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LEFT-DEFINITE VARIATIONS OF THE CLASSICAL FOURIER EXPANSION THEOREM*

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Abstract. In a recent paper, Littlejohn and Wellman developed a general left-definite theory for arbitrary self-adjoint operators in a Hilbert space that are bounded below by a positive constant. We apply this theory and construct the sequences of left-definite Hilbert spaces $\{H_n\}_{n\in\mathbb{N}}$ and left-definite self-adjoint operators $\{A_n\}_{n\in\mathbb{N}}$ associated with the classical, regular self-adjoint boundary value problem consisting of the Fourier equation with periodic boundary conditions. As a particular consequence of our analysis, we obtain a Fourier expansion theorem in each left-definite space H_n as well as an explicit representation of the domain of $A^{n/2}$ for each positive integer n.

Key words. self-adjoint operator, Hilbert space, left-definite Hilbert space, left-definite operator, regular self-adjoint boundary value problem, Fourier series

AMS subject classification. 34B24, 33B10

1. Introduction. For a self-adjoint operator A in a Hilbert space H, which is bounded below by a positive constant, Littlejohn and Wellman [8] construct a continuum of unique Hilbert spaces $\{H_r\}_{r>0}$ and a continuum of self-adjoint operators $\{A_r\}_{r>0}$ from the pair (H, A). For each $r > 0$, H_r is called the r^{th} left-definite Hilbert space associated with (H, A) and A_r is called the r^{th} left-definite operator associated with (H, A). Some information of this theory and the constructions of these spaces and operators are given below in Section 2.

This general theory has been applied to several classical singular second-order differential equations, including the Jacobi [2], Hermite [3], Legendre [4], and Laguerre [8] equations. In these papers, the authors construct sequences - but not the full continua - of left-definite spaces and left-definite operators associated with the special self-adjoint operator A that has the corresponding classical orthogonal polynomials (Jacobi, Hermite, Legendre, and Laguerre, respectively) as eigenfunctions.

In this paper, we determine the sequences of left-definite spaces $\{H_n \mid n \in \mathbb{N}\}$ and left-definite operators $\{A_n\}_{n\in\mathbb{N}}$ associated with the regular self-adjoint operator A in $H = L^2[a, b]$ obtained from the Fourier boundary value problem

$$\begin{cases} \ell[y](x) = -y''(x) + ky(x) = \lambda y(x) \quad (x \in [a, b]) \\ \qquad y(a) = y(b);\ y'(a) = y'(b), \end{cases} \tag{1.1}$$

where $[a, b]$ is a compact interval of the real line and k is a fixed, positive constant. This boundary value problem is both well-known and important; indeed, the eigenfunction expansion in this case produces the classical Fourier series expansion for $f \in H$. We extend this expansion result to each of the left-definite spaces associated with this self-adjoint boundary value problem. As a consequence of this analysis, for each positive integer n, we obtain explicit characterizations of the domains $\mathcal{D}(A^{n/2})$ of $A^{n/2}$ and of the domains $\mathcal{D}(A_n)$ for each of the left-definite operators A_n associated with (H, A). Each of these domains is in the space in which the operator acts; in particular, we emphasize that this analysis produces new results for the *original* operator A.

The terminology *left-definite* is due to Schäfke and Schneider [12] but the origins of left-definite theory go back to at least the work of Hermann Weyl [13] in the early 1900's.

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The interest in left-definite theory originated, at least in part, in the study of classical Sturm-Liouville equations with a weight function that changes sign. The associated operator of such a problem, *when studied in the usual* L^2 spaces, is not bounded below. There is a vast literature for such problems; see Kong, Wu, and Zettl [9] for some recent work and further references.

The contents of this paper are as follows. In Section 2, we review the left-definite theory developed by Littlejohn and Wellman. Section 3 deals with the self-adjoint operator A generated from (1.1) and its properties, including information about its spectrum, its eigenfunctions and the fact that A is bounded below in $L^2[a,b]$ by kI, where k is the constant appearing in the differential expression in (1.1). The left-definite analysis of A - specifically, the construction of the sequence of left-definite spaces $\{H_n\}_{n\in\mathbb{N}}$ and left-definite operators $\{A_n\}_{n\in\mathbb{N}}$ - is developed in Section 4. In addition, we develop a Fourier expansion theorem in each left-definite space H_n in Section 4. Lastly, in Section 5, some special cases of these left-definite spaces and left-definite operators are discussed. Indeed, we determine the explicit domains of the powers $A^{n/2}$ and the domains of each of the left-definite operators A_n for each $n \in \mathbb{N}$.

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ denotes, respectively, the fields of real and complex numbers. The natural numbers $\{1, 2, \ldots\}$ are denoted by $\mathbb{N}$ and the non-negative integers by $\mathbb{N}_0$. For a compact interval I, the terminology $AC(I)$ denotes the space of all complex valued functions $f : I \to \mathbb{C}$ that are absolutely continuous on I. If A is a linear operator, $\mathcal{D}(A)$ denotes its domain. Lastly, a word is in order regarding displayed, bracketed information. For example,

$$f \text{ has property } P \quad (x \in I),$$

and

$$g_m \text{ has property } Q \quad (m \in \mathbb{N}_0)$$

mean, respectively, that f has property P *for all* $x \in I$ and g_m has property Q *for all* $m \in \mathbb{N}_0$.

2. A Review of Left-Definite Theory. Let V denote a vector space (over the complex field $\mathbb{C}$) and suppose that $(\cdot,\cdot)$ is an inner product with norm $\|\cdot\|$ generated from $(\cdot,\cdot)$ such that $H = (V, (\cdot,\cdot))$ is a Hilbert space. Suppose V_r (the subscripts will be made clear shortly) is a linear manifold (vector subspace) of the vector space V and let $(\cdot,\cdot)_r$ and $\|\cdot\|_r$ denote an inner product and its associated norm, respectively, over V_r (quite possibly different from $(\cdot,\cdot)$ and $\|\cdot\|$). We denote the resulting inner product space by $H_r = (V_r, (\cdot,\cdot)_r)$.

Throughout this section, we assume that $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI for some $k > 0$; that is,

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

It follows that A^r, for each $r > 0$, is a self-adjoint operator that is bounded below in H by $k^r I$.

We now define an r^{th} left-definite space associated with (H, A).

