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APPROXIMATING EIGENVALUES OF BOUNDARY VALUE PROBLEMS BY USING THE HERMITE-GAUSS SAMPLING METHOD*

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Abstract. The Hermite-Gauss sampling operator was introduced by Asharabi and Prestin (2015) to approximate a function from a wide class of entire functions, using few samples from the function and its first derivative. This operator converges at the rate $e^{-(2\pi-\sigma h)N}/\sqrt{N}$, and has been applied to construct a new sampling method for approximating the eigenvalues of boundary value problems whose eigenvalues are real and simple. In this paper, we use the first derivative of this operator to approximate non-real and non-simple eigenvalues of boundary value problems. For this task, we estimate two types of errors associated with the first derivative of the Hermite-Gauss operator. These error estimates give us the possibility to establish the error analysis when the eigenvalues are not real or not algebraically simple. Illustrative examples are discussed and show the effectiveness of the proposed method. Our numerical results are compared with the results of sinc-Gaussian sampling method.

Key words. sinc methods, approximating eigenvalues, boundary value problems, error bounds, rate of convergence

AMS subject classifications. 34L16, 94A20, 65L15, 65N15

1. Introduction.

Consider the following Sturm-Liouville problem

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \quad x \in [0, b], \quad \lambda \in \mathbb{C}, \tag{1.1}$$

with mixed-type boundary conditions

$$U_1(y) := \alpha_1 y(0, \lambda) + \beta_1 y(b, \lambda) = 0, \quad U_2(y) := \alpha_2 y'(0, \lambda) + \beta_2 y'(b, \lambda) = 0, \tag{1.2}$$

where q is a complex-valued function satisfying $q \in L^1[0, b]$ and α_i, β_i $(i = 1, 2)$ are complex numbers satisfying $|\alpha_i|, |\beta_i| > 0$ for all $i = 1, 2$. When $\alpha_i = -\beta_i = 1$ $(i = 1, 2)$, conditions (1.2) are called periodic and when $\alpha_i = \beta_i = 1$ they are called antiperiodic. For the spectral theory of periodic second-order differential equations, we refer the reader to [12, 17]. The eigenvalues of the problem (1.1)–(1.2) are in general complex numbers and are not necessarily simple, as in the case of separated boundary conditions, and this is a major difficulty.

Let $y_1(\cdot, \lambda)$ and $y_2(\cdot, \lambda)$ be the solutions of (1.1) satisfying the initial conditions

$$y_1(0, \lambda) = y_2'(0, \lambda) = 1, \qquad y_1'(0, \lambda) = y_2(0, \lambda) = 0. \tag{1.3}$$

The characteristic function, which is also called Hill's discriminant, of problem (1.1)–(1.2) can be written as follows:

$$\mathcal{D}(\lambda) := \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix} = \alpha_1\alpha_2 + \beta_1\beta_2 + \alpha_2\beta_1 y_1(b, \lambda) + \alpha_1\beta_2 y_2'(b, \lambda); \tag{1.4}$$

cf., e.g., [15]. Its zeroes are the square roots of eigenvalues of the problem (1.1)–(1.2). Observe that $\mathcal{D}(\lambda)$ is an entire function of λ because the solutions $y_1(\cdot, \lambda)$ and $y_2(\cdot, \lambda)$ are entire functions in λ, [12, p. 19]. As it is well known, we cannot compute the eigenvalues of boundary value problems exactly. We are thus forced to establish approximation methods to compute these eigenvalues. Methods for approximating eigenvalues of second-order boundary

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value problems have received much attention, and they are classified into several categories. An example of these categories is based on sampling theory and contains five methods: classical sinc (1996), regularized sinc (2007), sinc-Gaussian (2008), Hermite (2012) and Hermite-Gauss (2016), cf., e.g., [1–4, 10, 11, 13] and their references. All these sampling methods can be used for approximating non-real and non-simple eigenvalues except the last one, i.e., Hermite-Gauss, although it has the highest rate of convergence compared with the rest of the sampling methods, cf. [6]. The author of [4] uses the Hermite-Gauss operator, which is established by Asharabi and Prestin (2105), to construct a new sampling method. This method is established to approximate eigenvalues of second-order boundary value problems which have only real and simple eigenvalues. Likewise, Asharabi and Tharwat (2017) use this method to approximate eigenvalues of Dirac systems with discontinuities at several points, cf. [8]. The eigenvalues of the Dirac problem were also real and simple. It seems that the Hermite-Gauss method gives more accurate results than the other sampling methods. However, we can confidently say that the Hermite-Gauss method will be the best sampling method because it has the highest rate of convergence, cf. [4].

In this paper, we employ the Hermite-Gauss sampling operator to approximate double and complex eigenvalues of boundary value problems. More specifically, we apply this method to approximate eigenvalues of the problem (1.1)–(1.2). For this task, we estimate truncation and amplitude errors associated with the derivative of the Hermite-Gauss operator. The bounds of these errors allow us to establish the error analysis when the eigenvalues are neither real nor algebraically simple. Moreover, this method will be used for computing the complex eigenvalues of boundary value problems as non-self-adjoint eigenvalue problems with separate type conditions, cf. [10].

The rest of the paper is organized as follows: the next section is devoted to introducing two newer results on the derivative of the Hermite-Gauss operator. Moreover, we state two known results on the Hermite-Gauss operator. In Section 3, we describe in detail our method and we provide the error analysis of the present method in Section 4. Section 5 deals with the illustrative examples which show the efficiency and accuracy of the present method. Lastly, Section 6 concludes the paper.

2. Hermite-Gauss operator.

In this section, we state two known results on bounds of approximating entire functions using a Hermite-Gauss operator. We also introduce two new results on bounds of approximating the first derivative of entire functions using the derivative of the Hermite-Gauss operator in complex domains. These results will be used when we establish the error analysis of the present method. Let $\mathcal{E}_\sigma$, $\sigma > 0$, be the class of all entire functions that satisfy the condition

$$|f(z)| \le M\, \mathrm{e}^{\sigma|\Im z|}, \quad z \in \mathbb{C}, \tag{2.1}$$

where M is a positive real number. On the class $\mathcal{E}_\sigma$, Asharabi and Prestin [6], introduced the Hermite-Gauss localization operator, i.e., $\mathcal{H}_{h,N} : \mathcal{E}_\sigma \longrightarrow \mathcal{E}_\sigma$, as follows

$$\begin{aligned}\mathcal{H}_{h,N}[f](z) := \sum_{n\in\mathbb{Z}_N(z)} &\left\{\left(1+\frac{2\beta(z-nh)^2}{h^2N}\right) f(nh) + (z-nh)f'(nh)\right\} \\ &\times \operatorname{sinc}^2(h^{-1}z-n)\exp\left(-\frac{\beta}{N}\left(h^{-1}z-n\right)^2\right),\end{aligned} \tag{2.2}$$

where $h \in (0, 2\pi/\sigma]$, $\beta := (2\pi - h\sigma)/2$ and the sinc function is defined via

$$\operatorname{sinc}(t) := \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}.$$

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The index of the summation (2.2) belongs to the set

$$\mathbb{Z}_N(z) := \left\{ n \in \mathbb{Z} : |\lfloor h^{-1}\Re\zeta + 1/2 \rfloor - n| \le N \right\},$$

which depends only on the real part of z, where $N \in \mathbb{Z}^+$ and $\lfloor x \rfloor$ denotes the integer part of x. The Hermite-Gauss operator is extended in different ways, cf. [5, 7]. The error bound of approximating $\mathcal{E}_\sigma$-functions by the Hermite-Gauss operator is introduced in [6, Corollary 2.3]. If $f \in \mathcal{E}_\sigma$, then we have for $z \in \mathbb{C}$, $|\Im z| < N$,

$$|f(z) - \mathcal{H}_{h,N}[f](z)| \le 2 \left|\sin^2(h^{-1}\pi z)\right| \|f\|_\infty A_N \left(h^{-1}\Im z\right) \frac{\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta N}}, \tag{2.3}$$

where

$$\begin{aligned} A_N(t) &:= \frac{4\mathrm{e}^{\beta t^2/N}}{\sqrt{\pi\beta N}\left(1 - (t/N)^2\right)} + \frac{\mathrm{e}^{-2\beta t}}{(1 - \mathrm{e}^{-2\pi(N+t)})^2} + \frac{\mathrm{e}^{2\beta t}}{(1 - \mathrm{e}^{-2\pi(N-t)})^2} \\ &= 2\cosh(2\beta t) + O(N^{-1/2}), \quad \text{as} \quad N \to \infty. \end{aligned} \tag{2.4}$$

The following theorem is devoted to estimate a bound for the error associated with the derivative of the Hermite-Gauss operator, i.e., $\left|f'(z) - \mathcal{H}'_{h,N}[f](z)\right|$, where $f \in \mathcal{E}_\sigma$. From the definition of the class $\mathcal{E}_\sigma$, we can verify that if $f \in \mathcal{E}_\sigma$, then $f' \in \mathcal{E}_\sigma$.

