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APPROXIMATION OF WEAKLY SINGULAR INTEGRAL EQUATIONS BY SINC PROJECTION METHODS*

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Abstract. In this paper, two numerical schemes for a nonlinear integral equation of Fredholm type with weakly singular kernel are studied. These numerical methods blend collocation, convolution, and approximations based on sinc basis functions with iterative schemes like the steepest descent and Newton's method, involving the solution of a nonlinear system of equations. Exponential rate of convergence for the convolution scheme is shown and collocation method is analyzed. Numerical experiments are presented to illustrate the sharpness of the theoretical estimates and the sensitivity of the solutions with respect to some parameters in the equations. The comparison between the schemes indicates that the sinc convolution method is more effective.

Key words. Fredholm integral equation, Urysohn integral operator, weak singularity, convolution method, collocation method

AMS subject classifications. 45B05, 45E99, 65J15, 65R60.

1. Introduction.

The aim of this paper is to study the numerical solution of the nonlinear Fredholm integral equation

$$u(t) = g(t) + \int_a^b f(|t-s|)k(t,s)\psi(s,u(s))\mathrm{d}s, \qquad -\infty < a \le t \le b < \infty, \tag{1.1}$$

where $u(t)$ is an unknown function to be determined and $k(t,s)$, $\psi(s,u)$, and $g(t)$ are given functions. Equation (1.1) is an algebraic weakly singular integral equation whenever $f(t)$ is given by $t^{-\lambda}$, $0 < \lambda < 1$. A more general type of this equation, the so-called Urysohn weakly singular integral equation [25], is defined as

$$u(t) = g(t) + \int_a^b f(|t-s|)k(t,s,u(s))\mathrm{d}s, \qquad -\infty < a \le t \le b < \infty. \tag{1.2}$$

Linear and nonlinear integral equations with weakly singular kernels arise in various applications such as astrophysics [2]. In potential theory, the boundary integral equations of the Laplace and Helmholtz operators can be expressed as linear combinations of weakly singular operators [16].

It is well known that the solution of (1.1) has some singularities near the boundaries. This is an important property that should be considered in the design of numerical solution methods. There has been considerable interest in the numerical analysis of linear and nonlinear integral equations with weakly singular kernels. This interest was followed by the development of some projection schemes such as Galerkin, collocation, and product integration methods with singularity-preserving approaches, which find approximate solutions with optimal error bounds [1, 5, 6, 7, 8, 15, 26]. It is worth mentioning that the numerical solution of (1.1) with a smooth kernel is comprehensively studied; for more information, see [3, 11, 14].

In the current study, we propose two reliable schemes in order to achieve appropriate approximations for the nonlinear weakly singular integral equation (1.1). The methods are designed to take the singular behavior of the solution into account. For the sake of comparison,

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we present two algorithms based on sinc approximation methods. In what follows, we will elucidate the relevant characteristics and convergence rates of these schemes.

The first objective of this study is to investigate the analysis of sinc collocation methods for nonlinear weakly singular Fredholm integral equations. In [12], the authors have studied this subject and obtained the rate of convergence $\mathcal{O}(\|A^{-1}\|_\infty(3+\log(N))\sqrt{N}\exp(-\sqrt{\pi d\lambda N}))$. Here, we consider a sinc collocation method with different basis functions by adding two fractional polynomials $(t-a)^\lambda$ and $(b-t)^\lambda$ in the finite-dimensional space utilized as the approximation space. We propose an error analysis for the approximate solution chosen from an appropriate finite-dimensional space built from shifted sinc functions, while considering the singular properties of the exact solution. However, we encounter a term ξ_N in the upper bound, which depends on N and is unavoidable due to the nature of the projection methods.

For the second objective, we present and analyze a numerical scheme using as the key idea an appropriate approximation of the following nonlinear convolution

$$r(x) = \int_a^x k(x, x-t, t)\mathrm{d}t,$$

which is called sinc convolution method. Here, we replace the independent variables with the single exponential transformation introduced in Section 2.

