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ON THE PERIODIC QUOTIENT SINGULAR VALUE DECOMPOSITION *

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Abstract. The periodic Schur decomposition has been generally seen as a tool to compute the eigenvalues of a product of matrices in a numerically sound way. In a recent technical report, it was shown that the periodic Schur decomposition may also be used to accurately compute the singular value decomposition (SVD) of a matrix. This was accomplished by reducing a periodic pencil that is associated with the standard normal equations to eigenvalue revealing form. If this technique is extended to the periodic QZ decomposition, then it is possible to compute the quotient singular value decomposition (QSVD) of a matrix pair. This technique may easily be extended further to a sequence of matrix pairs, thus computing the "periodic" QSVD.

Key words. singular value decomposition, periodic Schur decomposition, periodic QR algorithm, periodic QZ algorithm, QSVD, SVD.

AMS subject classifications. 15A18, 65F05, 65F15.

1. Introduction. In a recent technical report [8], a method was proposed by which to compute the singular value decomposition (SVD) of a matrix via the periodic Schur decomposition. This method applied the periodic QR algorithm [1] [6] to the matrices in the periodic linear system of equations

$$\begin{aligned} x_2 &= Ax_1, \\ x_3 &= A^T x_2, \end{aligned} \tag{1.1}$$

thereby computing the orthogonal matrices and the non-negative definite diagonal matrix that comprise the SVD of the matrix A. While numerically not as efficient as the standard SVD algorithm, it was shown that the periodic QR algorithm could compute the SVD of a matrix with comparable accuracy. In fact, if the standard periodic QR algorithm is modified to perform a single (real) implicit shift instead of the standard double (complex-conjugate) implicit shift, the result is essentially the Golub–Kahan SVD algorithm [3] applied to both A and A^T, doubling the normal operations count.

The technical report further showed that it was possible to compute the singular values of a matrix operator defined by a "quotient" of matrices. Here we speak of the operator $\Xi : X_1 \to X_2$, $X_1 \subseteq \mathbb{R}^n$ $X_2 \subseteq \mathbb{R}^n$ defined by the equations

$$Ex_2 = Fx_1, \tag{1.2}$$

where $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n}$. This method accurately extracted the singular values of Ξ by examining the eigenvalues of a related operator.

In this paper, we develop the idea further by relating the results to the quotient singular value decomposition (QSVD) [11] [13]. The QSVD reduces the matrices E and F in (1.2) to (diagonal) singular value revealing form. More precisely, via the

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QSVD it is possible to compute a non–orthogonal matrix X, non–negative definite diagonal matrices Φ and Θ, and orthogonal matrices U and V such that

$$\begin{array}{rcl} X^TEU & = & \Phi, \\ X^TFV & = & \Theta. \end{array} \tag{1.3}$$

Furthermore, if E is nonsingular, the ordering of the diagonal elements of Φ and Θ may be chosen to reflect the ordering of the singular values in the SVD of the explicitly formed operator $\hat{\Xi} = E^{-1}F$, *i.e.,*

$$\frac{\theta_i}{\phi_i} \geq \frac{\theta_{i+1}}{\phi_{i+1}}. \tag{1.4}$$

In this paper, as in [7], we will show that it is possible to compute the matrices X, Φ, Θ, U and V via the periodic QZ decomposition. Further, we will show how this technique may be extended to compute the periodic QSVD of a sequence of matrix pairs. First, however, we review from [9] the periodic QZ decomposition.

2. The Periodic QZ Decomposition. The periodic QZ decomposition [1] [6] [9] simultaneously triangularizes by orthogonal equivalences a sequence of matrices associated with the generalized periodic eigenvalue problem. Consider the linear algebraic map $\Pi_k : \mathrm{X_k} \rightarrow \mathrm{X_{k+p}}$ where $\mathrm{X_k} \subseteq \mathbb{R}^n$ and $\mathrm{X_{k+p}} \subseteq \mathbb{R}^n$ and $x_{k+p} = \Pi_k x_k$ is defined by the set of linear equations

$$\begin{array}{rcl} E_k x_{k+1} & = & F_k x_k, \\ E_{k+1} x_{k+2} & = & F_{k+1} x_{k+1}, \\ \vdots & & \vdots \\ E_{k+p-1} x_{k+p} & = & F_{k+p-1} x_{k+p-1}. \end{array} \tag{2.1}$$

It is possible to operate on the matrices E_k and F_k with orthogonal matrices Q_k and Z_k so as to triangularize the system associated with the generalized periodic eigenvalue problem

$$\begin{array}{rcl} E_k x_{k+1} & = & F_k x_k, \\ E_{k+1} x_{k+2} & = & F_{k+1} x_{k+1}, \\ \vdots & & \vdots \\ E_{k+p-2} x_{k+p-1} & = & F_{k+p-2} x_{k+p-2}, \\ \lambda E_{k+p-1} x_k & = & F_{k+p-1} x_{k+p-1}. \end{array} \tag{2.2}$$

More precisely, the following products

$$\begin{array}{rcl} Q_k^T E_k Z_{k+1} & , & Q_k^T F_k Z_k, \\ Q_{k+1}^T E_{k+1} Z_{k+2} & , & Q_{k+1}^T F_{k+1} Z_{k+1}, \\ \vdots & & \vdots \\ Q_{k+p-2}^T E_{k+p-2} Z_{k+p-1} & , & Q_{k+p-2}^T F_{k+p-2} Z_{k+p-2}, \\ Q_{k+p-1}^T E_{k+p-1} Z_{k+p} & , & Q_{k+p-1}^T F_{k+p-1} Z_{k+p-1}, \end{array} \tag{2.3}$$

have the properties that $Q_k^T E_k Z_{k+1}$ is a quasi–upper triangular matrix H_k, and the remaining matrices $Q_{k+\ell}^T E_{k+\ell} Z_{k+\ell+1}$ and $Q_{k+\ell}^T F_{k+\ell} Z_{k+\ell}$ are upper triangular matrices $H_{k+\ell}$ and $T_{k+\ell}$, respectively, with $Z_{k+p} = Z_k$. This permits the periodic pencil in (2.2) to be expressed as the triangularized periodic pencil that is associated with the linear algebraic map $\Gamma_k : \mathrm{Y_k} \rightarrow \mathrm{Y_{k+p}}$ where $\mathrm{Y_k} \subseteq \mathbb{R}^n$ and $\mathrm{Y_{k+p}} \subseteq \mathbb{R}^n$, and where $y_{k+p} = \Gamma_k y_k$, as follows:

$$
\begin{aligned}
H_k y_{k+1} &= T_k y_k, \\
H_{k+1} y_{k+2} &= T_{k+1} y_{k+1}, \\
&\vdots \\
H_{k+p-2} y_{k+p-1} &= T_{k+p-2} y_{k+p-2}, \\
\mu H_{k+p-1} y_k &= T_{k+p-1} y_{k+p-1},
\end{aligned}
\tag{2.4}
$$

