \qquad \text{and} \qquad \lim_{x\to -1^+} f(x)u(x)=0, \quad \text{if } \delta>0.$$

C_u equipped with the uniform norm

$$\|f\|_{C_u} := \|fu\|_\infty = \max_{x\in[-1,1]} |f(x)|u(x)$$

is a Banach space, i.e.,

$$\lim_{m\to\infty} E_m(f)_u = 0 \Longleftrightarrow f\in C_u,$$

where $E_m(f)_u$ is the error of best polynomial approximation of f defined as

$$E_m(f)_u := \inf_{P\in\mathbb{P}_m} \|f-P\|_{C_u};$$

see, e.g., [20]. In the case $\gamma=\delta=0$, C_u coincides with the space C^0 of all continuous functions in $[-1,1]$. For any $f\in C_u$, setting $\varphi(x):=\sqrt{1-x^2}$, by the main part of the *φ-modulus of smoothness* [11, p. 90] of order $k\in\mathbb{N}$,

$$\Omega_\varphi^k(f,t)_u = \sup_{0<h\le t} \|\Delta_{h\varphi}^k fu\|_{I_{kh}}, \qquad r\in\mathbb{N},$$

where

$$\Delta_{h\varphi}^k f(x) = \sum_{j=0}^k (-1)^j \binom{k}{j} f\left(x+(k-2j)\frac{h}{2}\varphi(x)\right), \quad I_{kh}=[-1+(2kh)^2, 1-(2kh)^2],$$

it is possible to define the Hölder–Zygmund space $Z_r(u)$ of order $r\in\mathbb{R}^+$ as

$$Z_r(u) = \left\{ f\in C_u \ : \ \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_u}{t^r} < \infty, \quad k>r \right\},$$

endowed with the norm

$$\|f\|_{Z_r(u)} = \|f\|_{C_u} + \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_u}{t^r}.$$

For any $f\in Z_r(u)$ and m sufficiently large, say $m>m_0$, we have

$$E_m(f)_u \le \mathcal{C}\, \frac{\|f\|_{Z_r(u)}}{m^r}, \quad \mathcal{C}\ne\mathcal{C}(f). \tag{2.1}$$

Finally, in what follows $DT(u)\subset C_u$ will denote the set of functions satisfying the Dini-type condition

$$DT(u) = \left\{ f\in C_u : \int_0^1 \frac{\Omega_\varphi(f,\sigma)_u}{\sigma} d\sigma < \infty \right\}.$$

The following lemma will be useful:

LEMMA 2.1 ([22, Lemma 2.1]). *For any $f\in C_u$ s.t. $f\in DT(u)$, then*

$$\int_0^{\frac{1}{m}} \frac{\Omega_\varphi(f-P_m,t)_u}{t} dt \le \|fu\|_\infty + \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^r(f,t)_u}{t} dt,$$

where $\mathcal{C}\ne\mathcal{C}(m,f)$.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

2.2. Orthonormal polynomials and Lagrange interpolating polynomials. Denote by $\{p_m(\tau)\}_{m\in\mathbb{N}}$, the system of the orthonormal polynomials for the Jacobi weight $\tau(x)=v^{\rho,\sigma}(x)$, $\rho,\sigma>-1$, $x\in(-1,1)$, with positive leading coefficients, i.e.,

$$p_m(\tau,x)=\gamma_m(\tau)x^m+\text{terms of lower degree},\quad \gamma_m(\tau)>0.$$

Setting $\{x_k\}_{k=1}^{m+1}$ the zeros of $p_{m+1}(\tau)$ and denoting by $\{\lambda_{m+1,k}(\tau)\}_{k=1}^{m+1}$ the Christoffel numbers of order $(m+1)$ w.r.t. τ, the Lagrange polynomial interpolating f at the zeros of $p_{m+1}(\tau)$ can be represented as

$$\begin{aligned} L_{m+1}(\tau,f,x)&=\sum_{k=1}^{m+1} l_{m+1,k}(\tau,x)f(x_k),\\ l_{m+1,k}(\tau,x)&=\lambda_{m+1,k}(\tau)\sum_{j=0}^{m} p_j(\tau,x)p_j(\tau,x_k). \end{aligned} \tag{2.2}$$

The next theorem states necessary and sufficient conditions such that the norm of the operator $L_{m+1}(\tau):C_u\to C_u$ diverges with optimal order.

THEOREM 2.2 ([20, Theorem 4.3.1]). *Let $\tau=v^{\rho,\sigma}$ and $u=v^{\gamma,\delta}$ with $\rho,\sigma>-1$, and $\gamma,\delta\geq 0$. Then, for all $f\in C_u$*

$$\|L_{m+1}(\tau,f)\|_{C_u}\leq \mathcal{C}\log m\|f\|_{C_u},\quad \mathcal{C}\neq\mathcal{C}(m,f),$$

if and only if

$$\begin{cases} \frac{\rho}{2}+\frac14\leq\gamma\leq\frac{\rho}{2}+\frac54,\\ \frac{\sigma}{2}+\frac14\leq\delta\leq\frac{\sigma}{2}+\frac54. \end{cases} \tag{2.3}$$

It follows that

$$\|f-L_{m+1}(\tau,f)\|_{C_u}\leq\mathcal{C}\log m\,E_m(f)_u,\quad \mathcal{C}\neq\mathcal{C}(m,f). \tag{2.4}$$

About the norm of the operator $L_{m+1}(\tau):Z_r(u)\to Z_s(u)$, $r\geq s>0$, from a more general result stated in [21] (see also [34]) the next theorem follows.

THEOREM 2.3. *For any function $f\in Z_r(u)$, under the assumptions (2.3),*

$$\|f-L_{m+1}(\tau,f)\|_{Z_s(u)}\leq\mathcal{C}\log m\frac{\|f\|_{Z_r(u)}}{m^{r-s}},\quad \mathcal{C}\neq\mathcal{C}(m,f),\quad r\geq s>0.$$

Setting $\overline{\tau}(x)=\tau(x)(1-x^2)$ and denoted by $\{p_n(\overline{\tau})\}_n$ the corresponding sequence of orthonormal polynomials, we recall the extended interpolation process based on the zeros of $Q_{2m+1}:=p_{m+1}(\tau)p_m(\overline{\tau})$. Denoting by $\{y_k\}_{k=1}^m$ the zeros of $p_m(\overline{\tau})$ and by $\{\lambda_{m,k}(\overline{\tau})\}_{k=1}^m$ the corresponding Christoffel numbers, the extended Lagrange polynomial interpolating f at the zeros of Q_{2m+1} takes the form

$$\begin{aligned} \mathcal{L}_{2m+1}(\tau,\overline{\tau},f,x)=&\sum_{k=1}^{m+1} f(x_k)\frac{Q_{2m+1}(x)}{Q'_{2m+1}(x_k)(x-x_k)}\\ &+\sum_{k=1}^{m} f(y_k)\frac{Q_{2m+1}(x)}{Q'_{2m+1}(y_k)(x-y_k)}. \end{aligned} \tag{2.5}$$

REMARK 2.4. We observe that in view of (2.5), once the polynomial $L_{m+1}(\tau, f)$ has been obtained, the construction of the polynomial $\mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)$ requires only m new samples of f at $\{y_k\}_{k=1}^{m}$.

Similarly as the Lagrange polynomial based on the zeros of $p_{m+1}(\tau)$, also the norm of the operator $\mathcal{L}_{2m+1}(\tau, \overline{\tau}) : C_u \to C_u$ behaves as $\log m$ under suitable assumptions on the parameters $\rho, \sigma, \gamma, \delta$. Indeed, the following theorem holds:

THEOREM 2.5 ([33, Theorem 2.2]). *Let be $\tau = v^{\rho,\sigma}$, $\overline{\tau} = v^{\rho+1,\sigma+1}$, with $\rho, \sigma > -1$. For any $f \in C_u$, $u = v^{\gamma,\delta}$, under the assumptions*

$$\begin{cases} \rho + 1 \le \gamma \le \rho + 2, \\ \sigma + 1 \le \delta \le \sigma + 2, \end{cases} \tag{2.6}$$

then

$$\|\mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)\|_{C_u} \le \mathcal{C} \log m \|f\|_{C_u}, \quad \mathcal{C} \neq \mathcal{C}(m, f).$$

Moreover, the following error estimate holds true:

$$\|f - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)\|_{C_u} \le \mathcal{C} \log m \, E_{2m}(f)_u, \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{2.7}$$

REMARK 2.6. In [6] it was introduced the polynomial $\mathcal{L}_{2m+1,r,s}(\tau, \overline{\tau}, f)$ interpolating f at the zeros of $Q_{2m+1}A_r B_s$, being the zeros of $A_r \in \mathbb{P}_r,\ B_s \in \mathbb{P}_s$. The r and s additional knots "close" to the endpoints ± 1 are chosen in such a way that the norm of the operator $\mathcal{L}_{2m+1}(\tau, \overline{\tau}) : f \in C^0 \to C^0$ logarithmically diverges.

Theorems 2.2 and 2.5 assure that both the sequences $\{L_m(\tau, f)\}_m$ and $\{\mathcal{L}_{2m+1}(\tau,\overline{\tau},f)\}_m$ can be selected, so that their norms in C_u diverge as $\log m$. In addition, according to Remark 2.4, with fixed m and computed $L_{m+1}(\tau, f)$, by the extended polynomial we double the degree of the approximant of f with only m new samples of f instead of $2m + 1$. Supported by these properties, the following *mixed sequence* of Lagrange interpolating polynomials was introduced in [33],

$$\mathbf{L}_n(f, x) = \begin{cases} L_{2^n+1}(\tau, f, x), & n = 0, 2, 4, \dots, \\ \mathcal{L}_{2^n+1}(\tau, \overline{\tau}, f, x), & n = 1, 3, 5, \dots, \end{cases}$$

i.e., a suitable sequence composed of the ordinary and the extended Lagrange polynomials, defined in (2.2) and (2.5), respectively, proving the following result:

THEOREM 2.7 ([33, Theorem 2.3]). *Under the assumptions*

$$\begin{cases} \rho + 1 \le \gamma \le \frac{\rho}{2} + \frac{5}{4}, \\ \sigma + 1 \le \delta \le \frac{\sigma}{2} + \frac{5}{4}, \end{cases} \tag{2.8}$$

for any $f \in Z_r(u)$, we have

$$\|[f - \mathbf{L}_n(f)]u\|_\infty \le \mathcal{C} \frac{\log m}{m^r} \|f\|_{Z_r(u)}, \quad m = 2^n,\ \mathcal{C} \neq \mathcal{C}(m, f).$$

We conclude this section, by stating two theorems that we need in the sequel to provide error estimates of the new numerical methods.