THEOREM 2.1. *Let $f \in \mathcal{E}_\sigma$, $\sigma > 0$. Then we have for $z \in \mathbb{C}$, $|\Im\zeta| < N$,*

$$\left|f'(z) - \mathcal{H}'_{h,N}[f](z)\right| \le 2\|f\|_\infty \psi_N(z) A_N \left(h^{-1}\Im z\right) \frac{\mathrm{e}^{-\beta N}}{h\sqrt{\pi\beta N}}, \tag{2.5}$$

where $h \in (0, 2\pi/\sigma]$, $\beta := (2\pi - h\sigma)/2$, A_N is defined in (2.4), and ψ_N is defined by

$$\psi_N(z) := \left(2\pi \left|\cos(\pi h^{-1} z)\right| + \left(N^{-1} + 2\beta\right) \left|\sin(\pi h^{-1} z)\right|\right) \left|\sin(\pi h^{-1} z)\right|. \tag{2.6}$$

Proof. Since $f \in \mathcal{E}_\sigma$, the authors of [6, p. 423] used the residue theorem to prove that

$$f(z) - \mathcal{H}_{h,N}[f](z) = \frac{\sin^2(\pi h^{-1} z)}{2\pi\mathrm{i}} \int_{\mathcal{R}} \frac{f(\zeta h)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z - \zeta\right)^2}}{(\zeta - h^{-1}z)\sin^2(\pi\zeta)}\, \mathrm{d}\zeta, \tag{2.7}$$

for all $z \in \mathbb{C}$, where $\mathcal{R}$ is the rectangle whose vertices are located at

$$\pm (N + 1/2) + \lfloor h^{-1}\Re z + 1/2 \rfloor + i(h^{-1}\Im z \pm N).$$

Differentiating (2.7) gives

$$\begin{aligned} f'(z) - \mathcal{H}'_{h,N}[f](z) = {} & \frac{\sin(\pi h^{-1} z)\cos(\pi h^{-1} z)}{h\mathrm{i}} \int_{\mathcal{R}} \frac{f(\zeta h)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z - \zeta\right)^2}}{(\zeta - h^{-1}z)\sin^2(\pi\zeta)}\, \mathrm{d}\zeta \\ & + \frac{\sin^2(\pi h^{-1} z)}{2\pi h\mathrm{i}} \int_{\mathcal{R}} \frac{f(\zeta h)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z - \zeta\right)^2}}{(\zeta - h^{-1}z)^2\sin^2(\pi\zeta)}\, \mathrm{d}\zeta \\ & + \frac{\beta\sin^2(\pi h^{-1} z)}{\pi h N\mathrm{i}} \int_{\mathcal{R}} \frac{f(\zeta h)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z - \zeta\right)^2}}{\sin^2(\pi\zeta)}\, \mathrm{d}\zeta. \end{aligned} \tag{2.8}$$

The first integral of (2.8) is estimated in [6, pp. 423–425] as follows

$$\left|\int_{\mathcal{R}} \frac{f(\zeta h)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z - \zeta\right)^2}}{(\zeta - h^{-1}z)\sin^2(\pi\zeta)}\, \mathrm{d}\zeta\right| \le 4\pi\|f\|_\infty A_N \left(h^{-1}\Im z\right) \frac{\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta N}}. \tag{2.9}$$

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Applying the same technique of [6], we obtain the following estimates

$$\left|\int_{\mathcal{R}} \frac{f(\zeta h)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-\zeta\right)^2}}{(\zeta - h^{-1}z)^2 \sin^2(\pi\zeta)}\, \mathrm{d}\zeta\right| \le 4\pi \|f\|_\infty A_N\left(h^{-1}\Im z\right) \frac{\mathrm{e}^{-\beta N}}{N\sqrt{\pi\beta N}}, \tag{2.10}$$

$$\left|\int_{\mathcal{R}} \frac{f(\zeta h)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-\zeta\right)^2}}{\sin^2(\pi\zeta)}\, \mathrm{d}\zeta\right| \le 4\pi \|f\|_\infty A_N\left(h^{-1}\Im z\right) \frac{\sqrt{N}\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta}}. \tag{2.11}$$

Applying the triangle inequality to (2.8) and combining the result with the inequalities (2.9)–(2.11), we obtain (2.5). ◻

As we mentioned above, the operators $\mathcal{H}_{h,N}$ and $\mathcal{H}'_{h,N}$ are approximating the function $f \in \mathcal{E}_\sigma$ and its derivative by using only few samples from the function f and its first derivative. However, sometimes these samples cannot be computed explicitly. For this reason the author of [4] established a bound for the amplitude error associated with the Hermite-Gauss operator $\mathcal{H}_{h,N}$. The amplitude error arises when the exact values $f^{(i)}(nh)$, $i = 0, 1$, of (2.2) are replaced by closer approximate ones. Let $\varepsilon > 0$ be sufficiently small such that

$$\sup_{n\in\mathbb{Z}_N(z)} |\widetilde{f}^{(i)}(nh) - f^{(i)}(nh)| < \varepsilon, \quad i = 0, 1, \tag{2.12}$$

where $\widetilde{f}^{(i)}(nh)$ is an approximation for the sample $f^{(i)}(nh)$. Then the amplitude error associated with (2.2) is defined by $\mathcal{H}_{h,N}[f](z) - \mathcal{H}_{h,N}[\widetilde{f}](z)$, $z \in \mathbb{C}$. Assume that (2.12) holds. Then for $z \in \mathbb{C}$, $|\Im z| < N$ we have [4, Theorem 2.1]

$$\begin{aligned} &\left|\mathcal{H}_{h,N}[f](z) - \mathcal{H}_{h,N}[\widetilde{f}](z)\right| \\ &\quad \le 2\varepsilon\left(1 + \frac{2\beta}{\pi^2 N} + \frac{h}{\pi}\right)\left(1 + \sqrt{N/\beta\pi}\right) \mathrm{e}^{\left(2\pi+\beta h^{-1}\right)h^{-1}|\Im z|}\mathrm{e}^{-\beta/4N}, \end{aligned} \tag{2.13}$$

where $h \in (0, 2\pi/\sigma]$, $\beta := (2\pi - h\sigma)/2$. Likewise, we define an amplitude error associated with the operator $\mathcal{H}'_{h,N}$ by $\mathcal{H}'_{h,N}[f](z) - \mathcal{H}'_{h,N}[\widetilde{f}](z)$, $f \in \mathcal{E}_\sigma$ and $z \in \mathbb{C}$. The following theorem provides a bound for this amplitude error, which will be required for studying the error analysis of our method when an approximated eigenvalue is double.