DEFINITION 2.1. *Let $r > 0$ and suppose V_r is a linear manifold of the Hilbert space $H = (H, (\cdot,\cdot))$ and $(\cdot,\cdot)_r$ is an inner product on V_r. Let $H_r = (V_r, (\cdot,\cdot)_r)$. We say that H_r is an r^{th}* ***left-definite space*** *associated with the pair (H, A) if each of the following conditions hold:*

(i) H_r is a Hilbert space,
(ii) $\mathcal{D}(A^r)$ is a linear manifold of V_r,
(iii) $\mathcal{D}(A^r)$ is dense in H_r,

(iv) $(x,x)_r \geq k^r (x,x) \quad (x \in V_r)$, *and*
(v) $(x,y)_r = (A^r x, y) \quad (x \in \mathcal{D}(A^r),\ y \in V_r)$.

It is not clear, from the definition, if such a self-adjoint operator A generates a left-definite space for a given $r > 0$. However, in [8], the authors prove the following theorem; the Hilbert space spectral theorem plays a prominent role in establishing this result. Notice that, in the case that A is a bounded operator, the left-definite theory is trivial but, when A is unbounded, the theory has substance.

THEOREM 2.2. (*see [8, Theorems 3.1 and 3.4]*) *Suppose* $A : \mathcal{D}(A) \subset H \to H$ *is a self-adjoint operator that is bounded below by* kI, *for some* $k > 0$. *Let* $r > 0$. *Define* $H_r = (V_r, (\cdot,\cdot)_r)$ *by*

$$V_r = \mathcal{D}(A^{r/2}), \tag{2.1}$$

and

$$(x,y)_r = (A^{r/2}x, A^{r/2}y) \quad (x,y \in V_r).$$

Then H_r *is a left-definite space associated with the pair* (H,A). *Moreover, suppose* $H_r' := (V_r', (\cdot,\cdot)_r')$ *is another* r^{th} *left-definite space associated with the pair* (H,A). *Then* $V_r = V_r'$ *and* $(x,y)_r = (x,y)_r'$ *for all* $x, y \in V_r = V_r'$; *i.e.* $H_r = H_r'$. *That is to say,* $H_r = (V_r, (\cdot,\cdot)_r)$ *is the unique left-definite space associated with* (H,A). *Moreover,*

(a) suppose A *is bounded. Then, for each* $r > 0$,
  (i) $V = V_r$;
  (ii) the inner products $(\cdot,\cdot)$ *and* $(\cdot,\cdot)_r$ *are equivalent.*

(b) suppose A *is unbounded. Then, for each* $r, s > 0$,
  (i) V_r *is a proper subspace of* V;
  (ii) V_s *is a proper subspace of* V_r *whenever* $0 < r < s$;
  (iii) the inner products $(\cdot,\cdot)$ *and* $(\cdot,\cdot)_r$ *are not equivalent for any* $r > 0$;
  (iv) the inner products $(\cdot,\cdot)_r$ *and* $(\cdot,\cdot)_s$ *are not equivalent for any* $r, s > 0$, $r \neq s$.

REMARK 2.3. *Although all five conditions in Definition 2.1 are used in the proof of Theorem 2.2, the most important property, in a sense, is the one given in part (v) of the definition. Indeed, this property asserts that the* r^{th} *left-definite inner product is generated from the* r^{th} *power of* A. *If* A *is generated from a Lagrangian symmetric differential expression* $\ell[\cdot]$, *we see that the* r^{th} *powers of* A *are then determined by the* r^{th} *powers of* $\ell[\cdot]$. *Consequently, in this case, it is possible to explicitly obtain these powers only when* r *is a positive integer. We refer the reader to [8] where an example of a self-adjoint operator* A *in* $\ell^2(\mathbb{N})$ *is discussed in which the entire continuum of left-definite spaces is explicitly obtained. In this example, we note that the explicit spectral resolution of the identity associated with* A *is constructed; this construction allows for a complete determination of the continua of left-definite spaces and left-definite operators.*

DEFINITION 2.4. *For* $r > 0$, *let* $H_r = (V_r, (\cdot,\cdot)_r)$ *denote the* r^{th} *left-definite space associated with* (H,A). *If there exists a self-adjoint operator* $A_r : \mathcal{D}(A_r) \subset H_r \to H_r$ *that is a restriction of* A; *that is,*

$$A_r f = Af \quad (f \in \mathcal{D}(A_r) \subset \mathcal{D}(A)),$$

we call such an operator an r^{th} ***left-definite operator associated with*** (H,A).

Again, it is not immediately clear that such an A_r exists for a given $r > 0$; in fact, however, as the next theorem shows, A_r exists and is unique for each $r > 0$.

THEOREM 2.5. (*see [8, Theorems 3.2 and 3.4]*) *Suppose* A *is a self-adjoint operator in a Hilbert space* H *that is bounded below by* kI, *for some* $k > 0$. *For any* $r > 0$, *let*

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$H_r = (V_r, (\cdot,\cdot)_r)$ be the r^{th} left-definite space associated with (H, A). Then there exists a unique left-definite operator A_r in H_r associated with (H, A); in fact,

$$\mathcal{D}(A_r) = V_{r+2}.$$

Moreover, from Theorem 2.2, we have the following results.

(a) Suppose A is bounded. Then, for each $r > 0$, $A = A_r$.

(b) Suppose A is unbounded. Then, for each $r, s > 0$,

(i) $\mathcal{D}(A_r)$ is a proper subspace of $\mathcal{D}(A)$ for each $r > 0$;

(ii) $\mathcal{D}(A_s)$ is a proper subspace of $\mathcal{D}(A_r)$ whenever $0 < r < s$.

The last theorem that we state in this section shows that the point spectrum, continuous spectrum, and resolvent set of a self-adjoint operator A and each of its associated left-definite operators A_r $(r > 0)$ are identical.

THEOREM 2.6. *(see [8, Theorem 3.6]) For each $r > 0$, let A_r denote the r^{th} left-definite operator associated with the self-adjoint operator A that is bounded below by kI, where $k > 0$. Then*

(a) *the point spectra of A and A_r coincide; i.e. $\sigma_p(A_r) = \sigma_p(A)$;*

(b) *the continuous spectra of A and A_r coincide; i.e. $\sigma_c(A_r) = \sigma_c(A)$;*

(c) *the resolvent sets of A and A_r are equal; i.e. $\rho(A_r) = \rho(A)$.*

We refer the reader to [8] for other theorems, and examples, associated with the general left-definite theory of self-adjoint operators A that are bounded below.