THEOREM 2.8. *For any function $f \in Z_r(u)$, under the assumptions (2.6),*

$$\|f - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)\|_{Z_s(u)} \le \mathcal{C} \log m \frac{\|f\|_{Z_r(u)}}{m^{r-s}}, \quad \mathcal{C} \neq \mathcal{C}(m, f), \quad r \ge s > 0. \tag{2.9}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Proof. To prove (2.9), we note that

$$E_k(f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f))_u \begin{cases} = E_k(f)_u, & \text{if } k \geq 2m, \\ \leq \|[f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f)]u\|_\infty, & \text{if } k < 2m. \end{cases} \tag{2.10}$$

Consequently, for all $s>0$, we have

$$\begin{aligned} &\|f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f)\|_{Z_s(u)} \\ &\qquad = \|[f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f)]u\|_\infty + \sup_{i\geq 0}(i+1)^s E_i(f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f))_u \\ &\qquad \leq \mathcal{C}E_{2m}(f)_u \log m + \mathcal{C}\|[f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f)]u\|_\infty (2m+1)^s, \end{aligned}$$

and hence, by (2.7) and (2.1) under the assumption $f\in Z_r(u)$, it follows

$$\|f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f)\|_{Z_s(u)} \leq \mathcal{C}\log m \frac{\|f\|_{Z_r(u)}}{m^{r-s}},$$

and the theorem is proved. ∎

By using Theorems 2.3 and 2.8, the following result can be deduced:

THEOREM 2.9. *For any function $f\in Z_r(u)$, under the assumptions (2.8), for any $0<s\leq r$ it is*

$$\|f-\mathbf{L}_n(f)\|_{Z_{s(u)}} \leq \mathcal{C}\log m \frac{\|f\|_{Z_r(u)}}{m^{r-s}}, \quad \mathcal{C}\neq\mathcal{C}(m,f).$$

2.3. The Hilbert transform. We conclude by recalling a result in [34] (see [18]) about the boundedness of the Hilbert transform. To this end, denote by

$$B_0(u) := \left\{ f\in C_u \ : \ \sum_{k=1}^{\infty} \frac{E_k(f)_u}{k+1} < \infty \right\}$$

the Besov-type space (introduced in [18]) which is the "correct and minimal" space to study the boundedness of the Hilbert transform, equipped with the norm

$$\|f\|_{B_0(u)} := \|f\|_{C_u} + \sum_{m=1}^{\infty} \frac{E_m(f)_u}{m+1}.$$

THEOREM 2.10 ([34]). *Let $w=v^{\alpha,\beta}$, with $\alpha,\beta>-1$, be the Jacobi weight defining $w_+(x) := v^{\alpha_+,\beta_+}(x)$ and $w_-(x) := v^{\alpha_-,\beta_-}(x)$, and*

$$h(t) := \frac{1}{1+(w_+w_-)(-1)|\log(1+t)|+(w_+w_-)(1)|\log(1-t)|}.$$

Then for all $t\in(-1,1)$ and any $f\in B_0(w_+)$, it is

$$|\mathcal{H}_0^w(f,t)|w_-(t) \leq \mathcal{C}\left(|f(t)|w_+(t)+\|f\|_{B_0(w_+)}\right), \quad \mathcal{C}\neq\mathcal{C}(f,t).$$

In conclusion, the last theorem provides conditions assuring the boundedness of $\mathcal{H}_0^w$ as a map from $B_0(w_+)$ into $C^0_{w_-}$.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

2.4. The ordinary product integration rules. A well-known product integration rule is obtained by approximating the density function f by the Lagrange polynomial $L_{m+1}(\tau, f)$ defined in (2.2) with $\tau = v^{\rho,\sigma}$, i.e.,

$$\mathcal{H}_0^w(f,t) = \mathcal{H}_{0,m+1}^w(f,t) + e_{0,m+1}^w(f,t), \tag{2.11}$$

where the rule takes the form

$$\mathcal{H}_{0,m+1}^w(f,t) := \mathcal{H}_0^w(L_{m+1}(\tau,f),t) = \sum_{k=1}^{m+1} f(x_k)\mathcal{D}_k^{(0)}(t), \tag{2.12}$$
$$\mathcal{D}_k^{(0)}(t) := \mathcal{H}_0^w(l_{m+1,k}(\tau),t),$$

and the remainder term is defined as

$$e_{0,m+1}^w(f,t) := \mathcal{H}_0^w(f - L_{m+1}(\tau,f),t).$$

Formula (2.11) has been considered by several authors (for instance [4]); see also [20, 28] and the references therein. In particular, we recall the error estimate given in [34]. Denoting by

$$c_+ := \max\{0, c\}, \quad c_- := \max\{0, -c\},$$

the following theorem holds:

THEOREM 2.11 ([34]). *Let $w = v^{\alpha,\beta}$, with $\alpha, \beta > -1$, be the Jacobi weight defining the Hilbert transform (1.1), and set $w = \frac{w_+}{w_-}$ with $w_+(x) := v^{\alpha_+,\beta_+}(x)$ and $w_-(x) := v^{\alpha_-,\beta_-}(x)$, and*

$$h(t) := \frac{1}{1 + (w_+w_-)(-1)|\log(1+t)| + (w_+w_-)(1)|\log(1-t)|}.$$

Then under the assumptions

$$\begin{cases} 2\alpha_+ - \frac{5}{2} \le \rho \le 2\alpha_+ - \frac{1}{2}, \\ 2\beta_+ - \frac{5}{2} \le \sigma \le 2\beta_+ - \frac{1}{2}, \end{cases}$$

for all $t \in (-1,1)$ and any $f \in Z_r(w_+)$, we have

$$|e_{0,m+1}^w(f,t)|w_-(t)h(t) \le \mathcal{C}\frac{\log^2 m}{m^r}\|f\|_{Z_r(w_+)}, \qquad \mathcal{C} \ne \mathcal{C}(f,t). \tag{2.13}$$

3. Main results: approximation of the Hilbert transform.

3.1. The extended product rule. Now, we introduce the *extended product rule* obtained by replacing the function f in the Hilbert transform in (1.1) by the polynomial $\mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)$, i.e.,

$$\mathcal{H}_0^w(f,t) = \widetilde{\mathcal{H}}_{0,2m+1}^w(f,t) + \xi_{0,2m+1}^w(f,t), \tag{3.1}$$

where the product rule takes the form

$$\widetilde{\mathcal{H}}_{0,2m+1}^w(f,t) = \mathcal{H}_0^w(\mathcal{L}_{2m+1}(\tau,\overline{\tau},f),t) = \sum_{k=1}^{m+1} f(x_k)\mathcal{A}_k^{(0)}(t) + \sum_{k=1}^{m} f(y_k)\mathcal{B}_k^{(0)}(t),$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

with

$$\begin{aligned} \mathcal{A}_k^{(0)}(t) &= \mathcal{H}_0^w \left(\frac{Q_{2m+1}}{Q'_{2m+1}(x_k)(\cdot - x_k)}, t \right), \\ \mathcal{B}_k^{(0)}(t) &= \mathcal{H}_0^w \left(\frac{Q_{2m+1}}{Q'_{2m+1}(y_k)(\cdot - y_k)}, t \right), \end{aligned} \tag{3.2}$$

and the remainder term is

$$\xi_{0,2m+1}^w(f,t) = \mathcal{H}_0^w(f - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f), t).$$

About the convergence of the extended rule, we can prove the following:

THEOREM 3.1. *Let $w = v^{\alpha,\beta}$, with $\alpha, \beta > 0$, be the Jacobi weight defining the Hilbert transform (1.1), and set $w = \frac{w_+}{w_-}$ with $w_+(x) := v^{\alpha_+,\beta_+}(x)$ and $w_-(x) := v^{\alpha_-,\beta_-}(x)$. Then, under the assumptions*

$$\begin{cases} \alpha - 2 \le \rho \le \alpha - 1, \\ \beta - 2 \le \sigma \le \beta - 1, \end{cases} \tag{3.3}$$

for all $t \in (-1,1)$ and any $f \in Z_r(w)$, we have

$$|\xi_{0,2m+1}^w(f,t)| \le \mathcal{C} \frac{\log^2 m}{m^r} \|f\|_{Z_r(w)}, \qquad \mathcal{C} \ne \mathcal{C}(f,t). \tag{3.4}$$

Proof. From (3.1) it follows that

$$|\xi_{0,2m+1}^w(f,t)| = |\mathcal{H}_0^w(f,t) - \widetilde{\mathcal{H}}_{0,2m+1}^w(f,t)| = |\mathcal{H}_0^w(f - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f), t)|.$$

First, we observe that under the assumptions on ρ, α_+ and σ, β_+ in Theorem 2.10 the exponents $\alpha_+ = \alpha, \ \beta_+ = \beta$, and consequently $w_+ = w$, $h(t) \equiv 1$ and $w_-(t) \equiv 1$. Hence, we have

$$\begin{aligned} |\xi_{0,2m+1}^w(f,t)| &\le \mathcal{C} \left(|(f(t) - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f, t))| w(t) + \|f - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)\|_{B_0(w)} \right) \\ &=: \mathcal{C} \left(\widehat{J}_1 + \widehat{J}_2 \right). \end{aligned}$$

Under the assumptions (3.3), by (2.7) it follows that

$$\widehat{J}_1 \le \mathcal{C} \log m \, E_{2m}(f)_w \le \mathcal{C} \log m \frac{\|f\|_{Z_r(w)}}{m^r}.$$