THEOREM 2.2. *Let $\sigma > 0$, $h \in (0, 2\pi/\sigma]$ and $\beta := (2\pi - h\sigma)/2$. Assume that (2.12) holds. Then we have for $z \in \mathbb{C}$, $|\Im z| < N$,*

$$\begin{aligned} &\left|\mathcal{H}'_{h,N}[f](z) - \mathcal{H}'_{h,N}[\widetilde{f}](z)\right| \\ &\quad \le 2\varepsilon\, C_{h,N}\left(1 + \sqrt{N/\beta\pi}\right) \mathrm{e}^{\left(2\pi+\beta h^{-1}\right)h^{-1}|\Im z|}\mathrm{e}^{-\beta/4N}, \end{aligned} \tag{2.14}$$

where $C_{h,N}$ is defined by

$$C_{h,N} := 1 + \frac{2}{h}\left(N\pi + \frac{\beta}{\pi}\right)\left(\frac{1}{N} + \frac{2\beta(h+1)}{N\pi h} + 1 + h\right). \tag{2.15}$$

Proof. According to the amplitude error associated with $\mathcal{H}_{h,N}$, we have

$$\begin{aligned} &\mathcal{H}_{h,N}[f](z) - \mathcal{H}_{h,N}[\widetilde{f}](z) := \\ &\sum_{n\in\mathbb{Z}_N(z)} \left\{f(nh) - \widetilde{f}(nh)\right\}\left(1 + \frac{2\beta(z-nh)^2}{h^2 N}\right)\operatorname{sinc}^2(h^{-1}z - n)\,\mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-n\right)^2} \\ &+ \sum_{n\in\mathbb{Z}_N(z)} \left\{f'(nh) - \widetilde{f}'(nh)\right\}(z - nh)\operatorname{sinc}^2(h^{-1}z - n)\,\mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-n\right)^2}. \end{aligned} \tag{2.16}$$

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Differentiating (2.16), we obtain

$$\begin{aligned}
&\mathcal{H}'_{h,N}[f](z) - \mathcal{H}'_{h,N}[\widetilde{f}](z) = \\
&2 \sum_{n\in\mathbb{Z}_N(z)} \left\{ f(nh) - \widetilde{f}(nh) \right\} \varphi_{n,1}(z)\, \mathrm{sinc}(h^{-1}z - n)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-n\right)^2} \\
&\quad + \frac{4\beta}{Nh\pi} \sum_{n\in\mathbb{Z}_N(z)} \left\{ f(nh) - \widetilde{f}(nh) \right\} \varphi_{n,2}(z) \sin(\pi h^{-1}z - n\pi)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-n\right)^2} \\
&\quad + \sum_{n\in\mathbb{Z}_N(z)} \left\{ f'(nh) - \widetilde{f}'(nh) \right\} \varphi_{n,3}(z) \mathrm{sinc}(h^{-1}z - n)\, \mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-n\right)^2},
\end{aligned} \tag{2.17}$$

where the functions $\varphi_{n,j}(z)$, $j = 1, 2, 3$, are defined by

$$\begin{aligned}
\varphi_{n,1}(z) &:= \mathrm{sinc}'(h^{-1}z - n) + \beta N^{-1}\pi^{-1}h^{-1} \sin(\pi h^{-1}z - n\pi), \\
\varphi_{n,2}(z) &:= (z - nh)\ \left(\mathrm{sinc}'(h^{-1}z - n) - \beta N^{-1}\pi^{-1}h^{-1} \sin(\pi h^{-1}z - n\pi)\right), \\
\varphi_{n,3}(z) &:= \mathrm{sinc}(h^{-1}z - n) + 2(z - nh)\left(\mathrm{sinc}'(h^{-1}z - n) - \beta N^{-1}\pi^{-1} \sin(\pi h^{-1}z - n\pi)\right).
\end{aligned}$$

Since

$$\mathrm{sinc}(h^{-1}z - n) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \mathrm{e}^{i\,\zeta(z-nh)} \mathrm{d}\zeta, \quad z \in \mathbb{C},\ n \in \mathbb{Z}, \tag{2.18}$$

we have

$$\left|\mathrm{sinc}(h^{-1}z - n)\right| \le \mathrm{e}^{\pi h^{-1}|\Im z|}, \quad z \in \mathbb{C},\ n \in \mathbb{Z}. \tag{2.19}$$

Differentiating (2.18), we obtain

$$\mathrm{sinc}'(h^{-1}z - n) = \frac{ih}{2\pi} \int_{-\pi/h}^{\pi/h} \zeta \mathrm{e}^{i\zeta(z-nh)} \mathrm{d}\zeta, \quad z \in \mathbb{C},\ n \in \mathbb{Z}.$$

Therefore

$$\left|\,\mathrm{sinc}'(h^{-1}z - n)\right| \le \pi h^{-1} \mathrm{e}^{\pi h^{-1}|\Im z|}, \quad z \in \mathbb{C},\ n \in \mathbb{Z}. \tag{2.20}$$

From the definition of the set $\mathbb{Z}_N(z)$, we can see that for all $z \in \mathbb{C}$, $|\Im z| < N$,

$$\sup_{n\in\mathbb{Z}_N(z)} |z - nh| \le (1 + h)\, N. \tag{2.21}$$

By using inequalities (2.19), (2.20), (2.21), and $|\sin(z)| \le \mathrm{e}^{|\Im z|}$, we get the following estimates

$$\begin{aligned}
&\sup_{n\in\mathbb{Z}_N(z)} \left|\varphi_{n,1}(z)\, \mathrm{sinc}(h^{-1}z - n)\right| \le \frac{1}{hN} \left(\pi N + \frac{\beta}{\pi}\right) \mathrm{e}^{2\pi h^{-1}|\Im z|}, \\
&\sup_{n\in\mathbb{Z}_N(z)} \left|\varphi_{n,2}(z)\ \sin(\pi h^{-1}z - n\pi)\right| \le \left(\frac{h+1}{h}\right) \left(N\pi + \frac{\beta}{\pi}\right) \mathrm{e}^{2\pi h^{-1}|\Im z|}, \\
&\sup_{n\in\mathbb{Z}_N(z)} \left|\varphi_{n,3}(z)\, \mathrm{sinc}(h^{-1}z - n)\right| \le \left(1 + \frac{2(h+1)}{h} \left(\pi N + \frac{\beta}{\pi}\right)\right) \mathrm{e}^{2\pi h^{-1}|\Im z|},
\end{aligned} \tag{2.22}$$

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where $z \in \mathbb{C}$ and $|\Im z| < N$. Applying the triangle inequality to (2.17) and in view of (2.12) and (2.22), we obtain

$$\left|\mathcal{H}'_{h,N}[f](z) - \mathcal{H}'_{h,N}[\widetilde{f}](z)\right| \leq \varepsilon\, C_{h,N} \mathrm{e}^{2\pi h^{-1}|\Im z|} \sum_{n\in\mathbb{Z}_N(z)} \left|\mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-n\right)^2}\right|, \tag{2.23}$$

where $C_{h,N}$ is defined in (2.15). The summation in (2.23) is estimated in [1, pp. 297–298] as follows

$$\sum_{n\in\mathbb{Z}_N(z)} \left|\mathrm{e}^{-\frac{\beta}{N}\left(h^{-1}z-n\right)^2}\right| \leq 2\left(1+\sqrt{N/\beta\pi}\right) \mathrm{e}^{\beta h^{-2}|\Im z|}\, \mathrm{e}^{-\beta/4N}. \tag{2.24}$$

From (2.24) and (2.23), we get (2.14). □

3. The method. This section is devoted to the construction of our method. The main idea is to establish an entire function using the Hermite-Gauss sampling operator which will be very close to the characteristic function of the problem (1.1)–(1.3), i.e., $\mathcal{D}(\lambda)$. The zeros of this function will be accurate approximations to the zeros of $\mathcal{D}(\lambda)$. Using the method of the variation of constants, we can write the solutions of the problem (1.1)–(1.3) as the fundamental solutions of the following Volterra integral equations

$$y_1(t,\lambda) := \cos(t\lambda) + \int_0^t \frac{\sin\left((t-x)\lambda\right)}{\lambda}\, q(x) y_1(x,\lambda) dx, \tag{3.1}$$

$$y_2(t,\lambda) := \frac{\sin(t\lambda)}{\lambda} + \int_0^t \frac{\sin\left((t-x)\lambda\right)}{\lambda}\, q(x) y_2(x,\lambda) dx. \tag{3.2}$$

Differentiating (3.2) with respect to t, we obtain

$$y_2'(t,\lambda) := \cos(t\lambda) + \int_0^t \cos((t-x)\lambda)\, q(x) y_2(x,\lambda) dx. \tag{3.3}$$

For convenience, we denote the two Volterra operator in (3.1) and (3.3), respectively, as follows

$$\mathcal{T}\left[y(x,\lambda)\right](t) := \int_0^t \frac{\sin\left((t-x)\lambda\right)}{\lambda}\, q(x) y(x,\lambda) dx,$$
$$\widetilde{\mathcal{T}}\left[y(x,\lambda)\right](t) := \int_0^t \cos((t-x)\lambda)\, q(x) y(x,\lambda) dx.$$

The operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are defined from $C[0,b]$ to $C[0,b]$. The method of successive approximations yields

$$y_1(t,\mu) = \sum_{n=0}^{\infty} \mathcal{T}^n \left[\cos(x\lambda)\right](t), \tag{3.4}$$

and

$$y_2'(t,\lambda) = \cos(t\lambda) + \sum_{n=0}^{\infty} \widetilde{\mathcal{T}}\mathcal{T}^n \left[\frac{\sin(x\lambda)}{\lambda}\right](t), \tag{3.5}$$

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where $\mathcal{T}^0$ is the identity operator. The convergence in (3.4) and (3.5) is uniform on $[0, b]$ for any $\lambda \in \mathbb{C}$. Combining (3.4) and (3.5) in (1.4), $\mathcal{D}(\lambda)$ can be written as

$$\mathcal{D}(\lambda) = \alpha_1\alpha_2 + \beta_1\beta_2 + \alpha_1\beta_2\cos(b\lambda) + \alpha_2\beta_1\sum_{n=0}^{\infty}\mathcal{T}^n\left[\cos(x\lambda)\right](b) + \alpha_1\beta_2\sum_{n=0}^{\infty}\widetilde{\mathcal{T}}\mathcal{T}^n\left[\frac{\sin(x\lambda)}{\lambda}\right](b).$$

Let us split $\mathcal{D}(\lambda)$ into two parts via

$$\mathcal{D}(\lambda) := \mathcal{K}_k(\lambda) + \mathcal{U}_k(\lambda), \quad k \in \mathbb{N}_0, \tag{3.6}$$

where $\mathcal{K}_k(\lambda)$ is the known part

$$\mathcal{K}_k(\lambda) := \alpha_1\alpha_2 + \beta_1\beta_2 + \alpha_1\beta_2\cos(b\lambda) + \alpha_2\beta_1\sum_{n=0}^{k}\mathcal{T}^n\left[\cos(x\lambda)\right](b) + \alpha_1\beta_2\sum_{n=0}^{k}\widetilde{\mathcal{T}}\mathcal{T}^n\left[\frac{\sin(x\lambda)}{\lambda}\right](b), \tag{3.7}$$

and $\mathcal{U}_k(\lambda)$ involves the infinite sum of integral operators

$$\mathcal{U}_k(\lambda) := \alpha_2\beta_1\sum_{n=k+1}^{\infty}\mathcal{T}^n\left[\cos(x\lambda)\right](b) + \alpha_1\beta_2\sum_{n=k+1}^{\infty}\widetilde{\mathcal{T}}\mathcal{T}^n\left[\frac{\sin(x\lambda)}{\lambda}\right](b). \tag{3.8}$$

In the following result, we prove that $\mathcal{U}_k(\cdot)$ belongs to the class $\mathcal{E}_b$ and then we will approximate $\mathcal{U}_k(\cdot)$ using the Hermite-Gauss formula (2.2).

LEMMA 3.1. *Let* $q(\cdot) \in L^1[0, b]$. *Then we have* $\mathcal{U}_k(\cdot) \in \mathcal{E}_b$ *for all* $k \in \mathbb{N}_0$.

Proof. Since $\mathcal{D}(\lambda)$ is an entire function, $\mathcal{U}_k(\lambda)$ is an entire function in λ for all $k \in \mathbb{N}_0$. To complete the proof it remains to prove that $\mathcal{U}_k(\lambda)$ satisfies the condition (2.1) of the class $\mathcal{E}_b$. Since $q(\cdot) \in L^1[0, b]$, we have, cf., e.g., [9], for all $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$,

$$\left|\sum_{n=k+1}^{\infty} T^n\left[\cos(x\lambda)\right](b)\right| \le \rho_{k+1}\mathrm{e}^{b|\Im\lambda|}, \tag{3.9}$$

$$\left|\sum_{n=k+1}^{\infty}\widetilde{T}T^n\left[\frac{\sin(x\lambda)}{\lambda}\right](b)\right| \le c\,b\,\tau\,\rho_{k+1}\mathrm{e}^{b|\Im\lambda|}, \tag{3.10}$$

where $\rho_k := \sum_{n=k}^{\infty}\frac{(c\,b\,\tau)^n}{n!}$, $\tau := \int_0^b |q(t)|dt$, and $c := 1.709$. Applying the triangle inequality to (3.8) and using estimates (3.9) and (3.10), we obtain $|\mathcal{U}_k(\lambda)| \le M_k\mathrm{e}^{b|\Im\lambda|}$, where

$$M_k := \left(|\alpha_2\beta_1| + cb\tau\,|\alpha_1\beta_2|\right)\rho_{k+1}, \quad k \in \mathbb{N}_0. \tag{3.11}$$

Therefore $\mathcal{U}_k(\cdot) \in \mathcal{E}_b$ for all $k \in \mathbb{N}_0$. ◻

Constructing our method requires the computation of the samples $\mathcal{U}_k(nh)$ and $\mathcal{U}_k'(nh)$, where $n \in \mathbb{Z}_N(\lambda)$ and $h \in (0, \pi/b]$. From now on, we restrict the parameter h to be in the interval $(0, \pi/b]$ because we will use sinc-Gaussian sampling, which is defined only in this interval. As predicted by the error estimates of the operators $\mathcal{H}_{h,N}$ and $\mathcal{H}'_{h,N}$, the accuracy of

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the approximations increases when h decreases and N is fixed. According to (3.6) and (1.4), we have

$$\begin{aligned}\mathcal{U}_k(nh) &= \mathcal{D}(nh) - \mathcal{K}_k(nh)\\ &= \alpha_1\alpha_2 + \beta_1\beta_2 + \alpha_2\beta_1 y_1(b,nh) + \alpha_1\beta_2 y_2'(b,nh) - \mathcal{K}_k(nh).\end{aligned} \tag{3.12}$$

It is clear that the samples $\mathcal{U}_k(nh)$ and $\mathcal{U}_k'(nh)$ cannot be determined explicitly in the general case. That is why the amplitude error usually appears. Let $\widetilde{\mathcal{U}}_k(nh)$ be the approximation of the samples $\mathcal{U}_k(nh)$ when $y_1(b,nh)$ and $y_2'(b,nh)$ are computed numerically at the nodes $\{nh\}_{n\in\mathbb{Z}_N(\lambda)}$. Also, let $\widetilde{\mathcal{U}}_k'(nh)$ be the approximations of the samples $\mathcal{U}_k'(nh)$ which are computed using the values $\widetilde{\mathcal{U}}_k(nh)$ through the sinc-Gaussian sampling, cf., e.g., [1, 16],

$$\widetilde{\mathcal{U}}_k'(nh) := \sum_{n\in\mathbb{Z}_N(\lambda)} \widetilde{\mathcal{U}}_k(nh) \left[\operatorname{sinc}(h^{-1}\lambda - n)\, \mathrm{e}^{-\frac{\alpha}{N}\left(h^{-1}\lambda - n\right)^2}\right]'_{\lambda=nh}. \tag{3.13}$$

Here the use of the sinc-Gaussian sampling operator (3.13) is guaranteed because $\mathcal{U}_k(\cdot) \in \mathcal{E}_b$. The error analysis associated with formula (3.13) is given in [1, Theorems 2.1, 2.2]. Now we are ready to define the following interesting function using the Hermite-Gauss operator $\mathcal{H}_{h,N}$,

$$\widetilde{\mathcal{D}}_{N,k}(\lambda) := \mathcal{K}_k(\lambda) + \mathcal{H}_{h,N}[\widetilde{\mathcal{U}}_k](\lambda), \quad h\in(0,\pi/b],\ k\in\mathbb{N}_0, \tag{3.14}$$

where $\mathcal{K}_k(\lambda)$ is defined in (3.7) and $\mathcal{H}_{h,N}$ is the Hermite-Gauss operator with the parameter $\beta := (2\pi - hb)/2$. The function $\widetilde{\mathcal{D}}_{N,k}(\lambda)$ is determined explicitly and will be very close to the characteristic function $\mathcal{D}(\lambda)$, as we will see in Theorem 3.2. Therefore, the zeros of $\widetilde{\mathcal{D}}_{N,k}(\lambda)$ will be very close to the desired zeros of $\mathcal{D}(\lambda)$. Finally, we assume that there exists a very small $\varepsilon > 0$ such that

$$\sup_{n\in\mathbb{Z}_N(\lambda)} \left|\mathcal{U}_k^{(i)}(nh) - \widetilde{\mathcal{U}}_k^{(i)}(nh)\right| < \varepsilon, \quad i=1,2, \quad k\in\mathbb{N}_0. \tag{3.15}$$