3. The Fourier Operator A and its Properties.

From here on, we let

$$H := L^2[a, b], \tag{3.1}$$

where $-\infty < a < b < \infty$, denote the classical Hilbert space of (equivalence classes of) Lebesgue measurable functions $f : [a, b] \to \mathbb{C}$ satisfying $\int_a^b |f(x)|^2 \, dx < \infty$ with inner product

$$(f, g) := \int_a^b f(x)\overline{g}(x) dx \quad (f, g \in H),$$

and associated norm

$$\|f\| = (f, f)^{1/2} \quad (f \in H).$$

Fix $k > 0$ and let $\ell[\cdot]$ denote the regular differential expression defined by

$$\ell[f](x) := -f''(x) + kf(x) \quad (x \in [a, b]). \tag{3.2}$$

The operator A that we deal with in this paper is defined as

$$\begin{cases} \mathcal{D}(A) = \{f : [a, b] \to \mathbb{C} \mid f, f' \in AC[a, b]; f'' \in H; f(a) = f(b); f'(a) = f'(b)\} \\ Af = \ell[f] \quad (f \in \mathcal{D}(A)). \end{cases} \tag{3.3}$$

It is well known (see, for example, [10] or [14]) that A is self-adjoint in H and has a discrete spectrum $\sigma(A)$. A calculation shows that the eigenvalues of A are given by

$$\lambda_m := \left(\frac{2m\pi}{b - a}\right)^2 + k \quad (m \in \mathbb{N}_0). \tag{3.4}$$

The eigenvalue $\lambda_0 = k$ is simple and each nonzero constant is an eigenfunction; we let

$$y_0(x) = 1/\sqrt{2} \tag{3.5}$$

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(an explanation for this choice of non-zero constant is made clear in (3.7) below). For $m \in \mathbb{N}$, the general solution of $\ell[f](x) = \lambda_m f(x)$ on $[a,b]$ is

$$f_m(x) = c_{m,1} \cos\left(\frac{2m\pi}{b-a}x\right) + c_{m,2} \sin\left(\frac{2m\pi}{b-a}x\right)$$

and

$$\left\{ \begin{array}{ll} y_{m,1}(x) = \cos\left(\frac{2m\pi}{b-a}x\right) & (m \in \mathbb{N}) \\ y_{m,2}(x) = \sin\left(\frac{2m\pi}{b-a}x\right) & (m \in \mathbb{N}) \end{array} \right. \tag{3.6}$$

form a basis for the eigenspace associated with λ_m for each $m \in \mathbb{N}$. It is well known (see [11, Chapter 4]) that the collection of eigenfunctions

$$\{y_0\} \cup \{y_{m,1}\}_{m\in\mathbb{N}} \cup \{y_{m,2}\}_{m\in\mathbb{N}}$$

forms a complete orthogonal set in H. In fact, a calculation shows that

$$\|y_0\| = \|y_{m,j}\| = \sqrt{\frac{b-a}{2}} \quad (m \in \mathbb{N};\ j = 1,2). \tag{3.7}$$

Consequently,

$$E := \{z_{m,1}\}_{m\in\mathbb{N}_0} \cup \{z_{m,2}\}_{m\in\mathbb{N}}, \tag{3.8}$$

where

$$\left\{ \begin{array}{l} z_{m,1} = \left\{ \begin{array}{ll} 1/\sqrt{b-a} & \text{if } m = 0 \\ \sqrt{\frac{2}{b-a}} \cos\left(\frac{2m\pi}{b-a}x\right) & \text{if } m \in \mathbb{N} \end{array} \right. \\ \\ z_{m,2} = \sqrt{\dfrac{2}{b-a}} \sin\left(\frac{2m\pi}{b-a}x\right) \quad (m \in \mathbb{N}) \end{array} \right. \tag{3.9}$$

is a complete orthonormal basis in $L^2[a,b]$. By re-ordering, for simplicity purposes, we write

$$E = \{e_m \mid m \in \mathbb{N}_0\} = \{z_{m,1}\}_{m\in\mathbb{N}_0} \cup \{z_{m,2}\}_{m\in\mathbb{N}} \tag{3.10}$$

as the complete set of *orthonormal* eigenvectors of A given in (3.8). Furthermore, when referring to $e_m \in E$, we shall assume that e_m is an eigenfunction of A corresponding to the eigenvalue $\widetilde{\lambda}_m \in \{\lambda_m \mid m \in \mathbb{N}_0\}$, where λ_m is defined in (3.4).

For later purposes, we note that for any eigenfunction $e_m \in E$, we have

$$e_m^{(j)}(a) = e_m^{(j)}(b) \quad (j = 0, 1, \ldots). \tag{3.11}$$

We remind the reader of the following classical expansion theorem (see [11, Chapter 4]) for functions $f \in L^2[a,b]$ in terms of the eigenfunctions of A.

THEOREM 3.1. *Let $f \in L^2[a,b]$; for each $N \in \mathbb{N}$, define the partial sums*

$$s_N(f)(x) = \sum_{m=0}^{N} a_m(f) \cos\left(\frac{2m\pi}{b-a}x\right) + \sum_{m=1}^{N} b_m(f) \sin\left(\frac{2m\pi}{b-a}x\right) \quad (x \in [a,b]),$$