About $\widehat{J}_2$, by (2.10) and (2.7) we have

$$\begin{aligned} \widehat{J}_2 &= \|f - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)\|_{C_w} + \sum_{k=1}^{\infty} \frac{E_k(f)_w}{k+1} \\ &\le \mathcal{C} \|f - \mathcal{L}_{2m+1}(\tau, \overline{\tau}, f)\|_{C_w} \left(1 + \sum_{k=1}^{2m-1} \frac{1}{k+1} \right) + \sum_{k=2m}^{\infty} \frac{E_k(f)_w}{k+1} \\ &\le \mathcal{C} \log^2 m \, E_{2m}(f)_w + \sum_{k=2m}^{\infty} \frac{E_k(f)_w}{k+1} \le \mathcal{C} \log^2 m \frac{\|f\|_{Z_r(w)}}{m^r} + \mathcal{C} \frac{\|f\|_{Z_r(w)}}{m^r}. \end{aligned}$$

Combining the two previous estimates, the thesis follows. $\square$

3.2. The mixed scheme for the approximation of the Hilbert transform. We consider the mixed scheme of product integration rules, obtained by combining the ordinary product rule $\mathcal{H}^w_{0,m+1}(f,t)$ and the extended product rule $\widetilde{\mathcal{H}}^w_{0,2m+1}(f,t)$, to take advantage of reducing the global number of required samples of f. To this end, we propose the following mixed scheme

$$\mathcal{H}^w_0(f,t) = \hat{\mathcal{H}}^w_{0,n}(f,t) + \zeta^w_{0,n}(f,t), \tag{3.5}$$

where

$$\hat{\mathcal{H}}^w_{0,n}(f) = \begin{cases} \mathcal{H}^w_{0,2^n+1}(f), & n = 0,2,4,\dots, \\ \widetilde{\mathcal{H}}^w_{0,2^n+1}(f), & n = 1,3,5,\dots. \end{cases} \tag{3.6}$$

Regarding the convergence of the mixed sequence, the following result holds:

THEOREM 3.2. *Assume that the weight functions $w = v^{\alpha,\beta}$, $\alpha,\beta > 0$, and $\tau = v^{\rho,\sigma}$ defining (3.5) satisfy*

$$\begin{cases} \max\left(2\alpha - \frac{5}{2}, \alpha - 2\right) \le \rho \le \alpha - 1, \\ \max\left(2\beta - \frac{5}{2}, \beta - 2\right) \le \sigma \le \beta - 1. \end{cases} \tag{3.7}$$

Then for all $f \in Z_r(w)$ and any $t \in (-1,1)$, we have

$$|\zeta^w_{0,n}(f,t)| \le \mathcal{C} \frac{\log^2 m}{m^r} \|f\|_{Z_r(w)}, \quad m = 2^n + 1,$$

being $\mathcal{C} \neq \mathcal{C}(m,f,t)$.

Proof. We observe that under assumptions (3.7), Theorems 2.11 and 3.1 are both satisfied. Hence, in view of (3.6), the thesis follows by combining the estimates (2.13) and (3.4). $\square$

4. Main results: approximation of the Hadamard transform. From now on, we assume that for $\mathcal{H}^w_1(f,t)$ defined in (1.2) the parameters defining the Jacobi weight satisfy $\alpha,\beta \ge 0$.

Our first result considers the boundedness of the Hadamard transform in (1.2).

THEOREM 4.1. *Let $w = v^{\alpha,\beta}$, with $\alpha,\beta \ge 0$ and $f' \in DT(w\varphi)$. Then, for any $t \in (-1,1)$ we have*

$$|\mathcal{H}^w_1(f,t)|\varphi^2(t) \le \mathcal{C}\left(\|fw\|_\infty + \int_0^1 \frac{\Omega_\varphi(f',y)_{w\varphi}}{y}dy\right), \qquad \mathcal{C} \neq \mathcal{C}(f,t).$$

REMARK 4.2. In the case $\alpha = \beta = 0$, Theorem 4.1 has been proved in [13].

To prove Theorem 4.1 we need two lemmas.

LEMMA 4.3. *For any $-1 < t \le -\frac{1}{2}$ and for any $f \in C_w$ s.t. $f' \in DT(w\varphi)$, we have*

$$= \!\!\!\!\!\!\! \int_{-1}^{2t+1} \frac{f(x)}{(x-t)^2} w(x)dx \le \mathcal{C}\left(\frac{1}{\varphi(t)} \int_0^1 \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma}d\sigma + \frac{\|fw\|_\infty}{1+t}\right),$$

where $\mathcal{C} \neq \mathcal{C}(f,t)$.

Proof. The proof starts from

$$\begin{aligned} = \!\!\!\!\!\!\! \int_{-1}^{2t+1} \frac{f(x)}{(x-t)^2} w(x)dx &= = \!\!\!\!\!\!\! \int_{-1}^{2t+1} \frac{f(x) - f(t) - f'(t)(x-t)}{(x-t)^2} w(x)dx \\ &\quad + f(t) = \!\!\!\!\!\!\! \int_{-1}^{2t+1} \frac{w(x)}{(x-t)^2}dx + f'(t) = \!\!\!\!\!\!\! \int_{-1}^{2t+1} \frac{w(x)}{x-t}dx \\ &=: A_1(t) + A_2(t) + A_3(t). \end{aligned} \tag{4.1}$$

Since $f' \in DT(w\varphi)$, the term $A_1(t)$ can be rewritten as

$$A_1(t) = \int_{-1}^{2t+1} \frac{\int_t^x [f'(y) - f'(t)]dy}{(x-t)^2} w(x)dx,$$

and therefore

$$A_1(t) = \int_{-1}^{t} \left[\int_x^t [f'(t) - f'(y)]dy\right] \frac{w(x)}{(x-t)^2} dx + \int_t^{2t+1} \left[\int_t^x [f'(y) - f'(t)]dy\right] \frac{w(x)}{(x-t)^2} dx.$$

Hence, introducing the changes of variable $x = t - \frac{\sigma}{2}\sqrt{1-t^2}$, $y = t - \frac{h}{2}\sqrt{1-t^2}$ in the first addendum and $x = t + \frac{\sigma}{2}\sqrt{1-t^2}$, $y = t + \frac{h}{2}\sqrt{1-t^2}$ in the second one, we get

$$A_1(t) = \int_0^{2\sqrt{\frac{1+t}{1-t}}} \left[\int_0^\sigma \left[f'(t) - f'\left(t - \frac{h}{2}\sqrt{1-t^2}\right)\right] dh\right] \frac{w(t - \frac{\sigma}{2}\sqrt{1-t^2})}{\sigma^2} d\sigma + \int_0^{2\sqrt{\frac{1+t}{1-t}}} \left[\int_0^\sigma \left[f'\left(t + \frac{h}{2}\sqrt{1-t^2}\right) - f'(t)\right] dh\right] \frac{w(t + \frac{\sigma}{2}\sqrt{1-t^2})}{\sigma^2} d\sigma.$$

Consequently,

$$|A_1(t)| \le \int_0^{2\sqrt{\frac{1+t}{1-t}}} \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma} \frac{w(t - \frac{\sigma}{2}\sqrt{1-t^2})}{w(t)\varphi(t)} d\sigma + \int_0^{2\sqrt{\frac{1+t}{1-t}}} \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma} \frac{w(t + \frac{\sigma}{2}\sqrt{1-t^2})}{w(t)\varphi(t)} d\sigma$$

and

$$|A_1(t)| \le \mathcal{C} \frac{1}{\varphi(t)} \int_0^1 \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma} d\sigma.$$

Since

$$|A_2(t)| \le |f(t)| \left| \int_{-1}^{2t+1} \frac{w(x)}{(x-t)^2} dx \right| \le \|fw\|_\infty \frac{2}{1+t}$$

and

$$|A_3(t)| \le \|fw\|_\infty \left| \int_{-1}^{2t+1} \frac{dx}{x-t} \right| = 0,$$

the statement follows by collecting the last three inequalities and (4.1). ◻

Similarly, one can prove the following:

LEMMA 4.4. *For any* $\frac{1}{2} \le t < 1$ *and for any* f *s.t.* $f' \in DT(w\varphi)$, *we have*

$$\int_{2t-1}^{1} \frac{f(x)}{(x-t)^2} w(x)dx \le \mathcal{C} \left(\frac{1}{\varphi(t)} \int_0^1 \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma} d\sigma + \frac{\|fw\|_\infty}{1-t} \right),$$

where $\mathcal{C} \ne \mathcal{C}(f,t)$.

Proof of Theorem 4.1. Firstly assume $-1 < t \le -\frac{1}{2}$. In this case $\varphi^2(t) \sim (1+t)$, and we have

$$\varphi^2(t)\left|\mathcal{H}_1^w(f,t)\right| \sim (1+t)\left|\ \fint_{-1}^{2t+1} \frac{f(x)}{(x-t)^2}w(x)dx + \int_{2t+1}^{1} \frac{f(x)}{(x-t)^2}w(x)dx\right|.$$

Since

$$(1+t)\left|\int_{2t+1}^{1} \frac{f(x)}{(x-t)^2}w(x)dx\right| \le \mathcal{C}\|fw\|_\infty,$$

the statement follows from Lemma 4.3 for any $-1 < t \le -\frac{1}{2}$.

Now, assume $\frac{1}{2} \le t < 1$, so that $\varphi^2(t) \sim (1-t)$. By using the decomposition

$$\varphi^2(t)\left|\mathcal{H}_1^w(f,t)\right| \sim (1-t)\left|\int_{-1}^{2t-1} \frac{f(x)}{(x-t)^2}w(x)dx + \fint_{2t-1}^{1} \frac{f(x)}{(x-t)^2}w(x)dx\right|,$$

and taking into account that

$$(1-t)\left|\int_{-1}^{2t-1} \frac{f(x)}{(x-t)^2}w(x)dx\right| \le \mathcal{C}\|fw\|_\infty,$$

the statement follows from Lemma 4.4 for any $\frac{1}{2} \le t < 1$.