THEOREM 3.2. *Let $N\in\mathbb{N}$ and $k\in\mathbb{N}_0$. Then for $|\Im\lambda| < N$, we have the following estimate*

$$\left|\mathcal{D}(\lambda) - \widetilde{\mathcal{D}}_{N,k}(\lambda)\right| \le \mathcal{T}_{N,k,h}(\lambda) + \mathcal{A}_{\varepsilon,h,N}\left(\Im\lambda\right), \tag{3.16}$$

where $\mathcal{D}$ is the characteristic function of the problem (1.1)–(1.2) and $\widetilde{\mathcal{D}}_{N,k}$ is defined in (3.14). The functions $\mathcal{T}_{N,k,h}$ and $\mathcal{A}_{\varepsilon,h,N}$ are defined by

$$\mathcal{T}_{N,k,h}(\lambda) := 2\left|\sin^2(h^{-1}\pi\lambda)\right| M_k A_N\left(h^{-1}\Im\lambda\right)\frac{\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta N}}, \tag{3.17}$$

$$\mathcal{A}_{\varepsilon,h,N}(t) := 2\varepsilon\left(1 + \frac{2\beta}{\pi^2 N} + \frac{h}{\pi}\right)\left(1 + \sqrt{N/\beta\pi}\right)\mathrm{e}^{\left(2\pi+\beta h^{-1}\right)h^{-1}|t|}\mathrm{e}^{-\beta/4N}, \tag{3.18}$$

where $h\in(0,\pi/b]$, $\beta := (2\pi - hb)/2$, and A_N and M_k are defined in (2.4) and (3.11), respectively. Moreover, $\widetilde{\mathcal{D}}_{N,k} \to \mathcal{D}$ uniformly on any compact subset of $\mathbb{C}$ when $N\to\infty$ and $\varepsilon\to 0$.

Proof. According to (3.6) and (3.14), we have

$$\begin{aligned}\left|\mathcal{D}(\lambda) - \widetilde{\mathcal{D}}_{N,k}(\lambda)\right| &= \left|\mathcal{U}_k(\lambda) - \mathcal{H}_{h,N}[\widetilde{\mathcal{U}}_k](\lambda)\right|\\ &\le \left|\mathcal{U}_k(\lambda) - \mathcal{H}_{h,N}[\mathcal{U}_k](\lambda)\right| + \left|\mathcal{H}_{h,N}[\mathcal{U}_k](\lambda) - \mathcal{H}_{h,N}[\widetilde{\mathcal{U}}_k](\lambda)\right|.\end{aligned} \tag{3.19}$$

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Since the function $\mathcal{U}_k \in \mathcal{E}_b$, we can approximate $\mathcal{U}_k$ by the Hermite-Gauss operator and then we have, cf. (2.3),

$$|\mathcal{U}_k(\lambda) - \mathcal{H}_{h,N}[\mathcal{U}_k](\lambda)| \le \mathcal{T}_{N,k,h}(\lambda), \quad |\Im\lambda| < N, \tag{3.20}$$

where $\mathcal{T}_{N,k,h}$ is defined in (3.17). Since condition (3.15) holds, we have, cf. (2.13),

$$\left|\mathcal{H}_{h,N}[\mathcal{U}_k](\lambda) - \mathcal{H}_{h,N}[\widetilde{\mathcal{U}}_k](\lambda)\right| \le \mathcal{A}_{\varepsilon,h,N}\left(\Im\lambda\right), \quad |\Im\lambda| < N, \tag{3.21}$$

where $\mathcal{A}_{\varepsilon,h,N}$ is defined in (3.18). Combining (3.21), (3.20), and (3.19) implies (3.16). In view of (3.17) and (3.18), the right-hand side of (3.16) goes to zero uniformly when $N \to \infty$ and $\varepsilon \to 0$. Therefore, $\widetilde{\mathcal{D}}_{N,k} \to \mathcal{D}$ uniformly on any compact subset of $|\Im\lambda| < N$ for all $k \in \mathbb{N}_0$. ◻

In the following theorem, we will prove that the function $\widetilde{\mathcal{D}}'_{N,k}(\lambda)$ will be very close to $\mathcal{D}'(\lambda)$. This result will be used in the next section to estimate the absolute error $|\lambda - \lambda_{N,k}|$ when $\lambda_{N,k}$ is a double eigenvalue of the problem (1.1)–(1.2).

THEOREM 3.3. *Let $N \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then for $|\Im\lambda| < N$, we have the following estimate*

$$\left|\mathcal{D}'(\lambda) - \widetilde{\mathcal{D}}'_{N,k}(\lambda)\right| \le \mathfrak{T}_{N,k,h}(\lambda) + \mathfrak{A}_{\varepsilon,h,N}\left(\Im\lambda\right), \tag{3.22}$$

where $\mathfrak{T}_{N,k,h}$ and $\mathfrak{A}_{\varepsilon,h,N}$ are defined by

$$\mathfrak{T}_{N,k,h}(\lambda) := 2M_k\psi_N(\lambda)A_N\left(h^{-1}\Im\lambda\right)\frac{\mathrm{e}^{-\beta N}}{h\sqrt{\pi\beta N}}, \tag{3.23}$$

$$\mathfrak{A}_{\varepsilon,h,N}(t) := \varepsilon\, C_{h,N}\, \mathrm{e}^{\left(2\pi+\beta h^{-1}\right)h^{-1}|t|}\mathrm{e}^{-\beta/4N}, \tag{3.24}$$

where $h \in (0, \pi/b]$, $\beta := (2\pi - hb)/2$, and A_N, M_k, ψ_N, and $C_{h,N}$ are defined in (2.4), (3.11), (2.6), and (2.15), respectively. Moreover, $\widetilde{\mathcal{D}}'_{N,k} \to \mathcal{D}'$ uniformly on any compact subset of $\mathbb{C}$ when $N \to \infty$ and $\varepsilon \to 0$.

Proof. According to (3.6) and (3.14), we have

$$\begin{aligned} &\left|\mathcal{D}'(\lambda) - \widetilde{\mathcal{D}}'_{N,k}(\lambda)\right| \\ &\quad \le \left|\mathcal{U}'_k(\lambda) - \mathcal{H}'_{h,N}[\mathcal{U}_k](\lambda)\right| + \left|\mathcal{H}'_{h,N}[\mathcal{U}_k](\lambda) - \mathcal{H}'_{h,N}[\widetilde{\mathcal{U}}_k](\lambda)\right|. \end{aligned} \tag{3.25}$$

Since the function $\mathcal{U}_k \in \mathcal{E}_b$, we can approximate $\mathcal{U}_k$ by the derivative of the Hermite-Gauss operator $\mathcal{H}'_{h,N}$, and then we have, cf. Theorem 2.1,

$$\left|\mathcal{U}'_k(\lambda) - \mathcal{H}'_{h,N}[\mathcal{U}_k](\lambda)\right| \le \mathfrak{T}_{N,k,h}(\lambda), \quad |\Im\lambda| < N, \tag{3.26}$$

where $\mathfrak{T}_{N,k,h}$ is defined in (3.23). Since condition (3.15) holds, we have, cf. Theorem 2.2,

$$\left|\mathcal{H}'_{h,N}[\mathcal{U}_k](\lambda) - \mathcal{H}'_{h,N}[\widetilde{\mathcal{U}}_k](\lambda)\right| \le \mathfrak{A}_{\varepsilon,h,N}\left(\Im\lambda\right), \quad |\Im\lambda| < N, \tag{3.27}$$

where $\mathfrak{A}_{\varepsilon,h,N}$ is defined in (3.24). Combining (3.27), (3.26), and (3.25) implies (3.22). In view of (3.23) and (3.24), the right-hand side of (3.22) goes to zero uniformly when $N \to \infty$ and $\varepsilon \to 0$, and therefore $\widetilde{\mathcal{D}}'_{N,k} \to \mathcal{D}'$ uniformly on any compact subset of $|\Im\lambda| < N$ for all $k \in \mathbb{N}_0$. ◻

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4. The error analysis. In this section, we derive error bounds for $|\lambda^* - \lambda_{N,k}|$, where $(\lambda^*)^2$ is an eigenvalue of the problem (1.1)–(1.2), i.e., $\mathcal{D}(\lambda^*) = 0$, and $(\lambda_{N,k})^2$ is its desired approximation, i.e., $\widetilde{\mathcal{D}}'_{N,k}(\lambda_{N,k}) = 0$. The computation of the error bounds for $|\lambda^* - \lambda_{N,k}|$ depends on whether λ^* is simple or double. Now we state the complex mean value theorem [14], which will be used in the proof of our results. Assume that f is holomorphic on an open convex set $\Omega \in \mathbb{C}$. Let a and b be distinct points in Ω and $\Gamma_{a,b}$ be the line joining a and b in Ω. Then there exist $z_1, z_2 \in \Gamma_{a,b}$ such that

$$\Re\left(\frac{f(b)-f(a)}{b-a}\right) = \Re(f'(z_1)), \qquad \Im\left(\frac{f(b)-f(a)}{b-a}\right) = \Im(f'(z_2)).$$

In the following result, we find a bound for the error $|\lambda^* - \lambda_{N,k}|$ when the zero λ^* of $\mathcal{D}(\lambda)$ is simple, i.e., $\mathcal{D}(\lambda^*) = 0,\ \mathcal{D}'(\lambda^*) \neq 0$.