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where

$$a_0(f) := (f, z_{0,1}) = \frac{1}{\sqrt{b-a}} \int_a^b f(x)dx, \tag{3.12}$$

$$a_m(f) := (f, z_{m,1}) = \sqrt{\frac{2}{b-a}} \int_a^b f(x) \cos\left(\frac{2m\pi}{b-a}x\right) dx \quad (m \in \mathbb{N}), \tag{3.13}$$

and

$$b_m(f) := (f, z_{m,2}) = \sqrt{\frac{2}{b-a}} \int_a^b f(x) \sin\left(\frac{2m\pi}{b-a}x\right) dx \quad (m \in \mathbb{N}) \tag{3.14}$$

are the Fourier coefficients of f corresponding to the orthonormal basis E. Then

$$\|f - s_N(f)\| \to 0 \textit{ as } N \to \infty,$$

and

$$\|f\|^2 = \sum_{m=0}^{\infty} |a_m(f)|^2 + \sum_{m=1}^{\infty} |b_m(f)|^2 .$$

For $f \in \mathcal{D}(A)$, we see from integration by parts and the boundary conditions in (3.3) that

$$\begin{aligned}(Af, f) &= \int_a^b [-f''(x) + kf(x)]\, \overline{f}(x)dx \\ &= -f'(x)\overline{f}(x) \mid_a^b + \int_a^b \left[|f'(x)|^2 + k\,|f(x)|^2\right] dx \\ &= \int_a^b \left[|f'(x)|^2 + k\,|f(x)|^2\right] dx \\ &\geq k \int_a^b |f(x)|^2\, dx = k(f, f);\end{aligned}$$

that is, A is bounded below by kI in H. Consequently, the left-definite theory discussed in the last section can be applied to this operator A. This analysis is made in the next section.

4. The Left-Definite Spaces and Operators Associated with (H, A). Let the Hilbert space H be given by (3.1) and let A be the self-adjoint differential operator in H defined in (3.3). In this section we use the theory given in Section 2 to explicitly construct the left-definite spaces H_n and the left-definite operators A_n associated with the pair (H, A), for positive integer values of n. We start with the determination of the integral powers of the differential expression $\ell[\cdot]$, defined in (3.2), given inductively by

$$\ell^2[y] = \ell[\ell[y]],\ \ \ell^n[y] = \ell[\ell^{n-1}[y]],\ \ n \in \mathbb{N}.$$

LEMMA 4.1. *For each $n \in \mathbb{N}$,*

$$\ell^n[y] = \sum_{j=0}^{n} (-1)^j \binom{n}{j} k^{n-j} y^{(2j)}. \tag{4.1}$$

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Proof. We prove (4.1) by induction on $n \in \mathbb{N}$. This formula is evident for $n = 1$ so assume that the formula holds for $n-1$. Then

$$\ell^n[y] = \ell[\ell^{n-1}[y]] = \ell\left(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}k^{n-1-j}y^{(2j)}\right)$$

$$(4.2)\qquad = -\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}k^{n-1-j}y^{(2j+2)} + \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}k^{n-j}y^{(2j)}.$$

Since $\binom{n-1}{j-1}+\binom{n-1}{j}=\binom{n}{j}$, we see that the coefficient of $y^{(2j)}$ $(0 \le j \le n)$ in (4.2) is

$$(-1)^j\binom{n-1}{j-1}k^{n-j} + (-1)^j\binom{n-1}{j}k^{n-j} = (-1)^j\binom{n}{j}k^{n-j};$$

this completes the proof. □

For example,

$$(4.3)\qquad \ell^2[y] = y^{(4)} - 2ky'' + k^2y$$

and

$$\ell^5[y] = -y^{(10)} + 5\,k\,y^{(8)} - 10k^2y^{(6)} + 10\,k^3\,y^{(4)} - 5k^4y'' + k^5y.$$

DEFINITION 4.2. *For $n \in \mathbb{N}$, define*

(i) $\widetilde{V}_n := \{f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b]\ (j = 0,1,\ldots,n-1);\ f^{(n)} \in L^2[a,b]\}$;

(ii) $V_n := \{f \in \widetilde{V}_n \mid f^{(j)}(a) = f^{(j)}(b)\ (j = 0,1,\ldots,n-1)\}$;

(iii) $(f,g)_n := \sum_{j=0}^{n}\binom{n}{j}k^{n-j}\int_a^b f^{(j)}(x)\overline{g}^{(j)}(x)dx \quad (f,g \in \widetilde{V}_n)$;

(iv) $\|f\|_n := (f,f)_n^{1/2}$;

(v) $\widetilde{H}_n := (\widetilde{V}_n, (\cdot,\cdot)_n)$;

(vi) $H_n := (V_n, (\cdot,\cdot)_n)$.

REMARK 4.3. *We note that both V_n and $\widetilde{V}_n$ are vector subspaces of $L^2[a,b]$ and $V_n \subset \widetilde{V}_n$. Furthermore, it is clear that $(\cdot,\cdot)_n$ is an inner product on both $V_n \times V_n$ and $\widetilde{V}_n \times \widetilde{V}_n$.*

REMARK 4.4. *Notice that the inner product $(\cdot,\cdot)_n$ is generated by the n^{th} integral power $\ell^n[\cdot]$ of the differential expression $\ell[\cdot]$; indeed, see (4.1) and item (v) in Definition 2.1.*

REMARK 4.5. *From (3.11), we see that E, the set of orthonormal eigenfunctions of A given in (3.10), is contained in H_n for each $n \in \mathbb{N}$. In Theorem 4.5 below we show that E is a complete orthogonal set in each space H_n. Theorem 4.9 shows that H_n is the n^{th} left-definite space associated with the pair (H, A).*

THEOREM 4.6. *For each $n \in \mathbb{N}$, $\widetilde{H}_n$ is a Hilbert space.*

Proof. We show that $\widetilde{H}_n$ is equivalent to the well-known (see [7]) Sobolev-Hilbert space $(W_n, \langle,\rangle_n)$, where

$$(4.4)\qquad W_n = \{f \in H \mid f^{(n-1)} \in AC[a,b],\ f^{(n)} \in H\},$$

and

$$(4.5)\qquad \langle f,g\rangle_n = \sum_{i=0}^{n}\int_a^b f^{(i)}(x)\overline{g^{(i)}(x)}dx,\ (f,g \in W_n).$$

Note that $\langle\cdot,\cdot\rangle$ is well defined since $f,\ f^{(n)} \in H$ implies $f^{(i)} \in H$ for $i = 1,...,n-1$ (see [6]) and it is an inner product on W_n. Hence the sets $\widetilde{H}_n$ and W_n are equal. Their

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equivalence as Hilbert spaces then follows from the equivalence of the inner products $\langle f,g\rangle_n$ and $(\cdot,\cdot)_n$ which is clear. □

We are now in position to prove the following theorem.