Finally, let $|t| < \frac{1}{2}$, for which $\varphi(t) \sim 1$. Fixing b s.t. $\frac{1}{4} < b < \frac{1}{2}$, consider the following decomposition

$$\begin{aligned}
&\varphi^2(t)\left|\mathcal{H}_1^w(f,t)\right| \\
&\sim \left|\int_{|x-t|\ge b} \frac{f(x)}{(x-t)^2}w(x)dx + \int_{t-b}^{t+b} \frac{f(x)-f(t)-f'(t)(x-t)}{(x-t)^2}w(x)dx\right. \\
&\qquad\left. + f(t)\fint_{t-b}^{t+b} \frac{w(x)}{(x-t)^2}dx + f'(t)\fint_{t-b}^{t+b} \frac{w(x)}{x-t}dx\right| \\
&=: B_1(t) + B_2(t) + B_3(t).
\end{aligned} \tag{4.2}$$

Since

$$\begin{aligned}
|B_2(t)| &\le \mathcal{C}\|fw\|_\infty \left|\fint_{t-b}^{t+b} \frac{dx}{(x-t)^2}\right| \le \frac{2}{b}\|fw\|_\infty, \\
|B_3(t)| &\le \|f'w\varphi\|_\infty \left|\fint_{t-b}^{t+b} \frac{dx}{x-t}\right| = 0,
\end{aligned} \tag{4.3}$$

we estimate $B_1(t)$. Following steps analogous to the ones used to estimate $A_1(t)$ in Lemma 4.3,

$$\begin{aligned}
|B_1(t)| \le &\left|\int_{t-b}^{t}\left[\int_{x}^{t}[f'(t)-f'(y)]dy\right]\frac{w(x)}{(x-t)^2}dx\right| \\
&+\left|\int_{t}^{t+b}\left[\int_{t}^{x}[f'(y)-f'(t)]dy\right]\frac{w(x)}{(x-t)^2}dx\right|,
\end{aligned}$$

and introducing the change of variables $x = t \pm \frac{\sigma}{2}\varphi(t)$ and $y = t \pm \frac{h}{2}\varphi(t)$, we obtain

$$\begin{aligned}
|B_1(t)| \le & \int_0^{\frac{2}{b\sqrt{1-t^2}}} \left[\int_0^\sigma \left[f'(t) - f'\left(t - \frac{h}{2}\sqrt{1-t^2}\right)\right] dh\right] \frac{w(t-\frac{\sigma}{2}\sqrt{1-t^2})}{\sigma^2} d\sigma \\
& + \int_0^{\frac{2}{b\sqrt{1-t^2}}} \left[\int_0^\sigma \left[f'\left(t + \frac{h}{2}\sqrt{1-t^2}\right) - f'(t)\right] dh\right] \frac{w(t+\frac{\sigma}{2}\sqrt{1-t^2})}{\sigma^2} d\sigma \\
\le & \int_0^{\frac{2}{b\sqrt{1-t^2}}} \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma} \frac{w(t-\frac{\sigma}{2}\sqrt{1-t^2})}{w(t)\varphi(t)} d\sigma \\
& + \int_0^{\frac{2}{b\sqrt{1-t^2}}} \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma} \frac{w(t+\frac{\sigma}{2}\sqrt{1-t^2})}{w(t)\varphi(t)} d\sigma \\
\le & \, \mathcal{C}\frac{1}{\varphi(t)} \int_0^1 \frac{\Omega_\varphi(f',\sigma)_{w\varphi}}{\sigma} d\sigma.
\end{aligned}$$

By combining the last estimate with (4.3) and (4.2), the theorem follows for $|t| < \frac{1}{2}$. Then, taking into account Lemmas 4.3 and 4.4, the theorem is completely proved. ∎

4.1. The product rules for the Hadamard transform. Following the same scheme used to treat the approximation of the Hilbert transform, we consider ordinary and extended product integration rules and their mixed sequence. By replacing f with $L_{m+1}(\tau, f)$, we have

$$\mathcal{H}_1^w(f,t) = \mathcal{H}_1^w(L_{m+1}(\tau,f),t) + e_{1,m+1}^w(f,t) =: \mathcal{H}_{1,m+1}^w(f,t) + e_{1,m+1}^w(f,t), \tag{4.4}$$

where

$$\mathcal{H}_{1,m+1}^w(f,t) := \sum_{k=1}^{m+1} f(x_k)\mathcal{D}_k^{(1)}(t), \quad \mathcal{D}_k^{(1)}(t) = \mathcal{H}_1^w(l_{m+1,k}(\tau),t), \tag{4.5}$$

and $e_{1,m+1}^w(f,t)$ is the remainder term. Moreover, approximating f by the extended polynomial $\mathcal{L}_{2m+1}(\tau,\overline{\tau},f)$ we get the extended rule for Hadamard integrals

$$\begin{aligned}
\mathcal{H}_1^w(f,t) &= \mathcal{H}_1^w(\mathcal{L}_{2m+1}(\tau,\overline{\tau},f),t) + \xi_{1,2m+1}^w(f,t) \\
&=: \widetilde{\mathcal{H}}_{1,2m+1}^w(f,t) + \xi_{1,2m+1}^w(f,t),
\end{aligned} \tag{4.6}$$

with

$$\widetilde{\mathcal{H}}_{1,2m+1}^w(f,t) := \sum_{k=1}^{m+1} f(x_k)\mathcal{A}_k^{(1)}(t) + \sum_{k=1}^{m} f(y_k)\mathcal{B}_k^{(1)}(t),$$

where

$$\begin{aligned}
\mathcal{A}_k^{(1)}(t) &= \mathcal{H}_1^w\left(\frac{Q_{2m+1}}{Q'_{2m+1}(x_k)(\cdot - x_k)}, t\right), \\
\mathcal{B}_k^{(1)}(t) &= \mathcal{H}_1^w\left(\frac{Q_{2m+1}}{Q'_{2m+1}(y_k)(\cdot - y_k)}, t\right).
\end{aligned} \tag{4.7}$$

About the convergence of the ordinary rule (4.5), the next theorem holds.

THEOREM 4.5. *Assume that the weight functions $w = v^{\alpha,\beta}$, $\alpha, \beta > 0$ and $\tau = v^{\rho,\sigma}$ defining (4.4) satisfy*

$$\begin{cases} 2\alpha - \frac{5}{2} \le \rho \le 2\alpha - \frac{1}{2}, \\ 2\beta - \frac{5}{2} \le \sigma \le 2\beta - \frac{1}{2}. \end{cases} \tag{4.8}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Then, for all $f' \in Z_r(w\varphi)$ *and any* $t \in (-1,1)$, *we have*

$$\varphi^2(t)|e^w_{1,m+1}(f,t)| \le \mathcal{C}\log^2 m \frac{\|f'\|_{Z_r(w\varphi)}}{m^r}. \tag{4.9}$$

Proof. Since $e^w_{1,m+1}(f,t)) = \mathcal{H}^w_1(f - L_{m+1}(\tau,f),t)$, by Theorem 4.1 it follows

$$\begin{aligned}\varphi^2(t)|e^w_{1,m+1}(f,t)| &\le \mathcal{C}\left(\|(f-L_{m+1}(\tau,f))w\|_\infty + \int_0^1 \frac{\Omega_\varphi((f-L_{m+1}(\tau,f))',t)_{w\varphi}}{t}dt\right)\\ &= J_1 + J_2.\end{aligned}$$

By (2.4), under the assumptions (4.8), we have

$$J_1 \le \mathcal{C}\log m E_m(f)_w. \tag{4.10}$$

To estimate J_2 we use Lemma 2.1,

$$\begin{aligned}&\int_0^{\frac{1}{m}} \frac{\Omega_\varphi((f-L_{m+1}(\tau,f))',t)_{w\varphi}}{t}dt\\ &\qquad \le \mathcal{C}\left(\|(f-L_{m+1}(\tau,f))'w\varphi\|_\infty + \int_0^{\frac{1}{m}} \frac{\Omega^r_\varphi(f',t)_{w\varphi}}{t}dt\right)\end{aligned}$$

and

$$\int_{\frac{1}{m}}^1 \frac{\Omega_\varphi((f-L_{m+1}(\tau,f))',t)_{w\varphi}}{t}dt \le \|(f-L_{m+1}(\tau,f))'w\varphi\|_\infty \log m$$

to conclude that

$$J_2 \le \mathcal{C}\left(\|(f-L_{m+1}(\tau,f))'w\varphi\|_\infty \log m + \int_0^{\frac{1}{m}} \frac{\Omega^r_\varphi(f',t)_{w\varphi}}{t}dt\right).$$

Now, by [20, Theorem 4.3.5] we have for any $f \in W_1(w)$

$$\|(f-L_{m+1}(\tau,f))'w\varphi\|_\infty \le \log m\, E_{m-1}(f')_{w\varphi},$$

by which, for $f' \in Z_r(w\varphi)$, it follows

$$J_2 \le \mathcal{C}\log^2 m \frac{\|f'\|_{Z_r(w\varphi)}}{m^r},$$

and combining the last inequality with (4.10), we obtain

$$\varphi^2(t)|e^w_{1,m+1}(f,t)| \le \mathcal{C}\log^2 m \frac{\|f'\|_{Z_r(w\varphi)}}{m^r},$$

and the theorem is completely proved. □

THEOREM 4.6. *Assume that the weight functions* $w = v^{\alpha,\beta}$ *and* $\tau = v^{\rho,\sigma}$ *defining* (4.6) *satisfy*

$$\begin{cases}\alpha - 2 \le \rho \le \alpha - 1,\\ \beta - 2 \le \sigma \le \beta - 1.\end{cases} \tag{4.11}$$

Then, for all $f' \in Z_r(w\varphi)$ *and any* $t \in (-1,1)$, *we have*

$$\varphi^2(t)|\xi^w_{1,2m+1}(f,t)| \le \mathcal{C}\log^2 m \frac{\|f'\|_{Z_r(w\varphi)}}{m^r}, \tag{4.12}$$

being $\mathcal{C} \neq \mathcal{C}(m,f,t)$.