In order to make the paper self-contained, the basic properties of the sinc approximation method are introduced in Section 2. Two numerical schemes based on sinc collocation and sinc convolution methods will be studied in Section 3. Furthermore, this section contains a complete convergence analysis for the proposed methods. Finally, Section 4 is devoted to some numerical experiments in order to show consistency with the theoretical estimates of the convergence rate.

In this work, we present numerical schemes based on sinc approximation, sinc convolution, and sinc collocation methods for nonlinear Fredholm weakly singular integral equations. Sinc convolution is introduced in [21] to collocate indefinite integrals of convolution type, and it can be interpreted as a special type of Nyström method. It will be shown that this method has an exponential rate of convergence. For a comprehensive study of sinc convolution methods and their applications to different kinds of equations, we refer to [20, 22]. Furthermore, sinc collocation methods and their properties in connection with nonlinear integral equation are studied in this paper.

Equations (1.1) and (1.2) can be expressed in operator form as

$$(I-\mathcal{K}_i)u = g, \qquad i = 1,2, \tag{1.3}$$

where

$$(\mathcal{K}_1 u)(t) = \int_a^b f(|t-s|)k(t,s)\psi(s,u(s))\mathrm{d}s,$$
$$(\mathcal{K}_2 u)(t) = \int_a^b f(|t-s|)k(t,s,u(s))\mathrm{d}s.$$

These operators are defined on the Banach space $X = \mathbf{H}^\infty(\mathcal{D}) \cap C(\bar{\mathcal{D}})$. In this notation, $\mathcal{D} \subset \mathbb{C}$ is a simply connected domain that satisfies $(a,b) \subset \mathcal{D}$, and $\mathbf{H}^\infty(\mathcal{D})$ denotes the family of all functions f that are analytic in the domain $\mathcal{D}$ and have finite uniform (supremum) norm. We assume that the unknown solution $u(t)$ to be determined is geometrically isolated [9, 13], which means that there is a ball

$$\mathfrak{B}(u,r) = \{x \in X : \|x-u\| \le r\},$$

with $r > 0$, where equation (1.1) has the only solution u.

2. Preliminaries. In order to make the paper self-contained, some basic definitions and theorems for sinc function, sinc interpolation, and quadrature are presented.

2.1. Sinc interpolation. The sinc function on the real line $\mathbb{R}$ is defined by

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

It is well known that a function f with suitable smoothness properties can be approximated by sinc functions as

$$f(t) \approx \sum_{j=-N}^{N} f(jh) S(j,h)(t), \qquad t \in \mathbb{R}, \tag{2.1}$$

where the basis function $S(j,h)(t)$ is given by

$$S(j,h)(t) = \operatorname{sinc}(\frac{t}{h} - j), \qquad j \in \mathbb{Z}. \tag{2.2}$$

Here, h is a step size appropriately chosen depending on a given positive integer N, and the function in (2.2) is called the jth sinc function. Equation (2.1) can be adjusted to approximate functions on general intervals by an accompanying variable transformation $t = \varphi(x)$. Appropriate single exponential and double exponential transformations can be used [20, 24] as a converting function $\varphi(x)$. The single exponential transformation and its inverse are given as

$$\varphi_{a,b}(x) = \frac{b-a}{2} \tanh(\frac{x}{2}) + \frac{b+a}{2},$$
$$\phi_{a,b}(t) = \log(\frac{t-a}{b-t}),$$

respectively. The subscripts a and b in the transformations play an important role in the application of sinc collocation methods for weakly singular integral equations. The strip domain is introduced

$$\mathcal{D}_d = \{z \in \mathbb{C} : |\Im z| < d\},$$

for some $d > 0$, in order to define a suitable function space. When it is incorporated into the transformation, we consider the transformed domain

$$\varphi(\mathcal{D}_d) = \left\{z \in \mathbb{C} : \left|\arg(\frac{z-a}{b-z})\right| < d\right\}.$$

The following definitions and theorems are presented for the sake of detailing the procedure.