with

$$
x_\ell = Z_\ell^T y_\ell. \tag{2.5}
$$

Clearly, since the systems in (2.2) and (2.4) are (orthogonally) equivalent, their eigenvalues are equal, *i.e.,* $\Lambda(\Pi_\ell) = \Lambda(\Gamma_\ell)$. Further, the triangularized system of linear equations in (2.4) is written in an eigenvalue revealing form; this allows the eigenvalues of the period map defined by (2.4) to be related to the diagonal elements of its constituent matrices. Proceeding, let t_{ijk} denote the ijth element of the matrix T_k, and similarly let h_{ijk} denote the ijth element of the matrix H_k. Also, let the matrices $\tilde{T}_{ik}$ and $\tilde{H}_{ik}$ be 2×2 matrices centered on the diagonals of T_k and H_k respectively, with the upper left–hand entries being t_{iik} and h_{iik}, if the system defined by $\tilde{T}_{ik}$ and $\tilde{H}_{ik}$ is associated with a 2×2 bulge in the quasi–upper triangular H_1.

Then the eigenvalues of the operator Γ_ℓ may be written as

$$
\Lambda(\Gamma_\ell) = \left\{ \begin{array}{c}
\left\{ \lambda_i = \prod_{j=1}^{p} t_{iij}/h_{iij} \text{ if } h_{iij} \neq 0 \text{ , } \lambda_i \in \mathbb{R} \right\} \\
\left\{ \begin{array}{c} \{\lambda_i, \lambda_{i+1}\} = \Lambda(\tilde{H}_{ip}^{-1}\tilde{T}_{ip} \cdots \tilde{H}_{i1}^{-1}\tilde{T}_{i1}) \\ \lambda_i = \bar{\lambda}_{i+1} \in \mathbb{C} \end{array} \right\} \\
\{ \ \lambda_i = \infty \text{ if } t_{iij} \neq h_{iij} = 0 \ \} \\
\{ \ \lambda_i \in \mathbb{C} \text{ if } t_{iij} = h_{iij} = 0 \ \}.
\end{array} \right\}, \quad \begin{array}{c} i = \{1, \ldots, n\} \\ \ell = \{1, \ldots, p\}. \end{array}
\tag{2.6}
$$

From the definition above, the eigenvalues fall into four categories. Since it is necessary to refer to these categories throughout the paper, we provide a table of nomenclature in the following definition.

DEFINITION 1. Let E and F in (1.2) be the scalars $E = \alpha$ and $F = \beta$. Then the adjectives[1] *enomorphic* and *medotropic* may be used to describe the four categories of possible eigenvalues as shown in Table 1.

3. Quotient Singular Value Decomposition. In this section, we show how the QSVD may be related to the periodic QZ decomposition. We start by restricting our attention to the case when E and F in (1.2) are nonsingular. In such a case, it is

[1] We introduce here the neologisms *enomorphic* and *medotropic*. These terms are derived from Greek meaning "of the form of one", and "changed by zero", respectively. The former term reflects the fact that an enomorphic number is equivalent to one for a finite scaling. The latter term emphasizes the fact that a medotropic number has a zero in its rational representation.

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TABLE 2.1
Table of Eigenvalue Categories

α	β	λ	CATEGORIES		
$\alpha \neq 0$	$\beta \neq 0$	$\frac{\beta}{\alpha}$	finite,	determinate,	enomorphic
$\alpha \neq 0$	$\beta = 0$	0	finite,	determinate,	medotropic
$\alpha = 0$	$\beta \neq 0$	∞	non-finite,	determinate,	medotropic
$\alpha = 0$	$\beta = 0$	$\mathbb{C}$	non-finite,	indeterminate,	medotropic

possible to write the matrices

$$\begin{aligned} \hat{\Delta}_1 &= F^T E^{-T} E^{-1} F, \\ \hat{\Delta}_2 &= F F^T E^{-T} E^{-1}, \\ \hat{\Delta}_3 &= E^{-1} F F^T E^{-T}, \\ \hat{\Delta}_4 &= E^{-T} E^{-1} F F^T = \hat{\Delta}_2^T, \end{aligned} \tag{3.1}$$

where the matrices $\hat{\Delta}_i$ are nonsingular. It turns out that the eigenvectors of these matrices are closely related to the QSVD, which we demonstrate formally in the following lemmas and theorems.

LEMMA 3.1. *Let $\hat{\Delta}_1$, $\hat{\Delta}_2$, $\hat{\Delta}_3$, and $\hat{\Delta}_4$ be defined by (3.1), with $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n}$ nonsingular. There exist matrices M_1, M_2, M_3, and M_4 such that the following sets of relations hold:*

$$\begin{aligned} S &= M_1^{-1} \hat{\Delta}_1 M_1, \\ &= M_2^{-1} \hat{\Delta}_2 M_2, \\ &= M_3^{-1} \hat{\Delta}_3 M_3, \\ &= M_4^{-1} \hat{\Delta}_4 M_4, \end{aligned} \tag{3.2}$$

where S is real, diagonal, positive definite and has arbitrarily ordered diagonal elements and

$$\begin{aligned} D_1 &= M_2^{-1} F \; M_1, \\ D_2 &= M_3^{-1} E^{-1} M_2, \\ D_3 &= M_4^{-1} E^{-T} M_3, \\ D_4 &= M_1^{-1} F^T \; M_4. \end{aligned} \tag{3.3}$$

where the matrices D_i are diagonal and positive definite.

Proof. Since $\hat{\Delta}_1$ is symmetric, we can find an orthogonal M_1 which diagonalizes $\hat{\Delta}_1$ in (3.1) and (3.2) with arbitrarily ordered eigenvalues, as above. Let

$$F * M_1 = \tilde{M}_2 \tilde{D}_1, \tag{3.4}$$

where the matrix $\tilde{D}_1$ is chosen to be an arbitrary positive definite diagonal matrix and where the matrix $\tilde{M}_2$ is nonsingular. Inserting the identity matrix $\tilde{M}_2 \tilde{M}_2^{-1}$ between E^{-1} and F in the equation for S in (3.1) yields the expression

$$\begin{aligned} M_1^{-1} F^T E^{-T} E^{-1} \tilde{M}_2 \quad \tilde{M}_2^{-1} F M_1 &= S, \\ \underbrace{M_1^{-1} F^T E^{-T} E^{-1} \tilde{M}_2}_{\square} \quad \underbrace{\tilde{D}_1}_{\diagdown} &= \underbrace{S}_{\diagdown}, \end{aligned} \tag{3.5}$$

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Since S and $\tilde{D}_1$ are diagonal, $M_1^{-1}F^TE^{-T}E^{-1}\tilde{M}_2$ must be diagonal as well. Noting that diagonal matrices commute, we write

$$\tilde{M}_2^{-1}FM_1M_1^{-1}F^TE^{-T}E^{-1}\tilde{M}_2 \quad = \quad S. \tag{3.6}$$

This implies that it is possible to write $\tilde{M}_2 = M_2$ and $\tilde{D}_1 = D_1$. Repeating this procedure for all of the matrices D_i in (3.3) completes the proof. □

In Lemma 3.1 the matrices M_2^{-1} and M_1 diagonalize F; however, we would like to diagonalize this matrix without forming an inverse. This indeed is possible, as we demonstrate in the next lemma.