THEOREM 4.7. *For each $n\in\mathbb{N}$, the space H_n, defined in Definition 4.2, is a Hilbert space.*

Proof. We need only show that V_n is closed in W_n. Suppose $\{f_k\}\subset V_n$ and

$$f_k\to f \text{ in } W.$$

Then

$$f_k^{(i)}\to f^{(i)} \text{ in } H,\ (i=0,1,...,n),$$

and

$$0=f_k^{(i)}(b)-f_k^{(i)}(a)=\int_a^b f_k^{(i+1)}(x)dx,\ (i=0,1,...,n-1).$$

From the Schwarz inequality

$$\left(\int_a^b |f_k^{(i)}(x)-f^{(i)}(x)|dx\right)^2\le(b-a)\int_a^b |f_k^{(i)}(x)-f^{(i)}(x)|^2dx,$$

and, from the convergence of $f_k^{(i)}\to f^{(i)}$ in H, it follows that $\int_a^b f_k^{(i)}(x)dx\to\int_a^b f^{(i)}(x)dx$. However $\int_a^b f_k^{(i)}(x)dx=0$ so that $\int_a^b f^{(i)}(x)dx=0$ for $i=0,1,...,n-1$. Hence $f\in V_n$ and thus V_n is closed in W_n. □

Let $e_m\in E$, where E is the set of eigenfunctions of A, defined in (3.10), and let $f\in V_n$. From (3.11), (4.1), integration by parts, and the definition of V_n, we see that

$$\begin{aligned}(A^n e_m,f)&=(\ell^n[e_m],f)\\&=\sum_{j=0}^n(-1)^j\binom{n}{j}k^{n-j}\int_a^b e_m^{(2j)}(x)\overline{f}(x)dx\\&=\sum_{j=0}^n(-1)^j\binom{n}{j}k^{n-j}\left[\sum_{s=0}^{j-1}(-1)^s e_m^{(2j-1-s)}(x)\overline{f}^{(s)}(x)\mid_a^b\right.\\&\left.+(-1)^j\int_a^b e_m^{(j)}(x)\overline{f}^{(j)}(x)dx\right]\\&=\sum_{j=0}^n\binom{n}{j}k^{n-j}\int_a^b e_m^{(j)}(x)\overline{f}^{(j)}(x)dx\\&=(e_m,f)_n.\end{aligned}\tag{4.6}$$

In particular, if $\widetilde{\lambda}_m\in\left\{\left(\frac{2m\pi}{b-a}\right)^2+k\mid m\in\mathbb{N}_0\right\}$ is the eigenvalue of A associated with the eigenfunction e_m, we see from (4.6) that $\widetilde{\lambda}_m^n(e_m,f)=(e_m,f)_n$ and, in particular, from (3.7)

$$(A^n e_m,e_r)=\widetilde{\lambda}_m^n(e_m,e_r)=\widetilde{\lambda}_m^n\left(\frac{b-a}{2}\right)\delta_{m,r}\quad(m,r=0,1,\ldots).\tag{4.7}$$

Comparing (4.6) (with $f = e_r$) and (4.7), we obtain the following theorem.

THEOREM 4.8. *The set E of eigenfunctions of A, defined in (3.9) and (3.10), are orthogonal in each Hilbert space H_n. In fact*

$$(e_m, e_r)_n = \widetilde{\lambda}_m^n \left(\frac{b-a}{2}\right) \delta_{m,r} \quad (m, r = 0, 1, \ldots),$$

where $\widetilde{\lambda}_m \in \left\{ \left(\frac{2m\pi}{b-a}\right)^2 + k \mid m \in \mathbb{N}_0 \right\}$ is the eigenvalue of A associated with the eigenfunction e_m. More specifically, with $y_0(x)$, $y_{n,1}(x)$, and $y_{n,2}(x)$ $(n \in \mathbb{N})$, defined in (3.5) and (3.6) respectively, it is the case that, for $m \in \mathbb{N}$ and $j = 1, 2$,

$$\|y_0\|_n = k^{n/2} \left(\frac{b-a}{2}\right)^{1/2}; \; \|y_{m,j}\|_n = \left(\left(\frac{2m\pi}{b-a}\right)^2 + k\right)^{n/2} \left(\frac{b-a}{2}\right)^{1/2}. \tag{4.8}$$

Consequently, for each $n \in N$, the set

$$E_n := \{Z_{m,n,1}\}_{m \in \mathbb{N}_0} \cup \{Z_{m,n,2}\}_{m \in \mathbb{N}}, \tag{4.9}$$

where

$$\begin{cases} Z_{m,n,1}(x) = \begin{cases} \dfrac{1}{k^{n/2}\sqrt{b-a}} & \text{if } m = 0 \\ \sqrt{\dfrac{2}{b-a}} \dfrac{1}{\sqrt{\left(\left(\frac{2m\pi}{b-a}\right)^2 + k\right)^n}} \cos\left(\frac{2m\pi}{b-a} x\right) & \text{if } m \in \mathbb{N} \end{cases} \\ Z_{m,n,2}(x) = \sqrt{\dfrac{2}{b-a}} \dfrac{1}{\sqrt{\left(\left(\frac{2m\pi}{b-a}\right)^2 + k\right)^n}} \sin\left(\frac{2m\pi}{b-a} x\right) \quad \text{if } m \in \mathbb{N} \end{cases} \tag{4.10}$$

is an orthonormal set in H_n.

Later in this section (see Theorem 4.10), we prove that E_n is, in fact, a *complete* orthonormal set in H_n for each $n \in \mathbb{N}$.

For later purposes, we need the following equality involving finite linear combinations of eigenfunctions of A - the so-called *trigonometric polynomials*. Let N_1, M_1, N, and M be non-negative integers with $N_1 \leq N$ and $M_1 \leq M$ and let $\alpha_m, \beta_r \in \mathbb{C}$ $(m = N_1, \ldots, N;$ $r = M_1, \ldots, M)$. Let

$$p(x) = \sum_{m=N_1}^{N} \alpha_m e_m(x), \; q(x) = \sum_{r=M_1}^{M} \beta_r e_r(x).$$

Then $p, q \in H_n$ for all $n \in \mathbb{N}$ and, by (4.6) and linearity, we see that

$$\begin{aligned} (A^n p, q) &= \sum_{m=N_1}^{N} \sum_{r=M_1}^{M} \alpha_m \overline{\beta}_r (A^n e_m, e_r) \\ &= \sum_{m=N_1}^{N} \sum_{r=M_1}^{M} \alpha_m \overline{\beta}_r (e_m, e_r)_n \\ &= (\sum_{m=N_1}^{N} \alpha_m e_m, \sum_{r=M_1}^{M} \beta_r e_r)_n \\ &= (p, q)_n. \end{aligned} \tag{4.11}$$

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We are now in position to prove the following main theorem; we remind the reader of the definitions of V_n, $(\cdot,\cdot)_n$, and H_n, given in Definition 4.2.