DEFINITION 2.1 ([20]). *Let α and C be positive constants, and let $\mathcal{D}$ be a bounded and simply connected domain which satisfies $(a,b) \subset \mathcal{D}$. Then $\mathcal{L}_\alpha(\mathcal{D})$ denotes the set of all functions $f \in \mathbf{H}^\infty(\mathcal{D})$ that satisfy*

$$|f(z)| \leq C|Q(z)|^\alpha, \tag{2.3}$$

for all z in $\mathcal{D}$, where $Q(z) = (z-a)(b-z)$.

The next theorem clarifies the exponential convergence rate of the sinc interpolation.

THEOREM 2.2 ([18]). *Let $f \in \mathcal{L}_\alpha(\varphi_{a,b}(\mathcal{D}_d))$ for d with $0 < d < \pi$. Suppose that h is given by the formula $h = \sqrt{\frac{\pi d}{\alpha N}}$, where N is a positive integer. Then there is a constant C independent of N such that*

$$\left\| f(t) - \sum_{j=-N}^{N} f(\varphi_{a,b}(jh)) S(j,h)(\phi(t)) \right\| \leq C\sqrt{N} \exp(-\sqrt{\pi d \alpha N}),$$

where

$$C = \frac{2K(b-a)^{2\alpha}}{\alpha} \left[\frac{2}{\pi d(1 - e^{-2\sqrt{\pi d \alpha}})(\cos(\frac{d}{2}))^{2\alpha}} + \sqrt{\frac{\alpha}{\pi d}} \right].$$

According to Theorem 2.2, in order to attain exponential convergence, the approximated function should be in $\mathcal{L}_\alpha(\mathcal{D})$. By condition (2.3), such a function is expected to be zero at the endpoints, which is too restrictive in practice. However, it can be related to the following function space $\mathcal{M}_\alpha(\mathcal{D})$ with $0 < \alpha \leq 1$ and $0 < d < \pi$.

DEFINITION 2.3 ([20]). *Let $\mathcal{D}$ be a simply connected and bounded domain which contains (a, b). The family $\mathcal{M}_\alpha(\mathcal{D})$ contains all analytical functions f that are continuous in $\bar{\mathcal{D}}$ such that the transformation*

$$G[f](t) = f(t) - [(\frac{b-t}{b-a})f(a) + (\frac{t-a}{b-a})f(b)],$$

is in $\mathcal{L}_\alpha(\mathcal{D})$.

2.2. Sinc quadrature. A sinc approximation incorporating a single exponential transformation can be applied to definite integration based on function approximations yielding a sinc quadrature. The following theorem provides an error bound for the sinc quadrature of f on (a, b).

THEOREM 2.4 ([18]). *Let $(fQ) \in \mathcal{L}_\alpha(\varphi_{a,b}(\mathcal{D}_d))$ for d with $0 < d < \pi$. Suppose that N is a positive integer and h is selected by the formula $h = \sqrt{\frac{\pi d}{\alpha N}}$. Then*

$$\left| \int_a^b f(s)\,\mathrm{d}s - h \sum_{j=-N}^{N} f(\varphi_{a,b}(jh))(\varphi_{a,b})'(jh) \right| \leq C(b-a)^{2\alpha-1} \exp(-\sqrt{\pi d \alpha N}), \tag{2.4}$$

where C is a constant independent of N.

3. Two numerical schemes.

3.1. Sinc collocation. In this section, sinc collocation and its application to nonlinear Fredholm integral equations with weakly singular kernels are discussed. A sinc approximation u_N to the solution $u \in \mathcal{M}_\lambda(\varphi_{a,b}(\mathcal{D}_d))$ of (1.1) is constructed in this section. For this aim, the interpolation operator $\mathcal{P}_N : \mathcal{M}_\lambda \to X_N$ is defined as

$$\mathcal{P}_N[u](t) = \mathfrak{L}u(t) + \sum_{j=-N}^{N} [u(t_j) - (\mathfrak{L}u)(t_j)] S(j,h)(\phi_{a,b}(t)),$$

where

$$\mathfrak{L}[u](t) = (\frac{b-t}{b-a})^\lambda u(a) + (\frac{t-a}{b-a})^{1-\lambda} u(b).$$