LEMMA 3.2. *Suppose M_2 and M_4 diagonalize $\hat{\Delta}_2$ and $\hat{\Delta}_4$, respectively, with the same eigenvalue ordering, as in (3.2). Then M_4^{-T} and M_2^{-T} diagonalize $\hat{\Delta}_2$ and $\hat{\Delta}_4$, respectively. Further, if the eigenvalues of the matrices $\hat{\Delta}_i$ are distinct, the matrices M_2 and M_4 may be related by the equation*

$$M_2^{-T} = M_4L, \tag{3.7}$$

with L diagonal.

Proof. Since M_2 diagonalizes $\hat{\Delta}_2$, M_4 diagonalizes $\hat{\Delta}_4$, and $S = S^T$,

$$\begin{aligned} S &= M_4^{-1}E^{-T}E^{-1}FF^TM_4, \\ &= S^T, \\ &= M_2^TE^{-T}E^{-1}FF^TM_2^{-T}. \end{aligned}$$

With the eigenvalues of S distinct, columns of the the matrices M_2^{-T} and M_4 are right eigenvectors associated with the eigenvalues of S. Since eigenvectors of distinct eigenvalues are equal up to a scaling, then $M_2^{-T} = M_4L$ with L nonsingular and diagonal. □

The lemmas above demonstrate a number of remarkable properties of the sets of matrices M_i and $\hat{\Delta}_i$, namely, that the matrices M_i diagonalize the matrices $\hat{\Delta}_i$, that the matrices M_i diagonalize the matrices E, E^T, F and F^T individually, and that under certain conditions *the inverses of all of the M_i's may be computed by appropriately weighting the columns of related M_j's.* These properties allow us to demonstrate our main result.

THEOREM 3.3. *Let the matrices $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n}$ be nonsingular with the eigenvalues of the matrices Δ_i in (3.1) distinct. Further, let M_1, M_2, M_3, and M_4 be defined as in (3.2) and (3.3). There exists a diagonal signature matrix P with*

$$P = \begin{bmatrix} \pm1 & & & \\ & \pm1 & & \\ & & \ddots & \\ & & & \pm1 \end{bmatrix}, \tag{3.8}$$

such that U, X, V Φ and Θ associated with the QSVD (1.3) of Ξ (1.2) may be written as

$$\begin{aligned} U &= M_3, \\ X &= M_4P, \\ V &= M_1, \\ \Phi &= PM_4^TEM_3, \\ \Theta &= PM_4^TFM_1. \end{aligned} \tag{3.9}$$

Proof. Since $\hat{\Delta}_1$ is symmetric, it is possible to choose M_1 to be orthogonal. Further, the matrices D_1, D_2, and D_3 may be chosen such that the columns of M_2, M_3 and M_4 are unit normalized. The fact that the columns of M_3 are unit normalized and that the matrix $\hat{\Delta}_3$ is symmetric imply that M_3 is orthogonal. Thus,

$$\begin{aligned} D_1 &= M_2^{-1} F M_1, \\ D_2^{-1} &= M_2^{-1} E M_3, \end{aligned}$$

with M_1 and M_3 are orthogonal, and with D_1 and D_2 diagonal and positive definite. Since the singular values of Ξ are distinct, then via Lemma 3.2,

$$\begin{aligned} LD_1 &= M_4^T F M_1, \\ LD_2^{-1} &= M_4^T E M_3. \end{aligned}$$

This implies that $M_4^T F M_1$ and $M_4^T E M_3$ are diagonal. Setting $P = \text{sgn}(L)$ completes the proof. ▯

Remark 1. In the proof of the above theorem, the condition that the singular values appearing on the diagonal of Φ and Θ be ordered as in (1.4) was not imposed. However, any ordering of the singular values in the QSVD may be imposed without loss of generality. $\diamond$

In general, we would like to be able to prove Theorem 3.3 when E and F are not restricted to be nonsingular with the singular values of Ξ in (1.2) distinct. Even in the case where E and F are nonsingular, the procedure outlined in the proof of Lemma 3.1 is undesirable from a numerical point of view since it requires forming the matrices $\hat{\Delta}_i$ explicitly. Fortunately, the periodic QZ decomposition allows us to view the matrices $\hat{\Delta}_i$ as period–maps Δ_i of certain related periodic systems, thereby eliminating the need for inversions and multiplications by non-orthogonal matrices. Three key properties of the periodic QZ decomposition are employed to generalize Theorem 3.3 to the case where E and F are singular. The first property allows the reduction of the matrices comprising a periodic system of equations to quasi–triangular (eigenvalue revealing) form without forming the period–map explicitly. Parenthetically, this property implies that the periodic QZ decomposition triangularizes each of the period–maps Δ_i separately. The second property allows the eigenvalues appearing along the diagonals of the triangularized period–map produced by the periodic QZ decomposition to be ordered arbitrarily. The third property asserts that the first columns (rows) of the matrices Z_i (Q_i^T) of the periodic QZ decomposition are right (left) eigenvectors of the period–maps Δ_i. By rotating each of the eigenvalues of the system triangularized by the periodic QZ decomposition in turn to the upper left–hand corner, it is possible to assemble the eigenvector matrices M_i associated with the $\hat{\Delta}_i$'s in (3.2) and (3.3) and thereby to compute all of the matrices related to the QSVD. This may be done irrespective of the rank of E and F. Proceeding, we propose a modified form for the periodic QZ decomposition, which is more closely related to the QSVD than the standard periodic QZ decomposition.

Lemma 3.4. *Let Q and Z be orthogonal matrices with the product QZ being upper-triangular. Then $QZ = P$ where P is a diagonal signature matrix.*

Proof. By definition, P is orthogonal which implies $P^T P = I$. Since P is upper triangular as well, $P^T P = I$ implies that P is diagonal. Finally, since the modulus of the eigenvalues of an orthogonal matrix must be one and since P is real, then P is a diagonal matrix with $p_{ii} = \pm 1$ ▯

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Next, the operator Δ_1 is defined in terms of a periodic system. Eventually it will be used to compute the QSVD of Ξ. The next lemma shows that its periodic QZ decomposition has a special form.