THEOREM 4.9. *For each* $n \in \mathbb{N}$, *let*

$$H_n = (V_n, (\cdot,\cdot)_n) \tag{4.12}$$

be defined by

$$V_n := \{f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b],\ f^{(j)}(a) = f^{(j)}(b)\ (j = 0, 1, \ldots, n-1); \quad f^{(n)} \in L^2[a,b]\}, \tag{4.13}$$

and

$$(f,g)_n := \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int_a^b f^{(j)}(x)\overline{g}^{(j)}(x)dx \quad (f,g \in V_n). \tag{4.14}$$

Then H_n *is the* n^{th} *left-definite space associated with the pair* (H, A).

Proof. Let $n \in \mathbb{N}$. We are required to establish properties (i)-(v) in Definition 2.1.

(i) H_n is a Hilbert space
This is proved in Theorem 4.7.

(ii) $\mathcal{D}(A^n) \subset V_n$
Let $f \in \mathcal{D}(A^n)$. Since the set $E = \{e_m \mid m \in \mathbb{N}_0\}$ of eigenfunctions of A is a complete orthonormal set in $L^2[a,b]$, we see that

$$p_j := \sum_{m=0}^{j} c_m e_m \to f \text{ as } j \to \infty \text{ in } L^2[a,b], \tag{4.15}$$

where $\{c_m\}$ are the Fourier coefficients of f in $L^2[a,b]$, defined by

$$c_m := \int_a^b f(t)e_m(t)dt = (f, e_m) \quad (m \in \mathbb{N}_0).$$

Since $A^n f \in L^2[a,b]$, we also have

$$\sum_{m=0}^{j} d_m e_m \to A^n f \text{ as } j \to \infty \text{ in } L^2[a,b], \tag{4.16}$$

where

$$d_m = (A^n f, e_m) \quad (m \in \mathbb{N}_0).$$

With $\widetilde{\lambda}_m$ denoting the eigenvalue of A associated with e_m, we see from the self-adjointness of A that

$$d_m = (A^n f, e_m) = (f, A^n e_m) = \widetilde{\lambda}_m^n (f, e_m) = \widetilde{\lambda}_m^n c_m.$$

Substituting this identity into (4.16), and using the linearity of A^n, we obtain

$$A^n p_j \to A^n f \text{ as } j \to \infty \text{ in } L^2[a,b], \tag{4.17}$$

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where p_j is defined in (4.15). From (4.11), (4.15), and (4.17), it follows that

$$\begin{aligned} \|p_j - p_r\|_n^2 &= (A^n(p_j - p_r), p_j - p_r) \\ &\to 0 \text{ as } j, r \to \infty; \end{aligned}$$

that is to say, $\{p_j\}_{j\in\mathbb{N}}$ is Cauchy in H_n. From the completeness of H_n, there exists $g \in V_n \subset L^2[a,b]$ such that

$$p_j \to g \text{ in } H_n.$$

From the inequality

$$\|p_j - g\|_n^2 \geq k^n \|p_j - g\|^2,$$

we see that

$$p_j \to g \text{ as } j \to \infty \text{ in } L^2[a,b]. \tag{4.18}$$

Comparing (4.15) and (4.18), we see that $f = g \in V_n$; consequently, $\mathcal{D}(A^n) \subset V_n$ as required.

(iii) $\underline{\mathcal{D}(A^n) \text{ is dense in } H_n}$
Since E, defined in (3.10), is contained in $\mathcal{D}(A^n)$, it suffices to show that E is a complete orthogonal set in H_n. From this, it will follow that the vector subspace $T \subset \mathcal{D}(A^n)$ of all trigonometric polynomials (that is, all finite linear combinations of elements from the set E defined in (3.10)) is dense in H_n and, consequently, $\mathcal{D}(A^n)$ is dense in H_n. To this end, suppose

$$(e_m, f)_n = 0 \quad (m \in \mathbb{N}_0)$$

for some $f \in H_n$. From (4.6), we see that

$$0 = (e_m, f)_n = (A^n e_m, f) = \widetilde{\lambda}_m^n (e_m, f),$$

where $\widetilde{\lambda}_m$ is the eigenvalue associated with e_m. Since $\widetilde{\lambda}_m > 0$, we see that

$$(e_m, f) = 0 \quad (m \in \mathbb{N}_0). \tag{4.19}$$

As remarked in Section 3, E is a complete orthonormal set in $L^2[a,b]$; consequently, (4.19) implies that $f = 0$ in $L^2[a,b]$. From this, it is clear that $f = 0$ in H_n, thereby completing the proof that E is a complete orthogonal set in H_n.

(iv) $\underline{(f,f)_n \geq k^n(f,f) \text{ for all } f \in V_n}$
This is clear from the definition of $(\cdot,\cdot)_n$:

$$\begin{aligned} (f,f)_n &= \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int_a^b \left| f^{(j)}(x) \right|^2 dx \\ &\geq k^n \int_a^b \left| f^{(j)}(x) \right|^2 dx = k^n (f,f). \end{aligned}$$

(v) $\underline{(A^n f, g) = (f,g)_n \text{ for all } f \in \mathcal{D}(A^n) \text{ and } g \in V_n}$
Let $f \in \mathcal{D}(A^n)$ and $g \in V_n$. From (4.11), we see that

$$(A^n p, q) = (p, q)_n \tag{4.20}$$

for all trigonometric polynomials

$$p = \sum_{m=0}^{N} \alpha_m e_m, \; q = \sum_{m=0}^{M} \beta_m e_m.$$

From part (iii) of this proof, we see that the space T of all trigonometric polynomials is dense in H_n. Hence there exists $\{p_j\}_{j\in\mathbb{N}_0}, \{q_j\}_{j\in\mathbb{N}_0} \subset T$ such that

$$p_j \to f, \; q_j \to g \text{ as } j \to \infty \text{ in } H_n. \tag{4.21}$$

Since convergence in H_n implies convergence in $L^2[a,b]$ (from (iv) in Definition 2.1), we see that

$$p_j \to f, \; q_j \to g \text{ as } j \to \infty \text{ in } L^2[a,b]. \tag{4.22}$$