In this formula, the sinc points t_j are determined by

$$t_j = \begin{cases} a, & j = -N-1, \\ \varphi_{a,b}(jh), & j = -N, \dots, N, \\ b, & j = N+1. \end{cases} \tag{3.1}$$

The approximate solution can be represented as

$$u_N(t) = c_{-N-1}(\frac{b-t}{b-a})^{\lambda} + \sum_{j=-N}^{N} c_j S(j,h)(\phi_{a,b}(t)) + c_{N+1}(\frac{t-a}{b-a})^{1-\lambda},$$

where the parameter λ indicating the exponent of the singularity is introduced in (1.1). It is worth noticing that the choice of these basis functions combined with sinc functions reflects the singularity of the exact solution well. Employing the operator $\mathcal{P}_N$ on both sides of (1.1) leads to the following approximate equation

$$u_N = \mathcal{P}_N g + \mathcal{P}_N \mathcal{K} u_N.$$

This equation can be rewritten as

$$u_N(t_i) = g(t_i) + \int_a^b f(|t_i - s|)k(t_i, s)\psi(s, u_N(s))\mathrm{d}s, \quad i = -N-1, \dots, N+1, \tag{3.2}$$

hence, the collocation method for solving (1.1) agrees with (3.2) for N sufficiently large. We utilize the theory of function spaces of holomorphic functions along with the singularity-preserving representation of the approximate solution to blend a mechanism for approximating the singular integrals that arise from the discretization of weakly singular integral operators. Let us start with the following representation of (3.2):

$$\begin{aligned} u_N(t_i) &= \int_a^{t_i} f(|t_i - s|)k(t_i, s)\psi(s, u_N(s))\mathrm{d}s \\ &\quad + \int_{t_i}^b f(|t_i - s|)k(t_i, s)\psi(s, u_N(s))\mathrm{d}s + g(t_i), \qquad i = -N-1, \dots, N+1. \end{aligned} \tag{3.3}$$

Due to the complexity of the integral kernel, we utilize the approximation of the integral operator in (3.3) by the quadrature formula presented in (2.4). We notice that in order to use the sinc quadrature method properly, the intervals $[a, t_i]$ and $[t_i, b]$ should be transformed to the whole real line. So, equation (3.3) can be written as

$$\begin{aligned} u_N(t_i) &= h|t_i - a|^{\lambda} \sum_{j=-N}^{N} \frac{k\Big(t_i, \varphi_{a,t_i}(jh)\Big)}{(1+e^{jh})^{\lambda}(1+e^{-jh})} \psi\Big(\varphi_{a,t_i}(jh), u_N(\varphi_{a,t_i}(jh))\Big) \\ &\quad + h|b - t_i|^{\lambda} \sum_{j=-N}^{N} \frac{k\Big(t_i, \varphi_{t_i,b}(jh)\Big)}{(1+e^{jh})^{\lambda}(1+e^{-jh})} \psi\Big(\varphi_{t_i,b}(jh), u_N(\varphi_{t_i,b}(jh))\Big) \\ &\quad + g(t_i), \qquad i = -N-1, \dots, N+1. \end{aligned} \tag{3.4}$$

This numerical procedure (3.4) can be rewritten in operator form as

$$u_N - \mathcal{P}_N \mathcal{K}_N u_N = \mathcal{P}_N g, \tag{3.5}$$

where the discrete operator $\mathcal{K}_N u$ is defined as

$$(\mathcal{K}_N u)(t) :=$$
$$h|t-a|^\lambda \sum_{j=-N}^{N} \frac{k\Big(t,\varphi_{a,t_i}(jh)\Big)}{(1+e^{jh})^\lambda(1+e^{-jh})}\psi\Big(\varphi_{a,t_i}(jh), u(\varphi_{a,t_i}(jh))\Big)$$
$$+\,h|b-t|^\lambda \sum_{j=-N}^{N} \frac{k\Big(t,\varphi_{t_i,b}(jh)\Big)}{(1+e^{jh})^\lambda(1+e^{-jh})}\psi\Big(\varphi_{t_i,b}(jh), u(\varphi_{t_i,b}(jh))\Big).$$

Equation (3.5) is the operator form of the discrete collocation method based on the sinc basis function. By solving the nonlinear system of equations (3.5), the unknown coefficients in u_N are determined.