LEMMA 3.5. *Let the operator* $\Delta_1 \ : \ \mathrm{X}_1 \rightarrow \mathrm{X}_4$ *be the operator defined by the equations*

$$\begin{aligned} Ex_2 &= Fx_1, \\ E^T x_3 &= x_2, \\ x_4 &= F^T x_3, \end{aligned} \tag{3.10}$$

with $E \in \mathbb{R}^{n\times n}$ *and* $F \in \mathbb{R}^{n\times n}$ *nonsingular. There exist orthogonal matrices* Q, U, V, *and* W *such that*

$$\begin{array}{rclcrcl} Q^T E U &=& H_1 &,& Q^T F V &=& T_1, \\ U^T E^T W &=& H_2 &,& V^T F^T W &=& T_3. \end{array} \tag{3.11}$$

where H_1, H_2, T_1, *and* T_3 *are upper triangular.*

Proof. Via the periodic Schur decomposition, there exist orthogonal matrices Q_1, Q_2, Q_3, Z_1, Z_2, and Z_3 such that

$$\begin{array}{rclcrcl} Q_1^T E Z_2 &=& H_1 &,& Q_1^T F Z_1 &=& T_1, \\ Q_2^T E^T Z_3 &=& H_2 &,& Q_2^T Z_2 &=& T_2, \\ Q_3^T Z_1 &=& H_3 &,& Q_3^T F^T Z_3 &=& T_3, \end{array} \tag{3.12}$$

where H_2, H_3, T_1, T_2, and T_3 are upper–triangular. Since the operator $\Delta_1 = \hat{\Delta}_1$ is symmetric, the eigenvalues of the operator will be real and therefore H_1 is upper-triangular as well. By Lemma 3.4, $Q_2^T Z_2 = P_1$ and $Q_3^T Z_1 = P_2$ where the diagonal elements of P_1 and P_2 are ± 1. This implies that the columns of Q_2 and Z_2 differ only by sign. This is equally the case for Q_3 and Z_1. By setting $Q = Q_1$, $U = Q_2 = Z_2 P_1$, $V = Q_3 = Z_1 P_2$, and $W = Z_3$, we complete the proof. ▯

REMARK 2. Since the operator $\Delta_1 = \hat{\Delta}_1$ with E and F nonsingular, the orthogonal matrix V resulting from the (modified) periodic QZ decomposition that diagonalizes the operator is the matrix V associated with the QSVD in (1.3), modulo the sign of the columns. Similarly, the orthogonal matrix U resulting from the periodic QZ decomposition is the matrix U associated with the same QSVD. ◇

In the following lemma, we show how the remaining matrices X, Φ and Θ may be computed via the periodic QZ decomposition.

LEMMA 3.6. *Let* Ξ *be defined as in (1.2), with* $E \in \mathbb{R}^{n\times n}$ *and* $F \in \mathbb{R}^{n\times n}$ *nonsingular. Further, let the eigenvalues of the matrices* Δ_i *in (3.1) be distinct. Then the matrices* X, U, V, Φ, *and* Θ *associated with the QSVD in (1.3) may be computed via the (modified) QZ decomposition in (3.11).*

Proof. Via Lemma 3.5, it is possible to find orthogonal matrices $\bar{Q}$, $\bar{U}$, $\bar{V}$, and $\bar{W}$ that diagonalize Δ_1 with a particular eigenvalue ordering. Let λ_i be the eigenvalue associated with the ith column of $\bar{V}$. Let the matrix $V_{[j]}$ be the orthogonal matrix that results from interchanging the first column of $\bar{V}$ and the jth column. Since the operator Δ_1 is diagonalized by the matrix $\bar{V}$, $V_{[j]}$ also diagonalizes the operator Δ_1 with the jth eigenvalue in the upper left–hand corner. Now, define the matrices $Q_{[j]}$, $U_{[j]}$, and $W_{[j]}$ as the matrices, along with $V_{[j]}$, that triangularize matrices E, F, E^T, and F^T in Δ_1 with the jth eigenvalue in the upper left–hand corner. Also define the vector $\tilde{x}_{[j]}$ to be the first column of $W_{[j]}$. Since Δ_4 is triangularized by $W_{[j]}$ with the

jth eigenvalue in the upper left–hand corner, then $\tilde{x}_{[j]}$ is a right eigenvector of Δ_4 and, via Lemma 3.2, $\tilde{x}_{[j]}^T$ is a left eigenvector of Δ_2. Thus, the matrix $\tilde{X} = [\tilde{x}_{[1]}, \ldots, \tilde{x}_{[n]}]$ diagonalizes Δ_4 from the right and $\tilde{X}^T$ diagonalizes Δ_2 from the left with the same eigenvalue ordering. This implies, via Lemma 3.1, that $\tilde{X}^T$ along with matrices U and V diagonalize E and F, albeit with the diagonal elements not necessarily positive. There exists, however, a diagonal signature matrix P as in (3.8), a matrix $X = P\tilde{X}$, and orthogonal matrices U and V such that X^T, U, and V diagonalize E and F with positive diagonal elements. □

In the proof above, we rely on the fact that the matrices E and F are nonsingular and with the singular values distinct. In the following theorem, we provide a proof for the existence of the QSVD with E and F possibly singular.

THEOREM 3.7. *Let $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n}$ and the operator Ξ be defined as in (1.2). There exist a non–orthogonal matrix X, non–negative definite diagonal matrices Φ and Θ, and orthogonal matrices U and V such that*

$$\begin{array}{rcl} X^T E U & = & \Phi, \\ X^T F V & = & \Theta. \end{array} \tag{3.13}$$

Proof. Let $\{E_{(k)}\}$ and $\{F_{(k)}\}$ be a sequence of nonsingular $n \times n$ matrices that converge pointwise to E and F respectively with the eigenvalues of the associated matrix $\hat{\Delta}_{(k)}$ distinct. Let the matrices $Q_{(k)}$, $U_{(k)}$, $V_{(k)}$, $W_{(k)}$, $H_{1(k)}$, $H_{2(k)}$, $T_{1(k)}$, and $T_{2(k)}$ be the matrices produced by the (modified) periodic QZ decomposition in (3.11). Further, let $X_{(k)}$, $\Phi_{(k)}$, and $\Theta_{(k)}$ be produced by the procedure in the proof in Lemma 3.6. For any given $E_{(k)}$ and $F_{(k)}$, the periodic QZ decomposition produces bounded $Q_{(k)}$, $U_{(k)}$, $V_{(k)}$, $W_{(k)}$. The boundedness of $Q_{(k)}$, $U_{(k)}$, $V_{(k)}$, $W_{(k)}$, $E_{(k)}$, and $F_{(k)}$ clearly imply the boundedness of $X_{(k)}$, $\Phi_{(k)}$, and $\Theta_{(k)}$. Finally, since these matrices are all elements of a compact metric space, the Bolzano-Weierstrass Theorem ensures that the bounded sequence $\{(U_{(k)}, X_{(k)}, V_{(k)}, \Phi_{(k)}, \Theta_{(k)})\}$ has a converging subsequence, k_i, *i.e.,*

$$\lim_{k_i\to\infty} \{(U_{(k_i}, X_{(k_i)}, V_{(k_i)}, \Phi_{(k_i)}, \Theta_{(k_i)})\} = (U, X, V, \Phi, \Theta).$$