THEOREM 4.1. *Let $(\lambda^*)^2$ be a simple eigenvalue of (1.1)–(1.2) and $(\lambda_{N,k})^2$ its approximation. Then, for $|\Im\lambda| < N$, we have the following estimate*

$$|\lambda^* - \lambda_{N,k}| < \frac{\mathcal{T}_{N,k,h}(\lambda_{N,k}) + \mathcal{A}_{\varepsilon,h,N}\left(\Im\lambda_{N,k}\right)}{\inf\limits_{z_1,z_2\in\Gamma_{\lambda^*,\lambda_{N,k}}} \sqrt{(\Re\mathcal{D}'(z_1))^2 + (\Im\mathcal{D}'(z_2))^2}}, \tag{4.1}$$

where $\Gamma_{\lambda^*,\lambda_{N,k}}$ is the line joining $\lambda^*, \lambda_{N,k}$ in the strip $|\Im\lambda| < N$. Moreover, $|\lambda^* - \lambda_{N,k}| \longrightarrow 0$ when $N \to \infty$ and $\varepsilon \to 0$.

Proof. Replacing λ by $\lambda_{N,k}$ in (3.16), we obtain

$$|\mathcal{D}(\lambda_{N,k}) - \mathcal{D}(\lambda^*)| < \mathcal{T}_{N,k,h}(\lambda_{N,k}) + \mathcal{A}_{\varepsilon,h,N}\left(\Im\lambda_{N,k}\right), \tag{4.2}$$

where we have used $\widetilde{\mathcal{D}}_{N,k}(\lambda_{N,k}) = \mathcal{D}(\lambda^*) = 0$. Using the complex mean value theorem above, there exist $z_1, z_2 \in \Gamma_{\lambda^*,\lambda_N}$ such that

$$\begin{aligned}
&\left|\frac{\mathcal{D}(\lambda_{N,k}) - \mathcal{D}(\lambda^*)}{\lambda^* - \lambda_{N,k}}\right| \\
&= \sqrt{\left(\Re\left\{\frac{\mathcal{D}(\lambda_{N,k}) - \mathcal{D}(\lambda^*)}{\lambda_{N,k} - \lambda^*}\right\}\right)^2 + \left(\Im\left\{\frac{\mathcal{D}(\lambda_{N,k}) - \mathcal{D}(\lambda^*)}{\lambda_{N,k} - \lambda^*}\right\}\right)^2} \\
&= \sqrt{(\Re\mathcal{D}'(z_1))^2 + (\Im\mathcal{D}'(z_2))^2}.
\end{aligned} \tag{4.3}$$

From (4.3) and (4.2), we get

$$|\lambda^* - \lambda_{N,k}| \sqrt{(\Re\mathcal{D}'(z_1))^2 + (\Im\mathcal{D}'(z_2))^2} < \mathcal{T}_{N,k,h}(\lambda_{N,k}) + \mathcal{A}_{\varepsilon,h,N}\left(\Im\lambda_{N,k}\right).$$

Since $\mathcal{D}'(\lambda^*) \neq 0$ and N is sufficiently large, we have

$$\inf_{z_1,z_2\in\Gamma_{\lambda^*,\lambda_N}} \sqrt{(\Re\mathcal{D}'(z_1))^2 + (\Im\mathcal{D}'(z_2))^2} > 0, \tag{4.4}$$

and hence dividing on the left hand side of (4.4), we obtain (4.1). The remaining part of the proof follows from the fact $\mathcal{T}_{N,k,h} \to 0$ as $N \to \infty$ and $\mathcal{A}_{\varepsilon,h,N} \to 0$ as $\varepsilon \to 0$. ◻

Now we estimate the error $|\lambda^* - \lambda_{N,k}|$ when λ^* is a double zero of $\mathcal{D}(\lambda)$, i.e., $\mathcal{D}(\lambda^*) = 0,\ \mathcal{D}'(\lambda^*) = 0$ and $\mathcal{D}''(\lambda^*) \neq 0$.

THEOREM 4.2. *Let $(\lambda^*)^2$ be a double eigenvalue of (1.1)–(1.2) and $(\lambda_{N,k})^2$ its approximation. Then, for $|\Im\lambda| < N$, we have the following estimate*

$$|\lambda^* - \lambda_{N,k}| < \frac{\mathfrak{T}_{N,k,h}(\lambda_{N,k}) + \mathfrak{A}_{\varepsilon,h,N}\left(\Im\lambda_{N,k}\right)}{\inf\limits_{z_1,z_2\in\Gamma_{\lambda^*,\lambda_{N,k}}} \sqrt{(\Re\mathcal{D}''(z_1))^2 + (\Im\mathcal{D}''(z_2))^2}}, \quad k \in \mathbb{N}_0, \tag{4.5}$$

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where $\Gamma_{\lambda^*,\lambda_{N,k}}$ *is the line joining* $\lambda^*, \lambda_{N,k}$ *in the strip* $|\Im\lambda| < N$. *Moreover,* $|\lambda^* - \lambda_{N,k}| \longrightarrow 0$ *when* $N \to \infty$ *and* $\varepsilon \to 0$.

Proof. Since λ^* is a double zero of $\mathcal{D}(\lambda)$, $\mathcal{D}'(\lambda^*) = \widetilde{\mathcal{D}}'_{N,k}(\lambda_{N,k}) = 0$. Therefore after replacing λ by $\lambda_{N,k}$, inequality (3.22) becomes

$$|\mathcal{D}'(\lambda_{N,k}) - \mathcal{D}'(\lambda^*)| \le \mathfrak{T}_{N,k,h}(\lambda_{N,k}) + \mathfrak{A}_{\varepsilon,h,N}\left(\Im\lambda_{N,k}\right). \tag{4.6}$$

Applying the complex mean value theorem and similarly treatments of (4.3), we get

$$|\mathcal{D}'(\lambda_{N,k}) - \mathcal{D}'(\lambda^*)| = |\lambda^* - \lambda_{N,k}| \sqrt{(\Re\mathcal{D}''(z_1))^2 + (\Im\mathcal{D}''(z_2))^2}. \tag{4.7}$$

Combining (4.6) and (4.7) implies

$$|\lambda^* - \lambda_{N,k}| \sqrt{(\Re\mathcal{D}''(z_1))^2 + (\Im\mathcal{D}''(z_2))^2} < \mathfrak{T}_{N,k,h}(\lambda_{N,k}) + \mathfrak{A}_{\varepsilon,h,N}\left(\Im\lambda_{N,k}\right), \tag{4.8}$$

where $z_1, z_2 \in \Gamma_{\lambda^*,\lambda_{N,k}}$. Since $\mathcal{D}''(\lambda^*) \neq 0$ and N is sufficiently large,

$$\inf_{z_1,z_2\in\Gamma_{\lambda^*,\lambda_{N,k}}} \sqrt{(\Re\mathcal{D}''(z_1))^2 + (\Im\mathcal{D}''(z_2))^2} > 0,$$

and hence (4.8) becomes (4.5). The remaining part of the proof follows from the fact that $\mathfrak{T}_{N,k,h} \to 0$ as $N \to \infty$ and $\mathfrak{A}_{\varepsilon,h,N} \to 0$ as $\varepsilon \to 0$. □