Proof. Note that from (4.6)

$$|\xi^w_{1,2m+1}(f,t)| = |\mathcal{H}^w_1(f,t) - \widetilde{\mathcal{H}}^w_{1,2m+1}(f,t)| = |\mathcal{H}^w_1(f - \mathcal{L}_{2m+1}(\tau,\overline{\tau},f),t)|.$$

By Theorem 4.1 we have

$$\begin{aligned}&\varphi^2(t)|\xi^w_{1,2m+1}(f,t)|\\ &\qquad\le \mathcal{C}\left(\|(f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f))w\|_\infty + \int_0^1 \frac{\Omega_\varphi((f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f))',t)_{w\varphi}}{t}dt\right)\\ &\qquad=: \widetilde{J}_1 + \widetilde{J}_2.\end{aligned}$$

Under the assumptions (4.11), by (2.7) it follows that

$$\widetilde{J}_1 \le \mathcal{C}\log m\, E_{2m}(f)_w.$$

Recalling that by [20, Theorem 4.3.5] we have for any $f \in W_1(w)$

$$\|(f-\mathcal{L}_{2m+1}(\tau,\overline{\tau},f))'w\varphi\|_\infty \le \log m\, E_{2m-1}(f')_{w\varphi},$$

and applying Lemma 2.1, we obtain the following bound for $\widetilde{J}_2$

$$\widetilde{J}_2 \le \mathcal{C}\log^2 m\,\frac{\|f'\|_{Z_r(w\varphi)}}{(2m)^r}.$$

Combining the two estimates of $\widetilde{J}_1$ and $\widetilde{J}_2$, the thesis follows. ∎

4.2. The mixed scheme for the approximation of the Hadamard transform. Under suitable assumptions, both sequences $\left\{\mathcal{H}^w_{1,m+1}(f)\right\}_m$ and $\left\{\widetilde{\mathcal{H}}^w_{1,2m+1}(f)\right\}_m$ converge uniformly to the Hadamard transform $\mathcal{H}^w_1(f)$. Thus, it makes sense to consider a mixed scheme combining the previously introduced methods. The mixed sequence is obtained as follows:

$$\mathcal{H}^w_1(f,t) = \hat{\mathcal{H}}^w_{1,n}(f,t) + \zeta^w_{1,n}(f,t),$$

where

$$\hat{\mathcal{H}}^w_{1,n}(f,t) = \begin{cases}\mathcal{H}^w_{1,2^n+1}(f,t), & n = 0,2,4,\ldots,\\ \widetilde{\mathcal{H}}^w_{1,2^n+1}(f,t), & n = 1,3,5,\ldots.\end{cases} \tag{4.13}$$

With regard to convergence the following result holds:

THEOREM 4.7. *Assume that the weight functions* $w = v^{\alpha,\beta}$ *and* $\tau = v^{\rho,\sigma}$ *satisfy*

$$\begin{cases}\max\left(2\alpha - \frac{5}{2}, \alpha - 2\right) \le \rho \le \alpha - 1,\\ \max\left(2\beta - \frac{5}{2}, \beta - 2\right) \le \sigma \le \beta - 1.\end{cases} \tag{4.14}$$

Then, for all $f' \in Z_r(w\varphi)$ *and any* $t \in (-1,1)$, *we have*

$$\varphi^2(t)|\zeta^w_{1,n}(f,t)| \le \mathcal{C}\log^2 m \frac{\|f'\|_{Z_r(w\varphi)}}{m^r}, \quad m = 2^n + 1,$$

with $\mathcal{C} \neq \mathcal{C}(m,f,t)$.

Proof. We observe that under assumptions (4.14), Theorems 4.5 and 4.6 are both satisfied. Hence, in view of (4.13), the thesis follows by combining estimates (4.9) and (4.12). ∎

ETNA
Kent State University and
Johann Radon Institute (RICAM)

5. Simultaneous approximation of Hilbert and Hadamard transforms. Summing up the results regarding both the mixed sequences, i.e., $\hat{\mathcal{H}}^w_{0,n}(f,t)$ and $\hat{\mathcal{H}}^w_{1,n}(f,t)$, we are now able to consider an appropriate scheme of work, useful in all the cases where we simultaneously want to approximate $\mathcal{H}^w_0(f)$ and $\mathcal{H}^w_1(f)$. Basically, we can obtain both the approximated values of $\mathcal{H}^w_0(f,t)$ and $\mathcal{H}^w_1(f,t)$, saving on the total amount of samples of f needed whenever we treat their approximation separately.

To be more precise, assuming $w = v^{\alpha,\beta}$ with $\alpha, \beta > 0$, for any $t \in [a,b] \subset (-1,1)$ we consider the following scheme obtained by merging both the mixed sequences of approximants of Hilbert and Hadamard transforms:

$$\begin{aligned} \widehat{\mathbf{H}}^w_n(f,t) &= \left\{\hat{\mathcal{H}}^w_{0,n}(f,t),\, \hat{\mathcal{H}}^w_{1,n}(f,t)\right\} \\ &= \begin{cases} \left\{\mathcal{H}^w_{0,2^n+1}(f,t),\, \mathcal{H}^w_{1,2^n+1}(f,t)\right\}, & n = 0,2,4,\dots, \\ \left\{\widetilde{\mathcal{H}}^w_{0,2^n+1}(f,t),\, \widetilde{\mathcal{H}}^w_{1,2^n+1}(f,t)\right\}, & n = 1,3,5,\dots, \end{cases} \end{aligned} \tag{5.1}$$

which allows us to gain many advantages. More precisely, at every even step of each compounded scheme, we save up to 33.2% on function evaluations, with comparable (or even better) performances w.r.t. the related sequence based on only the ordinary rule. This leads to a drastic reduction in CPU time required to create the approximant sequences. Moreover, we delay the difficulties of evaluating the modified moments for high values of m and the instability of their computation by recurrence relations. Finally, using the same set of function evaluations and avoiding the computation of the derivative f', we simultaneously approximate the values of $\mathcal{H}^w_0(f)$ and $\mathcal{H}^w_1(f)$.

About the convergence, by combining Theorems 3.2 and 4.7 the following result holds:

THEOREM 5.1. *Assume that the weight functions $w = v^{\alpha,\beta}$, $\alpha, \beta > 0$ and $\tau = v^{\rho,\sigma}$ satisfy*

$$\begin{cases} \max\left(2\alpha - \frac{5}{2}, \alpha - 2\right) \le \rho \le \alpha - 1, \\ \max\left(2\beta - \frac{5}{2}, \beta - 2\right) \le \sigma \le \beta - 1. \end{cases}$$

Then for any f s.t. $f' \in Z_r(w\varphi)$ and for any $t \in [a,b] \subset (-1,1)$,

$$|\widehat{\mathcal{H}}^w_{i,n}(f,t) - \mathcal{H}^w_i(f,t)| \le \mathcal{C} \log^2 m \frac{\|f'\|_{Z_r(w\varphi)}}{m^r}, \quad i \in \{0,1\},\ m = 2^n + 1,$$

$\mathcal{C} \neq \mathcal{C}(m,f,t)$.

6. Implementation details. This section is divided into two parts: in the first part we provide some details for the construction of the ordinary product rules and in the second one the details for the extended product rules.

6.1. Coefficients of the ordinary product rules. Recalling that the fundamental Lagrange polynomials of $L_{m+1}(\tau, f)$ can be expressed as

$$l_{m+1,k}(\tau,x) = \lambda_{m+1,k}(\tau) \sum_{i=0}^{m} p_i(\tau,x_k) p_i(\tau,x), \quad k = 1,2,\dots,m+1,$$

the functions $\mathcal{D}^{(0)}_k(t)$ in (2.12) take the form

$$\mathcal{D}^{(0)}_k(t) = \lambda_{m+1,k}(\tau) \sum_{i=0}^{m} p_i(\tau,x_k) \mathcal{H}^w_0(p_i(\tau),t), \tag{6.1}$$

and $\mathcal{D}_k^{(1)}(t)$ in (4.5) the expression

$$\mathcal{D}_k^{(1)}(t) = \lambda_{m+1,k}(\tau) \sum_{i=0}^{m} p_i(\tau, x_k) \mathcal{H}_1^w(p_i(\tau), t). \tag{6.2}$$

As it is known, the main effort in the construction of the coefficients in (6.1) and (6.2) is due to the evaluation of the so-called modified moments, i.e., the functions

$$M_i^{(0)}(t) := \mathcal{H}_0^w(p_i(\tau), t) = \fint_{-1}^{1} \frac{p_i(\tau, x)}{x - t} w(x) dx, \quad i = 0, 1, \dots,$$
$$M_i^{(1)}(t) := \mathcal{H}_1^w(p_i(\tau), t) = \fint_{-1}^{1} \frac{p_i(\tau, x)}{(x - t)^2} w(x) dx, \quad i = 0, 1, \dots.$$

They can be computed by recurrence relations, based on the three-term recurrence for the sequence of orthonormal Jacobi polynomials $\{p_m(\tau)\}_m$, $\tau(x) = v^{\rho,\sigma}(x)$, $\rho, \sigma > -1$, $x \in (-1, 1)$,

$$\begin{cases} p_{-1}(\tau, x) = 0, \qquad p_0(\tau, x) = \frac{1}{\sqrt{\mu_0}}, \\ b_{i+1} p_{i+1}(\tau, x) = (x - a_i) p_i(\tau, x) - b_i p_{i-1}(\tau, x), \quad i = 0, 1, \dots, \end{cases}$$

where we denote the Euler's Beta function by B,

$$\mu_0 = \int_{-1}^{1} \tau(x)\, dx = 2^{\rho+\sigma+1} \mathrm{B}(1 + \rho, 1 + \sigma),$$

and

$$\begin{cases} a_i = \frac{\sigma^2 - \rho^2}{(2i+\rho+\sigma)(2i+\rho+\sigma+2)}\,, & i = 0, 1, \dots, \\ b_i = \sqrt{\frac{4i(i+\rho)(i+\sigma)(i+\rho+\sigma)}{(2i+\rho+\sigma-1)(2i+\rho+\sigma)^2(2i+\rho+\sigma+1)}}\,, & i = 0, 1, \dots. \end{cases}$$

Hence, we compute the sequences $\left\{M_i^{(0)}(t)\right\}_{i \geq 0}$ and $\left\{M_i^{(1)}(t)\right\}_{i \geq 0}$ by means of the recursions

$$\begin{cases} M_1^{(0)}(t) = \dfrac{1}{b_1}\left(c_0 + (t - a_0) M_0^{(0)}(t)\right), \\ M_{i+1}^{(0)}(t) = \dfrac{1}{b_{i+1}}\left(c_i + (t - a_i) M_i^{(0)}(t) - b_i M_{i-1}^{(0)}(t)\right), & i = 1, 2, \dots, \\ M_1^{(1)}(t) = \dfrac{1}{b_1}\left(M_0^{(0)}(t) + (t - a_0) M_0^{(1)}(t)\right), \\ M_{i+1}^{(1)}(t) = \dfrac{1}{b_{i+1}}\left(M_i^{(0)}(t) + (t - a_i) M_i^{(1)}(t) - b_i M_{i-1}^{(1)}(t)\right), & i = 1, 2, \dots, \end{cases}$$

where the starting moments are

$$M_0^{(0)}(t) = \frac{1}{\sqrt{\mu_0}} \fint_{-1}^{1} \frac{w(x)}{x - t} dx, \quad M_0^{(1)}(t) = \frac{1}{\sqrt{\mu_0}} \fint_{-1}^{1} \frac{w(x)}{(x - t)^2} dx$$

and

$$c_i = \int_{-1}^{1} p_i(\tau, x) w(x) dx, \quad i = 0, 1, \dots.$$

Note that in the case $w = \tau$ we have $c_i = 0$, for all $i \geq 1$. In the other cases, the c_i can be exactly computed by the $\lfloor \frac{m+1}{2} \rfloor$-point Gauss–Jacobi quadrature rule w.r.t. w.