Moreover, from part (ii) of this proof, we see that

$$A^n p_j \to A^n f \text{ as } j \to \infty \text{ in } L^2[a,b]. \tag{4.23}$$

Consequently, from (4.20), (4.21), (4.22), and (4.23), we see that

$$(A^n f, g) = \lim_{j\to\infty} (A^n p_j, q_j) = \lim_{j\to\infty} (p_j, q_j)_n = (f,g)_n.$$

This completes the proof of (v) and finishes the proof of the theorem. □

The following result, part of which is proved in step (iii) of the above theorem, is the analogous result for the classical Fourier expansion theorem in $L^2[a,b]$ stated in Theorem 3.1 and further strengthens Theorem 4.8 for the Hilbert-Sobolev space setting H_n. In particular, note the identities in (4.26) and (4.27); these formulae relate the Fourier coefficients of f relative to the orthonormal basis E_n of $H_n \subset L^2[a,b]$ to the Fourier coefficients of f relative to the orthonormal basis E of $L^2[a,b]$.

THEOREM 4.10. (*Fourier Expansion Theorem in Left-Definite Spaces*) *For each $n \in \mathbb{N}$, let*

$$E_n = \{Z_{m,n,1}\}_{m\in\mathbb{N}_0} \cup \{Z_{m,n,2}\}_{m\in\mathbb{N}}$$

be as in (4.9) and (4.10). Then E_n is a complete orthonormal set in H_n. Furthermore, let $f \in H_n \subset L^2[a,b]$ and, for each $N \in \mathbb{N}$, define the partial sums

$$S_{N,n}(f)(x) = \sum_{m=0}^{N} A_{m,n}(f)\cos\left(\frac{2m\pi}{b-a}x\right) + \sum_{m=1}^{N} B_{m,n}(f)\sin\left(\frac{2m\pi}{b-a}x\right) \quad (x \in [a,b]),$$

where $\{A_{m,n}(f)\}_{m\in\mathbb{N}_0}$ and $\{B_{m,n}(f)\}_{m\in\mathbb{N}}$ are the Fourier coefficients of f relative to E_n defined by

$$A_{m,n}(f) := (f, Z_{m,n,1})_n \quad (m \in \mathbb{N}_0) \tag{4.24}$$

and

$$B_{m,n}(f) := (f, Z_{m,n,2})_n \quad (m \in \mathbb{N}). \tag{4.25}$$

Then

(a) $\|f - S_{N,n}(f)\|_n \to 0$ *as* $N \to \infty$;

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(b) $\|f\|_n^2 = \sum_{m=0}^{\infty} |A_{m,n}(f)|^2 + \sum_{m=1}^{\infty} |B_{m,n}(f)|^2$;
(c)

$$A_{m,n}(f) = a_m(f)\lambda_m^{n/2} \quad (m \in \mathbb{N}_0) \tag{4.26}$$

$$B_{m,n}(f) = b_m(f)\lambda_m^{n/2} \quad (m \in \mathbb{N}), \tag{4.27}$$

where $\{a_m(f)\}_{m\in\mathbb{N}_0}$ *and* $\{b_m(f)\}_{m\in\mathbb{N}}$ *are the Fourier coefficients of* f*, defined in* (3.12), (3.13)*, and* (3.14)*, relative to the orthonormal basis* E*, given in* (3.8) *and* (3.9)*, in* $L^2[a, b]$*, and where* $\{\lambda_m\}_{m\in\mathbb{N}_0}$ *are the eigenvalues of* A *defined in* (3.4)*.*

Proof. The proofs of parts (a) and (b) are standard for any complete orthonormal set in a Hilbert space; see [11, Theorem 4.18]. With regards to (c), a calculation shows

$$A_{0,n}(f) = \frac{k^{n/2}}{\sqrt{b-a}} \int_a^b f(x)dx = a_0(f)k^{n/2} = a_0(f)\lambda_0^{n/2}. \tag{4.28}$$

For $m \in \mathbb{N}$,

$$A_{m,n}(f) = \sqrt{\frac{2}{b-a}} \frac{1}{\sqrt{\left(\left(\frac{2m\pi}{b-a}\right)^2 + k\right)^n}} \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int_a^b f^{(j)}(x) \cos^{(j)}\left(\frac{2m\pi}{b-a}x\right) dx. \tag{4.29}$$

Moreover, since

$$\int_a^b f^{(j)}(x) \sin^{(j)}\left(\frac{2m\pi}{b-a}x\right) dx = \left(\frac{2m\pi}{b-a}\right)^{2j} \int_a^b f(x) \sin\left(\frac{2m\pi}{b-a}x\right) dx \quad (j \in \mathbb{N}_0),$$

$$\int_a^b f^{(j)}(x) \cos^{(j)}\left(\frac{2m\pi}{b-a}x\right) dx = \left(\frac{2m\pi}{b-a}\right)^{2j} \int_a^b f(x) \cos\left(\frac{2m\pi}{b-a}x\right) dx \quad (j \in \mathbb{N}_0),$$

and

$$\sum_{j=0}^{n} \binom{n}{j} k^{n-j} \left(\frac{2m\pi}{b-a}\right)^{2j} = \left(\left(\frac{2m\pi}{b-a}\right)^2 + k\right)^n = \lambda_m^n,$$

we see from (4.29) that

$$A_{m,n}(f) = \sqrt{\frac{2}{b-a}} \lambda_m^{n/2} \int_a^b f(x) \cos\left(\frac{2m\pi}{b-a}x\right) dx = a_m(f)\lambda_m^{n/2} \quad (m \in \mathbb{N}); \tag{4.30}$$

together, (4.28) and (4.30) establish (4.26); this completes the proof. □

REMARK 4.11. *From* (4.26) *and* (4.27) *it follows readily that*

$$a_m(f) = o(m^{-n}) \text{ as } m \to \infty, \tag{4.31}$$

and

$$b_m(f) = o(m^{-n}) \text{ as } m \to \infty \tag{4.32}$$

for any $f \in H_n$*. Indeed, to establish* (4.31)*, note that since*

$$\lim_{m\to\infty} A_{m,n}(f) = 0,$$

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we see, from (4.26), that

$$\begin{aligned} 0 = \lim_{m\to\infty} a_m(f)\lambda_m^{n/2} &= \frac{1}{\left(\frac{2\pi}{b-a}\right)^n}\lim_{m\to\infty} m^n a_m(f)\lambda_m^{n/2}m^{-n} \\ &= \frac{1}{\left(\frac{2\pi}{b-a}\right)^n}\lim_{m\to\infty} m^n a_m(f)\sqrt{\left(\left(\frac{2\pi}{b-a}\right)^2+\frac{k}{m^2}\right)^n} \\ &= \lim_{m\to\infty} m^n a_m(f). \end{aligned}$$