In the following corollary, we give special cases of the above bounds for $|\lambda^* - \lambda_{N,k}|$ when λ^* is real

COROLLARY 4.3. *If* $(\lambda^*)^2$ *is real and simple eigenvalue of problem* (1.1)–(1.2)*, then estimate* (4.1) *becomes*

$$|\lambda^* - \lambda_{N,k}| < \frac{\mathcal{T}_{N,k,h}(\lambda_{N,k}) + \mathcal{A}_{\varepsilon,h,N}\left(0\right)}{\inf\limits_{t\in I_{\lambda^*,\lambda_{N,k}}} |\mathcal{D}'(t)|},$$

where $I_{\lambda^*,\lambda_{N,k}} := \left[\min(\lambda^*, \lambda_{N,k}), \max(\lambda^*, \lambda_{N,k})\right]$. *Also, when* $(\lambda^*)^2$ *is real and a double eigenvalue, the estimate* (4.8) *turns out to be*

$$|\lambda^* - \lambda_{N,k}| < \frac{\mathfrak{T}_{N,k,h}(\lambda_{N,k}) + \mathfrak{A}_{\varepsilon,h,N}\left(0\right)}{\inf\limits_{t\in I_{\lambda^*,\lambda_{N,k}}} |\mathcal{D}''(t)|}.$$

5. Illustrative examples. In this section, we introduce four illustrative examples. In Examples 5.1–5.3, it is a simple task to compute the characteristic function, $\mathcal{D}(\lambda)$, explicitly, but we cannot compute the last example in a closed form. Therefore, the amplitude error does not appear in Examples 5.1–5.3, and we will replace the notation $\widetilde{\mathcal{D}}_{N,h}$ by $\mathcal{D}_{N,h}$. We can classify the sinc methods for approximating eigenvalues of boundary value problems into two groups. The convergence rate of the first group (classical sinc, regularized sinc and Hermite sampling methods) is of a polynomial order, see, e.g., [3, 11, 13], and for the second group (sinc-Gaussian and Hermite-Gauss methods) of an exponential order, see, e.g., [2, 6]. Therefore, our results in all examples are compared only with the results for the sinc-Gaussian method. Moreover, we would like to note that the convergence rate of the sinc-Gaussian method is $O\left(e^{-(\pi-h\sigma)N/2}/\sqrt{N}\right)$, while the Hermite-Gauss method has the order $O\left(e^{-(2\pi-h\sigma)N/2}/\sqrt{N}\right)$, where $h \in (0, \pi/\sigma]$ and σ is a positive number depending on the class of functions, $\mathcal{E}_\sigma$. Denote by E_G and E_H the absolute errors associated with results of the sinc-Gaussian and Hermite-Gauss operators, respectively. In all examples, we compute the bounds of the sinc-Gaussian and Hermite-Gauss operators which are denoted by B_G and B_H, respectively.

EXAMPLE 5.1. Consider the anti-periodic boundary value problem

$$-y''(x) - y(x) = \lambda^2 y(x), \quad 0 \le x \le 1,$$
$$y(0,\lambda) + y(1,\lambda) = 0, \qquad y'(0,\lambda) + y'(1,\lambda) = 0.$$

The eigenvalues of this problem are computed in [1, 2] using the classical sinc and sinc-Gaussian sampling methods, respectively. The characteristic function is

$$\mathcal{D}(\lambda) := 4\cos^2\left(\frac{\sqrt{1+\lambda^2}}{2}\right),$$

and thus the exact eigenvalues are $\lambda_k^2 = ((2k-1)\pi)^2 - 1,\ k \in \mathbb{Z}$. Obviously, all eigenvalues of this problem are algebraically double. Letting $k = 1$ in (3.14) and calculating $\mathcal{K}_1(\lambda)$ implies

$$\mathcal{D}_{N,1}(\lambda) = 2 + 2\cos(\lambda) - \frac{\sin(\lambda)}{\lambda} + \frac{\sin(\lambda) - \lambda\cos(\lambda)}{8\lambda^3} + \mathcal{H}_{h,N}[\mathcal{U}_1](\lambda),$$

where $h \in (0,\pi]$. Table 5.1 lists the first four approximate eigenvalues using the sinc-Gaussian and Hermite-Gauss methods with absolute errors and bounds when $N = 6$ and $h = 1$. The branch of the square roots is $0 \le \arg\lambda < \pi$.

TABLE 5.1
Comparison of the Hermite-Gauss and sinc-Gaussian methods, $N = 6$ and $h = 1$.

λ	Sinc-Gaussian	Hermite-Gauss	E_G	E_H	B_G	B_H
λ_1	2.978187294507986	2.978188107055666	8.12561×10^{-7}	1.36908×10^{-11}	1.91244×10^{-5}	4.76006×10^{-10}
λ_2	9.371576508390618	9.371576154030823	3.54413×10^{-7}	5.30829×10^{-11}	1.66062×10^{-5}	6.48980×10^{-9}
λ_3	15.676099832673613	15.676099962242917	1.29601×10^{-7}	3.19460×10^{-11}	1.73726×10^{-5}	6.32729×10^{-9}
λ_4	21.968400295432037	21.968400389046398	9.36126×10^{-8}	1.71596×10^{-12}	1.72394×10^{-5}	6.29020×10^{-10}

In Figure 5.1, we compare the characteristic function $\mathcal{D}(\lambda)$ and its approximation $\mathcal{D}_{6,1}(\lambda)$ on the interval $[0, 25]$.

EXAMPLE 5.2. The boundary value problem

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \quad 0 \le x \le 2,$$
$$2y(0,\lambda) + y(2,\lambda) = 0, \qquad y'(0,\lambda) + y'(2,\lambda) = 0,$$

is in the form of problem (1.1)–(1.2) when $\alpha_1 = 2$, $\alpha_2 = \beta_1 = \beta_2 = 1$ and

$$q(x) := \begin{cases} -1, & 0 \le x < 1, \\ 0, & 1 \le x \le 2. \end{cases}$$

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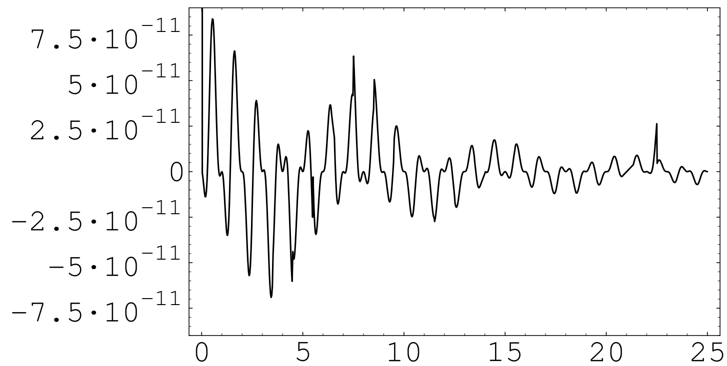

FIG. 5.1. $\mathcal{D}(\lambda) - \mathcal{D}_{6,1}(\lambda)$.

It is easy to see that

$$\mathcal{D}(\lambda) := 3 + 3\cos\left(2\sqrt{\lambda^2+1}\right).$$

Thus the exact eigenvalues are $\lambda_k^2 = ((k-1/2)\pi)^2 - 1,\ k \in \mathbb{Z}$ and all eigenvalues are algebraically double. Taking $k = 2$ in (3.14) and calculating the function $\mathcal{K}_2$ gives

$$\begin{aligned}\mathcal{D}_{N,2}(\lambda) = 3 + 3\cos(2\lambda) - \frac{3\sin(2\lambda)}{\lambda} + \frac{3(\sin(2\lambda) - 2\lambda\cos(2\lambda))}{4\lambda^3} \\ + \frac{6\lambda\cos(2\lambda) + (-3+4\lambda^2)\sin(2\lambda))}{12\lambda^5} + \mathcal{H}_{h,N}[\mathcal{U}_2](\lambda),\end{aligned}$$

where $\lambda \in \mathbb{C}$ and $h \in (0, \pi]$. Figure 5.2 shows the difference between the function $\mathcal{D}(\lambda)$ and its approximation $\mathcal{D}_{5,1/2}(\lambda)$ on the interval $[0, 12]$.