In the case $w = \tau$, all the coefficients $\left\{\mathcal{D}_k^{(0)}\right\}_{k=1}^{m+1}$ can be computed in $3m^2+4m$ floating point arithmetic operations, requiring in addition the computation of the zeros and Christoffel numbers $\{x_k, \lambda_{m+1,k}(\tau)\}_{k=1}^{m+1}$ through the Golub–Welsh procedure. The same cost is required for computing $\left\{\mathcal{D}_k^{(1)}\right\}_{k=1}^{m+1}$. However, in the case that both rules (2.12) and (4.5) have to be computed, the overall algorithm can be performed in $3m^2 + 10m$ floating point arithmetic operations.

6.2. Coefficients of the extended product rules. The extended Lagrange polynomial $\mathcal{L}_{2m+1}(\tau, \overline{\tau}, f, x)$ can be rewritten in the following form

$$\begin{aligned}\mathcal{L}_{2m+1}(\tau, \overline{\tau}, f, x) &= p_m(\overline{\tau}, x) L_{m+1}\left(\tau, \frac{f}{p_m(\overline{\tau})}, x\right) + p_{m+1}(\tau, x) L_m\left(\overline{\tau}, \frac{f}{p_{m+1}(\tau)}, x\right)\\ &= p_m(\overline{\tau}, x)\sum_{k=1}^{m+1}\lambda_{m+1,k}(\tau)\sum_{i=0}^{m} p_i(\tau, x)p_i(\tau, x_k)\frac{f(x_k)}{p_m(\overline{\tau}, x_k)}\\ &\quad + p_{m+1}(\tau, x)\sum_{k=1}^{m}\lambda_{m,k}(\overline{\tau})\sum_{i=0}^{m-1} p_i(\overline{\tau}, x)p_i(\overline{\tau}, y_k)\frac{f(y_k)}{p_{m+1}(\tau, y_k)}.\end{aligned}$$

Hence, the coefficients in (3.2) take the form

$$\mathcal{A}_k^{(0)}(t) = \frac{\lambda_{m+1,k}(\tau)}{p_m(\overline{\tau}, x_k)}\sum_{i=0}^{m} p_i(\tau, x_k)\mathcal{H}_0^w(p_m(\overline{\tau})p_i(\tau), t), \tag{6.3}$$

$$\mathcal{B}_k^{(0)}(t) = \frac{\lambda_{m,k}(\overline{\tau})}{p_{m+1}(\tau, y_k)}\sum_{i=0}^{m-1} p_i(\overline{\tau}, y_k)\mathcal{H}_0^w(p_{m+1}(\tau)p_i(\overline{\tau}), t), \tag{6.4}$$

and those in (4.7)

$$\begin{aligned}\mathcal{A}_k^{(1)}(t) &= \frac{\lambda_{m+1,k}(\tau)}{p_m(\overline{\tau}, x_k)}\sum_{i=0}^{m} p_i(\tau, x_k)\mathcal{H}_1^w(p_m(\overline{\tau})p_i(\tau), t),\\ \mathcal{B}_k^{(1)}(t) &= \frac{\lambda_{m,k}(\overline{\tau})}{p_{m+1}(\tau, y_k)}\sum_{i=0}^{m-1} p_i(\overline{\tau}, y_k)\mathcal{H}_1^w(p_{m+1}(\tau)p_i(\overline{\tau}), t).\end{aligned}$$

Denoting the ordinary modified moments w.r.t. $\overline{\tau}$ by

$$\overline{M}_i^{(0)}(t) := \mathcal{H}_0^w(p_i(\overline{\tau}), t), \quad \overline{M}_i^{(1)}(t) := \mathcal{H}_1^w(p_i(\overline{\tau}), t), \quad i = 0, 1, \ldots,$$

and the "extended modified moments" (see [32]) by

$$\begin{aligned}M_{m,i}^{(0)}(t) &:= \mathcal{H}_0^w(p_m(\overline{\tau})p_i(\tau), t), & M_{m,i}^{(1)}(t) &:= \mathcal{H}_1^w(p_m(\overline{\tau})p_i(\tau), t), \quad i = 0, 1, \ldots,\\ M_{m+1,i}^{(0)}(t) &:= \mathcal{H}_0^w(p_{m+1}(\tau)p_i(\overline{\tau}), t), & M_{m+1,i}^{(1)}(t) &:= \mathcal{H}_1^w(p_{m+1}(\tau)p_i(\overline{\tau}), t), \quad i = 0, 1, \ldots,\end{aligned}$$

the sequences $\left\{M^{(0)}_{m,i}(t), M^{(0)}_{m+1,i}(t)\right\}_{i\geq 0}$ and $\left\{M^{(1)}_{m,i}(t), M^{(1)}_{m+1,i}(t)\right\}_{i\geq 0}$ are computed by means of the following recursion schemes

$$(6.5)\qquad \begin{cases} M^{(0)}_{m,1}(t) = \dfrac{1}{b_1}\left(d_{m,0} + (t-a_0)M^{(0)}_{m,0}(t)\right), \\ M^{(0)}_{m,i+1}(t) = \dfrac{1}{b_{i+1}}\left(d_{m,i} + (t-a_i)M^{(0)}_{m,i}(t) - b_i M^{(0)}_{m,i-1}(t)\right) \\ \hfill i=1,2,\dots, \\ M^{(0)}_{m+1,1}(t) = \dfrac{1}{b_1}\left(d_{m+1,0} + (t-a_0)M^{(0)}_{m+1,0}(t)\right), \\ M^{(0)}_{m+1,i+1}(t) = \dfrac{1}{b_{i+1}}\left(d_{m+1,i} + (t-a_i)M^{(0)}_{m+1,i}(t) - b_i M^{(0)}_{m+1,i-1}(t)\right), \\ \hfill i=1,2,\dots, \end{cases}$$

$$(6.6)\qquad \begin{cases} M^{(1)}_{m,1}(t) = \dfrac{1}{b_1}\left(M^{(0)}_{m,0}(t) + (t-a_0)M^{(1)}_{m,0}(t)\right), \\ M^{(1)}_{m,i+1}(t) = \dfrac{1}{b_{i+1}}\left(M^{(0)}_{m,i}(t) + (t-a_i)M^{(1)}_{m,i}(t) - b_i M^{(1)}_{m,i-1}(t)\right), \\ \hfill i=1,2,\dots, \\ M^{(1)}_{m+1,1}(t) = \dfrac{1}{b_1}\left(M^{(0)}_{m+1,0}(t) + (t-a_0)M^{(1)}_{m+1,0}(t)\right), \\ M^{(1)}_{m+1,i+1}(t) = \dfrac{1}{b_{i+1}}\left(M^{(0)}_{m+1,i}(t) + (t-a_i)M^{(1)}_{m+1,i}(t) - b_i M^{(1)}_{m+1,i-1}(t)\right), \\ \hfill i=1,2,\dots, \end{cases}$$

where the starting moments are

$$M^{(0)}_{m,0}(t) = \frac{1}{\sqrt{\mu_0}}\overline{M}^{(0)}_m(t), \quad M^{(0)}_{m+1,0}(t) = \frac{1}{\sqrt{\overline{\mu}_0}}M^{(0)}_{m+1}(t),$$
$$M^{(1)}_{m,0}(t) = \frac{1}{\sqrt{\mu_0}}\overline{M}^{(1)}_m(t), \quad M^{(1)}_{m+1,0}(t) = \frac{1}{\sqrt{\overline{\mu}_0}}M^{(1)}_{m+1}(t),$$

with $\overline{\mu}_0 = \int_{-1}^1 \overline{\tau}(x)dx = 2^{\rho+\sigma+3}B(2+\rho, 2+\sigma)$, and the quantities

$$d_{m,i} = \int_{-1}^1 p_m(\overline{\tau}, x)p_i(\tau, x)w(x)dx, \qquad i = 0,1,\dots,$$
$$d_{m+1,i} = \int_{-1}^1 p_{m+1}(\tau, x)p_i(\overline{\tau}, x)w(x)dx, \qquad i = 0,1,\dots,$$

are exactly evaluated by means of $(m+1)$-point Gauss–Jacobi quadrature rules w.r.t w.

REMARK 6.1. From the recurrence schemes (6.5) and (6.6) we observe that the generalized modified moments sequences $\left\{M^{(0)}_{h,i}(t)\right\}_{i\geq 0}$ and $\left\{M^{(1)}_{h,i}(t)\right\}_{i\geq 0}$, $h = m, m+1$, can be obtained starting from the ordinary modified moments sequences $\left\{M^{(0)}_i(t), \overline{M}^{(0)}_i(t)\right\}_{i\geq 0}$ and $\left\{M^{(1)}_i(t), \overline{M}^{(1)}_i(t)\right\}_{i\geq 0}$.