We note that (4.31) and (4.32) can also be seen by n integrations by parts on (3.13) and (3.14) and an application of the Riemann-Lebesgue lemma (see [11, Section 5.14]). For general information on the order of magnitude of Fourier coefficients, see [5, Chapter I, Section 4].

Lastly, by combining Theorem 4.9 with Theorems 2.5 and 2.6, we obtain the following result concerning the sequence of left-definite operators $\{A_n\}_{n\in\mathbb{N}}$ associated with the pair (H, A).

THEOREM 4.12. *Let $n \in \mathbb{N}$ and let $H_n = (V_n, (\cdot,\cdot)_n)$ be the n^{th} left-definite operator associated with the pair (H, A). Define the operator $A_n : \mathcal{D}(A_n) \subset H_n \to H_n$ by*

$$\mathcal{D}(A_n) := \{f : [a, b] \to \mathbb{C} \mid f^{(j)} \in AC[a, b],\ f^{(j)}(a) = f^{(j)}(b)\ (j = 0, 1, \ldots, n+1); \\ f^{(n+2)} \in L^2[a, b]\}$$

and

$$A_n f := \ell[f] \quad (f \in \mathcal{D}(A_n)),$$

where $\ell[\cdot]$ is the differential expression given in (3.2). Then A_n is the n^{th} left-definite operator associated with the pair (H, A). In particular, A_n is self-adjoint in H_n and the spectrum $\sigma(A_n)$ is a purely discrete point spectrum given explicitly by

$$\sigma(A_n) = \sigma(A) = \left\{\left(\frac{2m\pi}{b-a}\right)^2 + k \mid m \in \mathbb{N}_0\right\}.$$

5. Concluding Remarks. In this last section, we focus on some special cases concerning the operator A and the sequences of left-definite spaces $\{H_n\}_{n\in\mathbb{N}}$ and left-definite operators $\{A_n\}_{n\in\mathbb{N}}$ obtained in the previous section.

REMARK 5.1. *For an arbitrary self-adjoint operator A in a Hilbert space that is bounded below by a positive constant we see, from Theorem 2.2, that the domain $\mathcal{D}(A^{1/2})$ of its positive square root $A^{1/2}$ is given by the first left-definite vector space V_1. For our specific operator A, defined in (3.3), we have the explicit characterization of this domain:*

$$\mathcal{D}(A^{1/2}) = \{f : [a, b] \to \mathbb{C} \mid f \in AC[a, b]; f(a) = f(b); f' \in L^2[a, b]\}. \tag{5.1}$$

REMARK 5.2. *From Theorem 4.12, the domain of the first left-definite operator A_1, which is a self-adjoint operator in the first left-definite space H_1, is given by*

$$\mathcal{D}(A_1) = \{f : [a, b] \to \mathbb{C} \mid f^{(j)} \in AC[a, b] \text{ and } f^{(j)}(a) = f^{(j)}(b)\ (j = 0, 1, 2); \\ f^{(3)} \in L^2[a, b]\}.$$

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Notice that $\mathcal{D}(A_1) = V_3$ *is also the domain of* $A^{3/2}$*. The domain of the second left-definite operator* A_2*, which is self-adjoint in the Hilbert space* $H_2 = (V_2, (\cdot,\cdot)_2)$*, defined by*

$$V_2 = \{f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b] \text{ and } f^{(j)}(a) = f^{(j)}(b)\ (j = 0,1);\\ f'' \in L^2[a,b]\}$$

and

$$(f,g)_2 = \int_a^b \left(f''(x)\overline{g}''(x) + 2kf'(x)\overline{g}'(x) + k^2 f(x)\overline{g}(x)\right) dx \quad (\text{see } (4.3)),$$

is given by

$$\mathcal{D}(A_2) = \{f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b] \text{ and } f^{(j)}(a) = f^{(j)}(b)\ (j = 0,1,2,3);\\ f^{(4)} \in L^2[a,b]\}.$$

Observe that $V_2 = \mathcal{D}(A)$*; see* (3.3)*. By way of another example, the domain of the third left-definite operator* A_3 *is the fifth left-definite vector space* V_5*, given explicitly by*

$$V_5 = \{\{f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b] \text{ and } f^{(j)}(a) = f^{(j)}(b)\ (j = 0,1,2,3,4);\\ f^{(5)} \in L^2[a,b]\}.$$

Furthermore, $V_5 = \mathcal{D}(A^{5/2})$*.*

REMARK 5.3. *In general, given an unbounded linear operator* A*, it is difficult to explicitly characterize the domains of its powers* $A^{n/2}$ $(n \in \mathbb{N})$*. However, using Theorem* 2.2 *together with Theorem* 4.9*, we quickly see that*

$$\mathcal{D}(A^{n/2}) = \{f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b],\ f^{(j)}(a) = f^{(j)}(b)\ (j = 0,1,\ldots,n-1);\\ f^{(n)} \in L^2[a,b]\}.$$

In particular observe that, as n *increases, the number of "boundary conditions" appearing in the definition of* $\mathcal{D}(A^{n/2})$ *increases accordingly. On the other hand, observe from* (5.1)*, that one less boundary condition is needed to describe the domain of the square root of* A*. Indeed, there are two boundary conditions* $f(a) = f(b)$ *and* $f'(a) = f'(b)$ *needed to ensure the self-adjointness of* A *but only* $f(a) = f(b)$ *is needed for* $\mathcal{D}(A^{1/2})$*.*

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