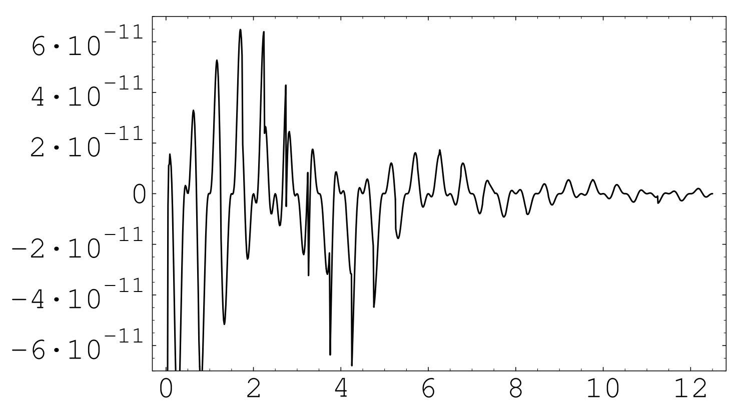

FIG. 5.2. $\mathcal{D}(\lambda) - \mathcal{D}_{5,1/2}(\lambda)$.

In Table 5.2, we summarize the results of this example with $N = 5$ and $h = 0.5$. The branch of the square roots is $0 \le \arg\lambda < \pi$.

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TABLE 5.2
Comparison of the Hermite-Gauss and sinc-Gaussian methods, $N = 5$ and $h = 1/2$.

λ	Sinc-Gaussian	Hermite-Gauss	E_G	E_H	B_G	B_H
λ_1	1.211359825514053	1.211363321461129	3.49747×10^{-6}	1.52349×10^{-10}	5.44287×10^{-5}	1.98233×10^{-7}
λ_2	4.605063072753784	4.605063506819465	4.34132×10^{-7}	6.63043×10^{-11}	4.05124×10^{-5}	9.39391×10^{-8}
λ_3	7.790059591346316	7.790059531803104	5.96862×10^{-8}	1.42997×10^{-10}	3.32204×10^{-5}	1.20897×10^{-7}
λ_4	10.950007012056991	10.950007027984602	1.59474×10^{-8}	1.97460×10^{-11}	3.92782×10^{-5}	4.30276×10^{-8}

EXAMPLE 5.3. Consider the non-self-adjoint boundary value problem

$$
\begin{aligned}
&-y''(x) - (1+\mathrm{i})y(x) = \lambda^2 y(x), \quad 0 \le x \le 1,\\
&y(0,\lambda) = y(1,\lambda), \qquad y'(0,\lambda) = y'(1,\lambda).
\end{aligned} \tag{5.1}
$$

The characteristic function is

$$\mathcal{D}(\lambda) = 4\sin^2\left(\frac{\sqrt{1+\mathrm{i}+\lambda^2}}{2}\right).$$

All eigenvalues of this problem are complex and double from the geometric and algebraic points of view. Indeed, $\lambda_k^2 = (2k\pi)^2 - 1 - \mathrm{i}$. Putting $k = 1$ in (3.14) yields, after calculating the function $\mathcal{K}_1$,

$$\mathcal{D}_{N,1}(\lambda) = 2 - 2\cos(\lambda) + \frac{(1+\mathrm{i})\sin(\lambda)}{\lambda} + \frac{(1+\mathrm{i})^2\,(\lambda\cos(\lambda) - \sin(\lambda))}{8\lambda^3} + \mathcal{H}_{h,N}[\mathcal{U}_1](\lambda),$$

where $\lambda \in \mathbb{C}$ and $h \in (0,\pi]$. In Table 5.3, we exhibit the first four approximate eigenvalues of the problem (5.1) using the sinc-Gaussian and Hermite-Gauss methods with $N = 6$ and $h = 1$. The branch of the square roots is $\pi \le \arg\lambda < 2\pi$.

TABLE 5.3
Comparison of the Hermite-Gauss and sinc-Gaussian methods, $N = 6$ and $h = 1$.

λ	Sinc-Gaussian	Hermite-Gauss	E_G	E_H	B_G	B_H
λ_1	6.20362385-0.08060084i	6.20362101-0.08059808i	3.96×10^{-6}	4.76×10^{-10}	5.06411×10^{-4}	2.17761×10^{-8}
λ_2	12.52658218-0.03991500i	12.52658228-0.03991511i	1.54×10^{-7}	7.67×10^{-11}	3.78228×10^{-4}	2.40723×10^{-8}
λ_3	18.82303014-0.02656355i	18.82303015-0.02656320i	3.51×10^{-7}	2.56×10^{-11}	4.80309×10^{-4}	1.64475×10^{-8}
λ_4	25.11284689-0.01991011i	25.11284687-0.01991012i	2.45×10^{-8}	1.30×10^{-11}	4.72228×10^{-4}	1.02584×10^{-8}

EXAMPLE 5.4. Consider the periodic boundary value problem

$$
\begin{aligned}
&-y''(x) + xy(x) = \lambda^2 y(x), \quad 0 \le x \le 1,\\
&y(0,\lambda) = y(1,\lambda), \qquad y'(0,\lambda) = y'(1,\lambda).
\end{aligned}
$$

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The characteristic function of this problem cannot be computed in a closed form and, thus, the amplitude error appears. It is given as a combination of four varieties of Airy functions. Moreover, all eigenvalues are real and double. Taking into account (3.14), a short calculation gives

$$\widetilde{\mathcal{D}}_{N,2}(\lambda) = 2 - 2\cos(\lambda) - \frac{\sin(\lambda)}{2\lambda} - \frac{(42\lambda - 5\lambda^3)\cos(\lambda) + (-42 + 19\lambda^2 + 3\lambda^4)\sin(\lambda)}{96\lambda^5} + \frac{\lambda(-105 + 7\lambda^2 + \lambda^4))\cos(\lambda) - (-105 + 42\lambda^2 + \lambda^4)\sin(\lambda)}{384\lambda^7} + \mathcal{H}_{h,N}[\widetilde{\mathcal{U}}_2](\lambda),$$

where $\lambda \in \mathbb{C}$ and $h \in (0, \pi]$. Table 5.4 shows the results for this example with $N = 6$, $h = 1/2$ and $\varepsilon = 10^{-10}$. The branch of the square roots is $0 \leq \arg\lambda < \pi$. To compute the absolute error in this example, the exact eigenvalues are computed approximately with *Mathematica*.

TABLE 5.4
Comparison of the Hermite-Gauss and sinc-Gaussian methods, $N = 6$, $h = 0.5$, and $\varepsilon = 10^{-10}$.

λ	Sinc-Gaussian	Hermite-Gauss	E_G	E_H	B_G	B_H
λ_1	6.322873989710145	6.322874725023604	7.35435×10^{-7}	1.21880×10^{-10}	2.36565×10^{-5}	4.83517×10^{-9}
λ_2	12.586254376912265	12.58625400251194	3.74457×10^{-7}	5.69713×10^{-11}	2.58510×10^{-5}	3.66186×10^{-9}
λ_3	18.862815572370014	18.86281565937659	8.69847×10^{-8}	2.18385×10^{-11}	2.52021×10^{-5}	4.53568×10^{-9}
λ_4	25.142687084718617	25.142687085313025	5.68598×10^{-8}	2.58105×10^{-11}	2.50414×10^{-5}	4.60061×10^{-9}

6. Conclusions. In this work we have employed the Hermite-Gauss sampling operator to approximate eigenvalues of Sturm-Liouville problems with mixed-type boundary conditions. Since the eigenvalues of this problem may be double, we studied bounds for truncation and amplitude errors associated with the derivative of this operator. The bounds of these errors allow us to establish the error analysis of this method when the eigenvalues are not algebraically simple. The main idea was to construct an entire function $\widetilde{\mathcal{D}}_{N,k}(\lambda)$ using the Hermite-Gauss operator $\mathcal{H}_{h,N}$, which is very close to the characteristic function $\mathcal{D}(\lambda)$. The zeros of $\widetilde{\mathcal{D}}_{N,k}(\lambda)$ will be very close to the zeros of $\mathcal{D}(\lambda)$. The Hermite-Gauss method that was applied above gives results which are more accurate than the sinc-Gaussian method because the convergence rate of our method is higher than of the sinc-Gaussian method. Up to now, we can confidently say that the presented method, i.e., Hermite-Gauss, is the best sampling method for approximating the eigenvalues of the boundary value problems because it has the highest rate of convergence

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