3.1.1. Convergence analysis. In this section we provide an error analysis for the sinc collocation method. We state the following lemmas which are used subsequently.

LEMMA 3.1 ([20]). *Let $h > 0$. Then it holds that*

$$\sup_{x\in\mathbb{R}} \sum_{j=-N}^{N} |S(j,h)(x)| \le \frac{2}{\pi}(3+\log(N)).$$

From this lemma, one may conclude that $\|\mathcal{P}_N\| \le C\log(N)$, where C is a constant independent of N and $\mathcal{P}_N$ is the interpolation operator constructed from the sinc points.

LEMMA 3.2 ([17]). *Let d be a constant with $0 < d < \pi$. Define the function*

$$\varphi_1(x) = \frac{1}{2}\tanh(\frac{x}{2}) + \frac{1}{2}.$$

Then there is a constant c_d such that for all $x \in \mathbb{R}$ and $y \in [-d, d]$,

$$|\{\varphi_{a,b}\}'(x+iy)| \le (b-a)c_d\varphi_1'(x), \tag{3.6}$$
$$|\varphi_{0,1}(x+iy)| \ge \varphi_1(x). \tag{3.7}$$

In addition, if $t \le x$, then

$$|\varphi_{a,b}(x+iy) - \varphi_{a,b}(t+iy)| \ge (b-a)\{\varphi_1(x) - \varphi_1(t)\}. \tag{3.8}$$

With the aid of Lemma 3.2, the analytical behavior of the solution is investigated for a general kernel function. It is convenient to define the following nonlinear operators, which will be used in the next theorem:

$$\begin{aligned}(\mathcal{K}^1 u)(t) &= \int_a^t |t-s|^{-\lambda}k(t,s,u(s))\mathrm{d}s,\\ (\mathcal{K}^2 u)(t) &= \int_t^b |t-s|^{-\lambda}k(t,s,u(s))\mathrm{d}s.\end{aligned} \tag{3.9}$$

THEOREM 3.3. *Let $\mathcal{D} = (\varphi_{a,b})^{-1}(\mathcal{D}_d)$ for a constant d with $0 < d < \pi$. Suppose that $k(z,.,v) \in \mathbf{H}^\infty(\mathcal{D})$ for all z and v in $\overline{\mathcal{D}}$, and $k(z,w,.) \in \mathbf{H}^\infty(\mathcal{D})$ for all z and w in $\overline{\mathcal{D}}$. Moreover, suppose that $k(.,v,w) \in \mathcal{M}_{1-\lambda}(\mathcal{D})$ for all $v,w \in \overline{\mathcal{D}}$, $k(z,v,w)$ is bounded for z, v, and w in $\overline{\mathcal{D}}$, and $y \in \mathcal{M}_\beta(\mathcal{D})$. Then the solution u of (1.1) belongs to $\mathcal{M}_\gamma(\mathcal{D})$, where $\gamma = \min(1-\lambda,\beta)$.*