In the case $w = \tau$, excluding the computation of the zeros and Christoffel numbers, the coefficients $\{\mathcal{A}^{(0)}_k\}_{k=1}^{m+1}$ in (6.3) and the coefficients $\{\mathcal{B}^{(0)}_k\}_{k=1}^m$ in (6.4) can be computed in

ETNA
Kent State University and
Johann Radon Institute (RICAM)

$3m^2 + 7m$ floating point arithmetic operations, for a total amount of $6m^2 + 14m$ floating point arithmetic operations. However, in case the extended rule is combined with the ordinary rule (2.12) (Path 1), then the additional global effort is only due to the computation of $\{\mathcal{B}_k^{(0)}\}_{k=1}^m$. This means that the compound scheme of ordinary and extended product rules results in a save in the construction of two consecutive elements of the sequence $\{\hat{\mathcal{H}}_{0,n}^w(f)\}_n$ in (3.6) and require a total amount of $6m^2$ floating point arithmetic operations instead of $9m^2$ floating point arithmetic operations needed if computed independently of each other. Moreover, in the global construction of two consecutive elements, the number of function evaluations will be around $2m$ instead of $3m$. Very similar considerations can be done for the implementation of the compound scheme $\{\hat{\mathcal{H}}_{1,n}^w(f)\}_n$ in (4.13) (Path 2). We conclude by observing that in the case of the simultaneous approximation of $\mathcal{H}_0^w(f)$ and $\mathcal{H}_1^w(f)$, the compound scheme $\{\widehat{\mathbf{H}}_n^w(f)\}_n$ in (5.1) requires only $2m$ evaluations of f for any 4 elements of the sequence, instead of $6m$ evaluations necessary when they are computed independently. As a final remark, we are aware that the proposed procedure is more expensive than other known methods in the literature (see, for example, the method in [1, 25] based on B-splines for the computation of the Hilbert transform on $\mathbb{R}$) having complexity $\mathcal{O}(m \log m)$, but this higher cost "repays" the fact that we deal with the more general case of weighted transforms, other than providing a procedure that computes both Hilbert and Hadamard transforms at the same time.

7. Numerical tests. In this section, we present some numerical examples to test the accuracy of the developed methods and confirm the theoretical estimates introduced in the previous sections. In all the considered examples, the exact values of the Hilbert and Hadamard transforms are unknown, and consequently we assume as exact values the results achieved by the ordinary product rules (2.12) and (4.5) for $m = 1024$ fixed. In this context, we compute the relative errors

$$\mathcal{E}_{Hil}^{Ord}(f,t) = \left| \frac{e_{0,m+1}^w(f,t)}{\mathcal{H}_0^w(f,t)} \right|, \qquad \mathcal{E}_{Had}^{Ord}(f,t) = \left| \frac{e_{1,m+1}^w(f,t)}{\mathcal{H}_1^w(f,t)} \right|,$$
$$\mathcal{E}_{Hil}^{Mix}(f,t) = \left| \frac{\zeta_{0,n}^w(f,t)}{\mathcal{H}_0^w(f,t)} \right|, \qquad \mathcal{E}_{Had}^{Mix}(f,t) = \left| \frac{\zeta_{1,n}^w(f,t)}{\mathcal{H}_1^w(f,t)} \right|.$$

More precisely, in the first example we report the relative errors attained at three different points in the interval $(-1, 1)$, while in the remaining examples we display the maximum relative errors obtained over a sufficiently dense set $\mathbb{I}$ of points in $(-1, 1)$, namely

$$\mathcal{E}_{Hil}^{Ord}(f) = \max_{t_i \in \mathbb{I}} \left| \frac{e_{0,m+1}^w(f,t_i)}{\mathcal{H}_0^w(f,t_i)} \right|, \qquad \mathcal{E}_{Had}^{Ord}(f) = \max_{t_i \in \mathbb{I}} \left| \frac{e_{1,m+1}^w(f,t_i)}{\mathcal{H}_1^w(f,t_i)} \right|,$$
$$\mathcal{E}_{Hil}^{Mix}(f) = \max_{t_i \in \mathbb{I}} \left| \frac{\zeta_{0,n}^w(f,t_i)}{\mathcal{H}_0^w(f,t_i)} \right|, \qquad \mathcal{E}_{Had}^{Mix}(f) = \max_{t_i \in \mathbb{I}} \left| \frac{\zeta_{1,n}^w(f,t_i)}{\mathcal{H}_1^w(f,t_i)} \right|.$$

In each test, we compare the results of the introduced mixed schemes (3.6) and (4.13) with the corresponding sequences based on the ordinary rules (2.12) and (4.5), respectively. All the computations are performed in double precision using Mathematica 13 software installed on a MacBook Pro laptop under the MacOS operating system. Moreover, we point out that the computation of the modified moments was carried out in quadruple precision to overcome potential instability issues of the recursive schemes based on the three-term recurrence relation for orthonormal polynomials.

EXAMPLE 7.1. Let us consider the following Hilbert and Hadamard transforms

$$\mathcal{H}_0^w(f,t) = \fint_{-1}^1 \frac{(x^2+25)^{-1}\sqrt{1-x^2}}{x-t}\, dx,$$

$$\mathcal{H}_1^w(f,t) = \,=\!\!\!\!\!\!\int_{-1}^{1} \frac{(x^2+25)^{-1}\sqrt{1-x^2}}{(x-t)^2}\,dx.$$

Here $f(x) = \frac{1}{x^2+25}$ and $w(x) = v^{\frac{1}{2},\frac{1}{2}}(x)$. Hence, the choice of exponents $\rho = \sigma = -\frac{1}{2}$ guarantees the convergence of both the mixed schemes approximating $\mathcal{H}_0^w(f,t)$ and $\mathcal{H}_1^w(f,t)$. Tables 7.1 and 7.2 report the relative errors of our product rules for the numerical approximation of the Hilbert and Hadamard transforms at the points $t = -\frac{3}{4}, \frac{1}{3}, \frac{3}{5}$, respectively. Since the function f is smooth, machine precision eps is easily reached, in agreement with the theoretical expectations.

TABLE 7.1
Example 7.1: Relative errors attained for the approximation of $\mathcal{H}_0^w(f,t)$.

m	$\mathcal{E}_{Hil}^{Ord}(f,-\frac{3}{4})$	$\mathcal{E}_{Hil}^{Mix}(f,-\frac{3}{4})$	$\mathcal{E}_{Hil}^{Ord}(f,\frac{1}{3})$	$\mathcal{E}_{Hil}^{Mix}(f,\frac{1}{3})$	$\mathcal{E}_{Hil}^{Ord}(f,\frac{3}{5})$	$\mathcal{E}_{Hil}^{Mix}(f,\frac{3}{5})$
5	1.13e-04	1.13e-04	2.83e-04	2.83e-04	1.20e-05	1.20e-05
9	1.19e-08	4.80e-11	2.28e-09	9.44e-12	7.20e-09	1.70e-10
17	eps	eps	eps	eps	eps	eps
33	eps	eps	eps	eps	eps	eps

TABLE 7.2
Example 7.1: Relative errors attained for the approximation of $\mathcal{H}_1^w(f,t)$.

m	$\mathcal{E}_{Had}^{Ord}(f,-\frac{3}{4})$	$\mathcal{E}_{Had}^{Mix}(f,-\frac{3}{4})$	$\mathcal{E}_{Had}^{Ord}(f,\frac{1}{3})$	$\mathcal{E}_{Had}^{Mix}(f,\frac{1}{3})$	$\mathcal{E}_{Had}^{Ord}(f,\frac{3}{5})$	$\mathcal{E}_{Had}^{Mix}(f,\frac{3}{5})$
5	3.37e-04	3.37e-04	6.77e-05	6.77e-05	6.10e-04	6.10e-04
9	2.81e-08	4.06e-10	8.79e-08	2.13e-10	9.36e-08	4.77e-10
17	eps	eps	eps	eps	eps	eps
33	eps	eps	eps	eps	eps	eps

EXAMPLE 7.2. Let us consider the following integrals

$$\mathcal{H}_0^w(f,t) = \int_{-1}^{1} \frac{e^x}{x-t}\left(1-x^2\right)^{\frac{1}{10}}\,dx,$$
$$\mathcal{H}_1^w(f,t) = \,=\!\!\!\!\!\!\int_{-1}^{1} \frac{e^x}{(x-t)^2}\left(1-x^2\right)^{\frac{1}{10}}\,dx.$$

Table 7.3 displays the maximum relative errors achieved by the ordinary and the mixed

TABLE 7.3
Example 7.2: Maximum relative errors attained for the approximation of $\mathcal{H}_0^w(f,t)$ and $\mathcal{H}_1^w(f,t)$.

m	$\mathcal{E}_{Hil}^{Ord}(f)$	$\mathcal{E}_{Hil}^{Mix}(f)$	$\mathcal{E}_{Had}^{Ord}(f)$	$\mathcal{E}_{Had}^{Mix}(f)$
5	2.13e-02	2.13e-02	1.84e-01	1.84e-01
9	1.68e-06	1.23e-07	7.83e-06	3.32e-07
17	eps	eps	eps	eps
33	eps	eps	eps	eps

sequences over a sufficiently dense set of points in $(-1,1)$. In this case, we have set $\rho = \sigma = -\frac{9}{10}$ to guarantee the convergence of both schemes. Even in this case, since $f(x) = e^x$ is analytic, machine precision is soon obtained.

EXAMPLE 7.3. Let us consider the Hilbert and Hadamard transforms

$$\mathcal{H}_0^w(f,t) = \fint_{-1}^{1} \frac{\left|x-\frac{1}{2}\right|^{\frac{15}{2}}}{x-t}(1-x)^{\frac{1}{4}}(1+x)^{\frac{1}{5}}\,dx,$$

$$\mathcal{H}_1^w(f,t) = \fint_{-1}^{1} \frac{\left|x-\frac{1}{2}\right|^{\frac{15}{2}}}{(x-t)^2}(1-x)^{\frac{1}{4}}(1+x)^{\frac{1}{5}}\,dx.$$

In this case $f(x) = \left|x-\frac{1}{2}\right|^{\frac{15}{2}}$, $f' \in Z_{\frac{13}{2}}(w\varphi)$ with $w(x) = v^{\frac{1}{4},\frac{1}{5}}(x)$ and $\tau(x) = v^{\rho,\sigma}(x)$, $\rho = -\frac{4}{5}$, $\sigma = -\frac{3}{4}$. Therefore, by Theorems 3.2 and 4.7, we expect that a larger number of nodes is required to achieve machine precision. Tables 7.4 and 7.5 display the sequences

TABLE 7.4
Example 7.3: Numerical results attained for the approximation of $\mathcal{H}_0^w(f,0)$.

m	Ordinary Sequence	Mixed Scheme
5	-3.463209284706466	-3.463209284706466
9	-3.542038534516906	-3.542697359167085
17	-3.542213963916695	-3.542213963916695
33	-3.542213959968340	-3.542213958262041
65	-3.542213959998572	-3.542213959998572
129	-3.542213959998260	-3.542213959998261
257	-3.542213959998261	-3.542213959998261
513	-3.542213959998261	-3.542213959998261

TABLE 7.5
Example 7.3: Numerical results attained for the approximation of $\mathcal{H}_1^w(f,0)$.

m	Ordinary Sequence	Mixed Scheme
5	14.81929011986100	14.81929011986100
9	5.035501928348207	4.995659780499221
17	4.995711526977758	4.995711526977758
33	4.995713956658255	4.995713937864166
65	4.995713935556930	4.995713935556930
129	4.995713936073610	4.995713936070872
257	4.995713936070761	4.995713936070761
513	4.995713936070774	4.995713936070774

of approximants of $\mathcal{H}_0^w(f,t)$ and $\mathcal{H}_1^w(f,t)$ using the ordinary and mixed rules at the point $t = 0$, respectively. We note that in the mixed scheme, at every even step we gain at least one significant digit more w.r.t. its ordinary counterpart, thanks to the good properties of the extended rule. This behavior is confirmed by the maximum relative errors reported in Table 7.6.