In the limit, the matrices U and V remain orthogonal and the matrices Φ and Δ are non–negative definite, and therefore are matrices associated with the QSVD in (1.3). □

REMARK 3. This proof of the existence of the QSVD for E and F singular requires the use of a converging subsequence, which is impractical from a computational point of view. If the requirement that Φ and Θ is diagonal is relaxed to be block upper triangular, where the non-diagonal blocks correspond to the non-distinct singular values, then the matrices resulting from the procedure in the proof in Lemma 3.6 suffices. In such a case, this algorithm requires approximately $162n^3$ flops, assuming 2 implicit QZ steps per eigenvalue. This compares with approximately $52n^3$ flops for the standard QSVD algorithm [13]. ◇

REMARK 4. In [12] and [13], Van Loan suggests the use of the VZ algorithm [12] applied to the pencil

$$E^T E - \lambda F^T F, \tag{3.14}$$

to compute the singular values of Ξ in (1.2). This idea is similar to the idea of this paper in a number of ways. First, the matrix pencil in (3.14) is closely related to

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the operator Δ_4. Second, the VZ decomposition is nearly identical to the modified periodic QZ decomposition proposed in Lemma 3.5. To our knowledge, however, the eludication of the relationships between the eigenvectors of the operators Δ_i and the matrices U, X, and V and the method of extracting the eigenvectors from the orthogonal matrices that comprise the modified QZ decomposition is novel to this paper. $\diamond$

4. Periodic Quotient Singular Value Decomposition. In this section, we examine the operator $\Pi_1 : X_1 \to X_{p+1}$, $X_1 \subseteq \mathbb{R}^n$ $X_{p+1} \subseteq \mathbb{R}^n$, where Π_1 is defined by the equations

$$\begin{array}{rcl} E_1 x_2 & = & F_1 x_1, \\ \vdots & = & \vdots \\ E_p x_{p+1} & = & F_p x_p, \end{array} \tag{4.1}$$

with $E_i \in \mathbb{R}^{n\times n}$ and $F_i \in \mathbb{R}^{n\times n}$. In this section, we intend to show that it is possible to implicitly reduce the matrices comprising the operator Π_1 to diagonal form as was the case for the non–periodic operator Ξ in (1.2). As in the previous section, we restrict our attention first to the case where the matrices E_i and F_i are nonsingular. Proceeding, we write definitions for the matrices $\hat{\Gamma}_i$ for $i = 1, \ldots, 4p$, where the matrices $\hat{\Gamma}_i$ are analogous to matrices $\hat{\Delta}_i$ in the non–periodic case

$$\begin{array}{lclllll} \hat{\Gamma}_1 & = & F_1^T \;\; E_1^{-T} F_2^T \;\; E_2^{-T} & \cdots & F_p^T E_p^{-T} E_p^{-1} F_p E_{p-1}^{-1} F_{p-1} & \cdots & E_1^{-1} F_1, \\ \hat{\Gamma}_2 & = & F_1 \;\; F_1^T \;\; E_1^{-T} F_2^T & \cdots & F_p^T E_p^{-T} E_p^{-1} F_p E_{p-1}^{-1} F_{p-1} & \cdots & F_2 \;\; E_1^{-1}, \\ \vdots & & & \vdots & & \vdots & \\ \hat{\Gamma}_{4p} & = & E_1^{-T} F_2^T \;\; E_2^{-T} F_3^T & \cdots & F_p^T E_p^{-T} E_p^{-1} F_p E_{p-1}^{-1} F_{p-1} & \cdots & F_1 \;\; F_1^T. \end{array} \tag{4.2}$$

The matrices $\hat{\Gamma}_i$, defined as above, provide a mechanism by which the existence of a periodic QSVD may be proved.

LEMMA 4.1. *Let $\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots, \hat{\Gamma}_{4p}$ be defined by (4.2), with $E_i \in \mathbb{R}^{n\times n}$ and $F_i \in \mathbb{R}^{n\times n}$ nonsingular for $i = 1, \ldots, p$. There exist matrices M_i for $i = 1, \ldots, 4p$ such that the following sets of relations hold:*

$$\begin{array}{rcl} S & = & M_1^{-1} \hat{\Gamma}_1 \;\; M_1, \\ & = & M_2^{-1} \hat{\Gamma}_2 \;\; M_2, \\ \vdots & & \vdots \\ & = & M_{4p}^{-1} \hat{\Gamma}_{4p} M_{4p}, \end{array} \tag{4.3}$$

where S is real, diagonal, positive definite and

$$\begin{array}{lcl} D_1 & = & M_2^{-1} F \quad M_1, \\ D_2 & = & M_3^{-1} E^{-1} M_2, \\ \vdots & & \vdots \\ D_{4p-1} & = & M_{4p}^{-1} E^{-T} M_{4p-1}, \\ D_{4p} & = & M_1^{-1} F^T \;\; M_{4p}. \end{array} \tag{4.4}$$

where the matrices D_i are diagonal and positive definite.

Proof. The proof is trivial extension of the proof of Lemma 3.1, with F_1 substituting for F in (3.4), and with the $\hat{\Gamma}_i$ playing the role of $\hat{\Delta}_i$ throughout. $\square$

Proceeding along the lines of the previous section, we generalize Lemma 3.2 to the periodic case.

LEMMA 4.2. *Suppose M_r diagonalizes $\hat{\Gamma}_r$ for $r = 2,\ldots,2p$ and $r = 2p+2,\ldots,4p$, with the same eigenvalue ordering, as in (4.3). Then M_{4p-r+2}^{-T} diagonalizes $\hat{\Gamma}_r$ for $r = 2,...,2p$ and $r = 2p+2,...,4p$. Further, if the eigenvalues of the matrices $\hat{\Gamma}_i$ are distinct, then the matrices M_r and M_{4p-r+2} may be related by the equation*

$$M_{4p-r+2}^{-T} = M_r L_r, \tag{4.5}$$

with L_r diagonal.

Proof. The key element of the proof for Lemma 3.2 was the observation that $\hat{\Delta}_2 = \hat{\Delta}_4^T$. In the periodic case, $\hat{\Gamma}_r = \hat{\Gamma}_{4p-r+2}^T$ for $r = 2,\ldots,2p$ and $r = 2p+2,\ldots,4p$. The remainder of the proof follows from this observation. □

Next, we prove the existence of the periodic QSVD in the case where the matrices E_i and F_i are nonsingular.