8. Conclusions. In this paper, we introduced a mixed scheme of product integration rules for the simultaneous approximation of weighted Hilbert and Hadamard transforms of a given function f. This approach allowed us to avoid the computation of the derivatives of the density function f and to delay the instability of the recurrence relations when evaluating the modified moments for high values of m. Moreover, using always the same samples of f for both

TABLE 7.6
Example 7.3: Maximum relative errors attained for the approximation of $\mathcal{H}_0^w(f,t)$ *and* $\mathcal{H}_1^w(f,t)$.

m	$\mathcal{E}_{Hil}^{Ord}(f)$	$\mathcal{E}_{Hil}^{Mix}(f)$	$\mathcal{E}_{Had}^{Ord}(f)$	$\mathcal{E}_{Had}^{Mix}(f)$
5	6.56e-01	6.56e-01	9.44e+01	9.44e+01
9	3.46e-03	1.28e-03	9.97e-01	3.55e-01
17	1.32e-06	1.32e-06	1.42e-04	1.42e-04
33	1.52e-09	1.51e-10	1.64e-06	1.71e-07
65	6.66e-12	6.66e-12	1.11e-09	1.11e-09
129	2.80e-14	6.10e-15	1.41e-11	5.29e-12
257	eps	eps	1.74e-13	1.74e-13
513	eps	eps	eps	eps

transforms led to a significant reduction in the computing time. Regarding the computational cost, cheaper methods exist in the literature, and hence our compounded algorithm is more expensive in terms of required long operations. Nevertheless, this higher cost is justified by the fact that we explore the more general case of weighted transforms, other than providing a procedure that computes both Hilbert and Hadamard transforms at the same time.

Acknowledgments. The authors are grateful to the anonymous reviewers for carefully reading the manuscript and for their precise and helpful suggestions which allowed us to improve the work.

This research has been partially supported by GNCS-INdAM 2023 project “Approssimazione ed integrazione multivariata con applicazioni ad equazioni integrali”. Domenico Mezzanotte has been partially supported by the Spoke 1 “FutureHPC & BigData” of ICSC - Centro Nazionale di Ricerca in High-Performance Computing, Big Data and Quantum Computing, funded by European Union - NextGenerationEU. Donatella Occorsio has been partially supported by PRIN 2022 PNRR project no. P20229RMLB financed by the European Union - NextGenerationEU and by the Italian Ministry of University and Research (MUR). The authors are members of the Gruppo Nazionale Calcolo Scientifico-Istituto Nazionale di Alta Matematica (GNCS-INdAM) and of the SIMAI Activity Group ANA&A. This research has been accomplished within “Research ITalian network on Approximation” (RITA).

REFERENCES

[1] R. Bilato, O. Maj, and M. Brambilla, *An algorithm for fast Hilbert transform of real functions*, Adv. Comput. Math., 40 (2014), pp. 1159–1168.

[2] H. Boche and V. Pohl, *Limits of calculating the finite Hilbert transform from discrete samples*, Appl. Comput. Harmon. Anal., 46 (2019), pp. 66–93.

[3] M. R. Capobianco, G. Criscuolo, P. Junghanns, and U. Luther, *Uniform convergence of the collocation method for Prandtl’s integro-differential equation*, ANZIAM J., 42 (2000), pp. 151–168.

[4] M. R. Capobianco, G. Mastroianni, and M. G. Russo, *Pointwise and uniform approximation of the finite Hilbert transform*, in Approximation and Optimization Vol. 1, D. D. Stancu, G. Coman, W. W. Breckner, and P. Blaga, eds., Transilvania Press, Cluj-Napoca, 1997, pp. 45–66.

[5] G. Criscuolo, G. Mastroianni, and P. Nevai, *Mean convergence of derivatives of extended Lagrange interpolation with additional nodes*, Math. Nachr., 163 (1993), pp. 73–92.

[6] G. Criscuolo, G. Mastroianni, and D. Occorsio, *Convergence of extended Lagrange interpolation*, Math. Comp., 55 (1990), pp. 197–212.

[7] M. C. De Bonis and G. Mastroianni, *Direct methods for CSIE in weighted Zygmund spaces with uniform norm*, Riv. Math. Univ. Parma (N.S.), 2 (2011), pp. 29–55.

[8] M. C. De Bonis and D. Occorsio, *On the simultaneous approximation of a Hilbert transform and its derivatives on the real semiaxis*, Appl. Numer. Math., 114 (2017), pp. 132–153.

[9] M. C. De Bonis, D. Occorsio, and W. Themistoclakis, *Filtered interpolation for solving Prandtl’s integro-differential equations*, Numer. Algorithms, 88 (2021), pp. 679–709.

[10] K. DIETHELM, *Uniform convergence of optimal order quadrature rules for Cauchy principal value integrals*, J. Comput. Appl. Math., 56 (1994), pp. 321–329.

[11] Z. DITZIAN AND V. TOTIK, *Moduli of Smoothness*, Springer, New York, 1987.

[12] S. EHRICH AND G. MASTROIANNI, *Stieltjes polynomials and Lagrange interpolation*, Math. Comp., 66 (1997), pp. 311–331.

[13] F. FILBIR, D. OCCORSIO, AND W. THEMISTOCLAKIS, *Approximation of finite Hilbert and Hadamard transforms by using equally spaced nodes*, Mathematics, 8 (2020), Art. 542 (23 pages).

[14] T. HASEGAWA, *Uniform approximations to finite Hilbert transform and its derivative*, J. Comput. Appl. Math., 163 (2004), pp. 127–138.

[15] A. I. KALANDIYA, *Mathematical Methods of Two-Dimensional Elasticity*, Mir Publishers, Moscow, 1973.

[16] F. W. KING, *Hilbert Transforms I & II*, Cambridge University Press, Cambridge, 2009.

[17] D. S. LUBINSKY AND G. MASTROIANNI, *Mean convergence of extended Lagrange interpolation with Freud weights*, Acta Math. Hungar., 84 (1999), pp. 47–63.

[18] U. LUTHER AND M. G. RUSSO, *Boundedness of the Hilbert transformation in some weighted Besov type spaces*, Integral Equations Operator Theory, 36 (2000), pp. 220–240.

[19] B. N. MANDAL AND A. CHAKRABARTI, *Applied Singular Integral Equations*, CRC Press, Boca Raton, 2011.

[20] G. MASTROIANNI AND G. V. MILOVANOVIĆ, *Interpolation Processes Basic Theory and Applications*, Springer, Berlin, 2008.

[21] G. MASTROIANNI AND M. G. RUSSO, *Lagrange interpolation in some weighted uniform spaces*, Facta Univ. Ser. Math. Inform., 12 (1997), pp. 185–201.

[22] G. MASTROIANNI, M. G. RUSSO, AND W. THEMISTOCLAKIS, *Numerical methods for Cauchy singular integral equations in spaces of weighted continuous functions*, in Recent Advances in Operator Theory and its Applications, M. A. Kaashoek, S. Seatzu, and C. van der Mee, eds., Birkhäuser, Basel, 2005, pp. 311–336.

[23] D. MEZZANOTTE AND D. OCCORSIO, *Compounded product integration rules on* $(0, +\infty)$, Dolomites Res. Notes Approx., 15 (2022), pp. 78–92.

[24] D. MEZZANOTTE, D. OCCORSIO, AND M. G. RUSSO, *Combining Nyström methods for a fast solution of Fredholm integral equations of the second kind*, Mathematics, 9 (2021), Art. 2652 (18 pages).

[25] C. A. MICCHELLI, Y. XU, AND B. YU, *On computing with the Hilbert spline transform*, Adv. Comput. Math., 38 (2013), pp. 623–646.

[26] G. MONEGATO, *The numerical evaluation of one-dimensional Cauchy principal value integrals*, Computing, 29 (1982), pp. 337–354.

[27] ———, *Numerical evaluation of hypersingular integrals*, J. Comput. Appl. Math., 50 (1994), pp. 9–31.

[28] G. MONEGATO AND L. SCUDERI, *High order methods for weakly singular integral equations with nonsmooth input functions*, Math. Comp., 67 (1998), pp. 1493–1515.

[29] D. OCCORSIO, *Extended Lagrange interpolation in weighted uniform norm*, Appl. Math. Comput., 211 (2009), pp. 10–22.

[30] D. OCCORSIO AND M. G. RUSSO, *The* L^p*-weighted lagrange interpolation on Markov-Sonin zeros*, Acta Math. Hungar., 112 (2006), pp. 57–84.

[31] ———, *A new quadrature scheme based on an extended Lagrange interpolation process*, Appl. Numer. Math., 124 (2018), pp. 57–75.

[32] ———, *A mixed scheme of product integration rules in* $(-1, 1)$, Appl. Numer. Math., 149 (2020), pp. 113–123.

[33] ———, *A mixed collocation scheme for solving second kind Fredholm integral equations in* $[-1, 1]$, Electron. Trans. Numer. Anal., 54 (2021), pp. 443–459. https://etna.math.kent.edu/vol.54.2021/pp443-459.dir/pp443-459.pdf.

[34] D. OCCORSIO, M. G. RUSSO, AND W. THEMISTOCLAKIS, *Filtered integration rules for finite weighted Hilbert transforms*, J. Comput. Appl. Math., 410 (2022), Art. 114166 (19 pages).