THEOREM 4.3. *Let the matrices $E_i \in \mathbb{R}^{n\times n}$ and $F_i \in \mathbb{R}^{n\times n}$ in (4.2) be nonsingular with the eigenvalues of the matrix $\hat{\Gamma}_1$ distinct. There exist non–orthogonal matrices $X_1,\ldots,X_p$ and $Y_2,\ldots,Y_p$, positive definite diagonal matrices $\Phi_1,\ldots,\Phi_p$ and $\Theta_1,\ldots,\Theta_p$, and orthogonal matrices U and V such that*

$$\begin{array}{rclcrcl}
X_1^T E_1 Y_2 & = & \Phi_1 & , & X_1^T F_1 V & = & \Theta_1, \\
X_2^T E_2 Y_3 & = & \Phi_2 & , & X_2^T F_2 Y_2 & = & \Theta_2, \\
\vdots & = & \vdots & , & \vdots & = & \vdots \\
X_{p-1}^T E_{p-1} Y_p & = & \Phi_{p-1} & , & X_{p-1}^T F_{p-1} Y_{p-1} & = & \Theta_{p-1}, \\
X_p^T E_p U & = & \Phi_p & , & X_p^T F_p Y_p & = & \Theta_p.
\end{array} \tag{4.6}$$

Further, the constituent matrices of the periodic QSVD may be related to the matrices M_i in (4.3) and (4.4) which diagonalize the matrices $\hat{\Gamma}_i$ in the following way:

$$\begin{array}{rclcrcl}
X_1 & = & M_{4p}P_{4p} & , & V & = & M_1, \\
X_2 & = & M_{4p-2}P_{4p-2} & , & Y_2 & = & M_{4p-1}P_{4p-1}, \\
X_3 & = & M_{4p-4}P_{4p-4} & , & Y_3 & = & M_{4p-3}P_{4p-3}, \\
\vdots & = & \vdots & , & \vdots & = & \vdots \\
X_p & = & M_{2p+2}P_{2p+2} & , & Y_p & = & M_{2p+3}P_{2p+3}, \\
 & & & , & U & = & M_{2p+1}, \\
\Phi_1 & = & P_{4p}M_{4p}^T E_1 M_{4p-1} P_{4p-1} & , & \Theta_1 & = & P_{4p}M_{4p}^T F_1 M_1, \\
\Phi_2 & = & P_{4p-2}M_{4p-2}^T E_2 M_{4p-3} P_{4p-3} & , & \Theta_2 & = & P_{4p-2}M_{4p-2}^T F_2 M_{4p-1} P_{4p-1}, \\
\vdots & = & \vdots & , & \vdots & = & \vdots \\
\Phi_{p-1} & = & P_{2p}M_{2p}^T E_{p-1} M_{2p+3} P_{2p+3} & , & \Theta_{p-1} & = & P_{2p}M_{2p}^T F_{p-1} M_{2p+1}, \\
\Phi_p & = & P_{2p+2}M_{2p+2}^T E_p M_{2p+1} & , & \Theta_p & = & P_{2p+2}M_{2p+2}^T F_p M_{2p+3} P_{2p+3}.
\end{array} \tag{4.7}$$

where the matrices P_i are diagonal signature matrices as in (3.8).

Proof. The proof is a straightforward extension of the proof of Theorem 3.3. By the substitution of the results of Lemmas 4.1 we can assure that the matrix S in (4.3) and the matrices D_i's (4.4) are diagonal and positive definite. Since $\hat{\Gamma}_1$ and $\hat{\Gamma}_{2p+1}$ are symmetric, M_1 and M_{2p+1} may be chosen to be orthogonal. With M_1 and M_{2p+1} orthogonal and S positive definite, Lemma 4.1 and Lemma 4.2 imply that there exist matrices P_i such that (4.7) holds, completing the proof. □

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As in the previous section, we show that the matrices in the periodic QSVD may be constructed from the matrices resulting from the periodic QZ decomposition of the matrices comprising a related operator $\Gamma_1 : X_1 \to X_{2p+2}$, $X_i \subseteq \mathbb{R}^n$ where Γ_1 is defined by the equations

$$\begin{array}{rcl} E_1 x_2 & = & F_1 x_1, \\ \vdots & = & \vdots \\ E_p x_{p+1} & = & F_p x_p, \\ E_p^T x_{p+2} & = & x_{p+1}, \\ E_{p-1}^T x_{p+3} & = & F_p^T x_{p+2}, \\ \vdots & = & \vdots \\ E_1^T x_{2p+1} & = & F_2^T x_{2p}, \\ x_{2p+2} & = & F_1^T x_{2p+1}. \end{array} \tag{4.8}$$

To prove the existence of the periodic QSVD, it is necessary to generalize the lemmas and theorems of the previous section.

LEMMA 4.4. *Let the operator Γ_1 be defined as in (4.8). There exist orthogonal matrices $Q_1, \ldots, Q_{2p-1}$, $W_2, \ldots, W_{2p}$, U and V such that the matrices H_k and T_k are upper triangular*

$$\begin{array}{rclcrcl} Q_1^T E_1 W_2 & = & H_1 & , & Q_1^T F_1 V & = & T_1, \\ & \vdots & & , & & \vdots & \\ Q_p^T E_p U & = & H_p & , & Q_p^T F_p W_p & = & T_p, \\ U^T E_p^T W_{p+1} & = & H_{p+1} & , & & & \\ Q_{p+1}^T E_{p-1}^T W_{p+2} & = & H_{p+2} & , & Q_{p+1}^T F_p^T W_{p+1} & = & T_{p+2}, \\ & \vdots & & , & & \vdots & \\ Q_{2p-1}^T E_1^T W_{2p} & = & H_{2p} & , & Q_{2p-1}^T F_2^T W_{2p-1} & = & T_{2p}, \\ & & & , & V^T F_1^T W_{2p} & = & T_{2p+1}. \end{array} \tag{4.9}$$

Proof. The proof is a trivial extension of that in Lemma 3.5. □

LEMMA 4.5. *Let the operator Γ_1 be defined as in (4.8), with the matrices $E_i \in \mathbb{R}^{n\times n}$ and $F_i \in \mathbb{R}^{n\times n}$ nonsingular and with the eigenvalues of Γ_1 distinct. The matrices X_i, U_i, V_i, Φ_i, and Θ_i associated with the periodic QSVD in (4.6) may be computed via the (modified) QZ decomposition in (4.9).*

Proof. The proof for the existence of the periodic QSVD with the matrices E_i and F_i nonsingular is an extension of Lemma 3.6, with the matrices X_k and Y_k constructed analogously. Lemma 4.4 ensures that there exist orthogonal matrices $\bar{Q}_1, \ldots, \bar{Q}_{2p-1}$, $\bar{U}$, $\bar{V}$, and $\bar{W}_2, \ldots, \bar{W}_{2p}$ that diagonalize Γ_1 with a particular eigenvalue ordering. Let λ_i be the eigenvalue of associated with the ith column of $\bar{V}$. Let the matrix $V_{[j]}$ be the orthogonal matrix that results from interchanging the first column of $\bar{V}$ and the jth column. Since the operator Γ_1 is diagonalized by the matrix $\bar{V}$, $V_{[j]}$ also diagonalizes the operator with the jth eigenvalue in the upper left–hand corner. Now, define the matrices $Q_{\ell\,[j]}$, U_ℓ, and $W_{\ell\,[j]}$ as the matrices, along with V_ℓ, that triangularize constituent matrices of Γ_1 with the jth eigenvalue in the upper left–hand corner. Also define the vector $\tilde{x}_{\ell\,[j]}$ to be the first column of $W_{2p-\ell+1\,[j]}$. Via Lemma 4.2 and in analogy with Lemma 3.6, $\tilde{x}_{\ell\,[j]}^T$ is a left eigenvector of the operator $\Gamma_{4p-2\ell}$. Similarly, define the vector $\tilde{y}_{\ell\,[j]}$ to be the first vector of $W_{\ell+1\,[j]}$. The vector $\tilde{y}_{\ell\,[j]}$ is a right eigenvector of the operator $\Gamma_{2\ell-2}$. Thus, the matrices $\tilde{X}_\ell = [\tilde{x}_{\ell\,[1]}, \ldots, \tilde{x}_{\ell\,[n]}]$ and

$\tilde{Y}_\ell = [\tilde{y}_{\ell\,[1]}, \ldots, \tilde{y}_{\ell\,[n]}]$ are matrices of eigenvectors of $\Gamma_{4p-2\ell}$ and $\Gamma_{2\ell-2}$, respectively, and therefore diagonalize the constituent matrices of the operator Γ_1 via Lemma 4.1, albeit with the diagonal elements not necessarily positive. There exists, however, diagonal signature matrices $P_{x\ell}$ and $P_{y\ell}$ as in (3.8) such that the matrices $X_\ell = \tilde{X}_\ell P_{x\ell}$ and $Y_\ell = \tilde{Y}_\ell P_{y\ell}$ diagonalize the constituent matrices of the operator Γ_1 with the diagonalized matrices positive definite. □

Finally, Theorem 3.7 is generalized for the periodic case.

THEOREM 4.6. *Let the operator Π be defined as in (4.1). There exist non–orthogonal matrices $X_1, \ldots, X_p$ and $Y_2, \ldots, Y_p$, non–negative definite diagonal matrices $\Phi_1, \ldots, \Phi_p$ and $\Theta_1, \ldots, \Theta_p$, and orthogonal matrices U and V such that the relations in (4.6) hold.*

Proof. The proof is a trivial extension of Theorem 3.7. □

5. Numerical Examples.

In this section we give some numerical examples to illustrate the points discussed in the previous sections.

Example 1 Consider the operator Ξ defined by (1.2) where the matrices E and F are

$$E = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \quad , \quad F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Since E is nonsingular, it is possible to write the map $\Xi = \hat{\Xi}$ directly:

$$\hat{\Xi} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

The singular value decomposition of $\hat{\Xi} = \hat{U}\hat{\Sigma}\hat{V}^T$ where

$$\hat{U} = \begin{bmatrix} 0.9239 & 0.3827 \\ -0.3827 & 0.9239 \end{bmatrix} \quad , \quad \hat{\Sigma} = \begin{bmatrix} 2.4142 & 0.0000 \\ 0.0000 & 0.4142 \end{bmatrix},$$

$$\hat{V} = \begin{bmatrix} 0.9239 & -0.3827 \\ 0.3827 & 0.9239 \end{bmatrix} \quad .$$

The matrices U, V, Q, W, H_i, and T_i that result from the periodic QZ decomposition of Δ_1 are:

$$U = \begin{bmatrix} 0.9239 & 0.3827 \\ -0.3827 & 0.9239 \end{bmatrix} \quad , \quad V = \begin{bmatrix} 0.9239 & -0.3827 \\ 0.3827 & 0.9239 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.3655 & 0.9308 \\ 0.9308 & -0.3655 \end{bmatrix} \quad , \quad W = \begin{bmatrix} -0.8669 & 0.4985 \\ 0.4985 & 0.8669 \end{bmatrix},$$

$$H_1 = Q^T E U = \begin{bmatrix} 1.9145 & 7.0174 \\ 0.0000 & 1.0446 \end{bmatrix} \quad , \quad H_2 = U^T E^T W = \begin{bmatrix} 0.2819 & 1.8937 \\ 0.0000 & 7.0947 \end{bmatrix},$$

$$T_1 = Q^T F V = \begin{bmatrix} 4.6221 & 2.9067 \\ 0.0000 & 0.4327 \end{bmatrix} \quad , \quad T_3 = V^T F^T W = \begin{bmatrix} 0.6806 & 4.5717 \\ 0.0000 & 2.9387 \end{bmatrix}.$$

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With a two by two system, it is possible to construct $X = [x_1, x_2]$ directly from Q and W:

$$\begin{aligned} x_1 &= w_1, \\ x_2 &= q_2, \end{aligned}$$

yielding

$$X = \begin{bmatrix} -0.8669 & 0.9308 \\ 0.4985 & -0.3655 \end{bmatrix}.$$

Computing Φ and Θ yields

$$\begin{aligned} \Phi &= X^T EU & \Theta &= X^T FV \\ &= \begin{bmatrix} 0.2819 & 0.0000 \\ 0.0000 & 1.0446 \end{bmatrix}, & &= \begin{bmatrix} 0.6806 & 0.0000 \\ 0.0000 & 0.4327 \end{bmatrix}. \end{aligned}$$

The product $\Sigma_{\phi\theta} = \Phi^{-1}\Theta$ is

$$\Sigma_{\phi\theta} = \begin{bmatrix} 2.4142 & 0.0000 \\ 0.0000 & 0.4142 \end{bmatrix},$$

which is equal to $\hat{\Sigma}$, as expected.

Example 2 Consider again the operator Ξ where

$$E = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Since E is singular it is not possible to write the map $\hat{\Xi}$ directly. However, it is possible to compute the QSVD of the system. The matrices U, V, Q, W, H_i, and T_i that result from the periodic QZ decomposition of Δ are:

$$U = \begin{bmatrix} 0.8321 & -0.5547 \\ -0.5547 & -0.8321 \end{bmatrix}, \quad V = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.3162 & -0.9487 \\ -0.9487 & 0.3162 \end{bmatrix}, \quad W = \begin{bmatrix} 0.8944 & -0.4472 \\ -0.4472 & -0.8944 \end{bmatrix},$$

$$\begin{aligned} H_1 &= Q^T EU & H_2 &= U^T E^T W \\ &= \begin{bmatrix} 0.0000 & 7.9812 \\ 0.0000 & 1.1402 \end{bmatrix}, & &= \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 8.0623 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} T_1 &= Q^T FV & T_3 &= V^T F^T W \\ &= \begin{bmatrix} 7.0711 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, & &= \begin{bmatrix} 1.0000 & 7.0000 \\ 0.0000 & 0.0000 \end{bmatrix}. \end{aligned}$$

As before, we construct $X = [x_1, x_2]$ directly from Q and W:

$$\begin{aligned} x_1 &= w_1, \\ x_2 &= q_2, \end{aligned}$$

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making

$$X = \begin{bmatrix} 0.8944 & -0.9487 \\ -0.4472 & 0.3162 \end{bmatrix}.$$

Computing Φ and Θ yields

$$\Phi = X^T E U = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 1.1402 \end{bmatrix}, \quad \Theta = X^T F V = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}.$$

Note that the system has an infinite and a zero singular value, and no enomorphic eigenvalues.

Example 3 Consider the operator Π in (4.1) defined by matrices E_i and F_i where

$$E_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}.$$

Since the matrices E_i are nonsingular, it is possible to write the map $\hat{\Pi}$ directly:

$$\hat{\Pi} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The singular value decomposition of $\hat{\Pi} = \hat{U}\hat{\Sigma}\hat{V}^T$ where

$$\hat{U} = \begin{bmatrix} 0.8507 & 0.5257 \\ 0.5257 & -0.8507 \end{bmatrix}, \quad \hat{\Sigma} = \begin{bmatrix} 2.6180 & 0.0000 \\ 0.0000 & 0.3820 \end{bmatrix},$$

$$\hat{V} = \begin{bmatrix} 0.5257 & -0.8507 \\ 0.8507 & 0.5257 \end{bmatrix}.$$

The matrices U, V, Q_k, W_k, H_k and T_k that result from the periodic QZ decomposition of the pencil of Γ are:

$$U = \begin{bmatrix} 0.8507 & 0.5257 \\ 0.5257 & -0.8507 \end{bmatrix}, \quad V = \begin{bmatrix} 0.5257 & -0.8507 \\ 0.8507 & 0.5257 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.4932 & 0.8699 \\ 0.8699 & -0.4932 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.3447 & -0.9387 \\ 0.9387 & 0.3447 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0.3568 & -0.9342 \\ 0.9342 & 0.3568 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -0.3568 & -0.9342 \\ 0.9342 & -0.3568 \end{bmatrix},$$

$$W_3 = \begin{bmatrix} -0.9156 & -0.4022 \\ 0.4022 & -0.9156 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0.9874 & -0.1583 \\ -0.1583 & -0.9874 \end{bmatrix},$$

$$H_1 = Q_1^T E_1 W_2 = \begin{bmatrix} 3.0649 & -4.4923 \\ 0.0000 & 0.6526 \end{bmatrix}, \quad H_2 = Q_2^T E_2 U = \begin{bmatrix} 5.5188 & -2.9173 \\ 0.0000 & 0.1812 \end{bmatrix},$$

$$H_3 = U^T E_2^T W_3 = \begin{bmatrix} 0.3421 & -5.5081 \\ 0.0000 & 2.9229 \end{bmatrix}, \quad H_4 = Q_3^T E_1^T W_4 = \begin{bmatrix} 1.4360 & -5.0988 \\ 0.0000 & 1.3928 \end{bmatrix},$$

$$T_1 = Q_1^T F_1 V = \begin{bmatrix} 7.3065 & -0.7345 \\ 0.0000 & 0.2737 \end{bmatrix}, \quad T_2 = Q_2^T F_2 W_2 = \begin{bmatrix} 6.0606 & -8.0772 \\ 0.0000 & 0.1650 \end{bmatrix},$$

$$T_4 = Q_3^T F_2^T W_3 = \begin{bmatrix} 0.5041 & -9.8899 \\ 0.0000 & 1.9838 \end{bmatrix}, \quad T_5 = V^T F_1^T W_4 = \begin{bmatrix} 2.5516 & -6.8465 \\ 0.0000 & 0.7838 \end{bmatrix}.$$

With a two by two system, it is possible to construct $X_1 = [x_{11}, x_{12}]$, $X_2 = [x_{21}, x_{22}]$, and $Y_2 = [y_{21}, y_{22}]$ directly from the Q_k's and W_k's:

$$\begin{aligned} x_{11} &= w_{41}, \\ x_{12} &= q_{12}, \\ x_{21} &= w_{31}, \\ x_{22} &= q_{22}, \\ y_{11} &= w_{21}, \\ y_{12} &= q_{32}. \end{aligned}$$

yielding

$$X_1 = \begin{bmatrix} 0.9874 & 0.8699 \\ -0.1583 & -0.4932 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -0.9156 & -0.9387 \\ 0.4022 & 0.3447 \end{bmatrix},$$

and

$$Y_2 = \begin{bmatrix} -0.3568 & -0.9342 \\ 0.9342 & 0.3568 \end{bmatrix}.$$

Computing Φ_1, Φ_2, Θ_1, and Θ_2 yields

$$\Phi_1 = X_1^T E_1 Y_2 = \begin{bmatrix} 1.0703 & 0.0000 \\ 0.0000 & 0.4864 \end{bmatrix}, \quad \Theta_1 = X_1^T F_1 V = \begin{bmatrix} 2.5516 & 0.0000 \\ 0.0000 & 0.2737 \end{bmatrix},$$

$$\Phi_2 = X_2^T E_2 U = \begin{bmatrix} 0.3421 & 0.0000 \\ 0.0000 & 0.1812 \end{bmatrix}, \quad \Theta_2 = X_2^T F_2 Y_2 = \begin{bmatrix} 0.3757 & 0.0000 \\ 0.0000 & 0.1230 \end{bmatrix}.$$

The product $\Sigma_{\phi\theta} = \Phi_2^{-1}\Theta_2\Phi_1^{-1}\Theta_1$ is

$$\Sigma_{\phi\theta} = \begin{bmatrix} 2.6180 & 0.0000 \\ 0.0000 & 0.3820 \end{bmatrix},$$

which is equal to $\hat{\Sigma}$, as expected.

6. Conclusion. In this paper, the relationship between the QSVD and the periodic QZ decomposition is elaborated. Specifically, the periodic QZ algorithm may be viewed as a method by which the QSVD may be computed as accurately, albeit half as efficiently, than the standard algorithm. Nevertheless, by using this technique the "periodic" QSVD of a sequence of matrix pairs may be readily computed. In this paper we have discussed the QSVD; however, other canonical forms, such as those discussed in previous papers [2] [3] [5] [8] [10] [11] [12], or the computation of principal angles and vectors [4], may be cast in the periodic QZ framework. This demonstrates the versatility of the QZ decomposition; it provides a theoretical basis for computing eigenvalue or singular value revealing matrix decompositions.

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