Proof. In [10, p. 83], sufficient conditions are stated to have a nonlinear analytic operator and thus an analytic solution. Hence, it is enough to show that $\mathcal{K}u$ is $(1-\lambda)$-Hölder continuous. For this aim, we show that the operators defined in (3.9) have this property. To demonstrate the $(1-\lambda)$-Hölder continuity of $\mathcal{K}^1 u$ and $\mathcal{K}^2 u$, the idea of Lemma A.2 in [17] is extended to the nonlinear case. Set $x = \mathrm{Re}[(\varphi_{a,b})^{-1}(z)]$, $y = \mathrm{Im}[(\varphi_{a,b})^{-1}(z)]$, and $v = \varphi_{a,b}(t+iy)$. Then

$$(\mathcal{K}^1 u)(z) - (\mathcal{K}^1 u)(a) = \int_a^z |z-v|^{-\lambda} k(z,v,u(v))\mathrm{d}v - 0$$
$$= \int_\infty^x |\varphi_{a,b}(x+iy) - \varphi_{a,b}(t+iy)|^{-\lambda} k(x+iy, t+iy, u(t+iy))(\varphi_{a,b})'(t+iy)\mathrm{d}t.$$

Applying the absolute value on both sides of the above equation and using equations (3.6) and (3.8), we have

$$\begin{aligned}|(\mathcal{K}^1 u)(z) - (\mathcal{K}^1 u)(a)| &\leq \int_\infty^x (b-a)^{-\lambda}(\varphi_1(x) - \varphi_1(t))^{-\lambda} M_k (b-a) c_d \varphi_1'(t)\mathrm{d}t \\ &\leq \frac{M_k c_d}{1-\lambda}\left((b-a)\varphi_1(x)\right)^{1-\lambda},\end{aligned}$$

where $M_k = \max_{\overline{\mathcal{D}}} |k(z,w,v)|$. In addition, by using property (3.7), the inequality $(b-a)\varphi_1(x) \leq |z-a|$ can be derived. Therefore,

$$|(\mathcal{K}^1 u)(z) - (\mathcal{K}^1 u)(a)| \leq \frac{M_k c_d}{1-\lambda}|z-a|^{(1-\lambda)}.$$

Now, the $(1-\lambda)$-Hölder continuity at the point b is considered:

$$\begin{aligned}(\mathcal{K}^1 u)(b) - (\mathcal{K}^1 u)(z) = &\int_a^b |b-v|^{-\lambda}\Big\{k(b,v,u(v)) - k(z,v,u(v))\Big\}\mathrm{d}v \\ &+ \int_a^b \Big\{|b-v|^{-\lambda} - |z-v|^{-\lambda}\Big\} k(z,v,u(v))\mathrm{d}v \\ &- \int_b^z |z-v|^{-\lambda} k(z,v,u(v))\mathrm{d}v.\end{aligned}$$

Since $k(.,v,w) \in \mathcal{M}_{1-\lambda}(\mathcal{D})$, there exists M_1 such that

$$\begin{aligned}\Big|\int_a^b |b-v|^{-\lambda}\Big\{k(b,v,u(v)) - k(z,v,u(v))\Big\}\mathrm{d}v\Big| &\leq M_1|b-z|^{(1-\lambda)}\int_a^b |b-v|^{-\lambda}\mathrm{d}v \\ &\leq \frac{M_1|b-a|^{1-\lambda}}{1-\lambda}|b-z|^{1-\lambda}.\end{aligned}$$

The third term is bounded by

$$\Big|\int_b^z |z-v|^{-\lambda} k(z,v,u(v))\mathrm{d}v\Big| \leq \frac{M_k c_d}{1-\lambda}|b-z|^{1-\lambda}.$$

Integration by part, the Hölder continuity of the function $F(z) = z^{1-\lambda}$, and the assumptions on $k(z,w,.)$ and $k(z,.,v) \in \mathbf{H}^\infty(\mathcal{D})$ result in the bound

$$|(\mathcal{K}^1 u)(b) - (\mathcal{K}^1 u)(z)| \leq \frac{M_2}{1-\lambda}|b-z|^{(1-\lambda)}.$$

The $(1-\lambda)$-Hölder continuity of the operator $\mathcal{K}^2(u)$ can be proved in a similar manner. □

The Fréchet derivative of the nonlinear operators $\mathcal{K}$ and $\mathcal{K}_N$ for all u is given by

$$(\mathcal{K}'u)x(t) = \int_a^b f(|t-s|)k(t,s)\frac{\partial \psi}{\partial u}(s,u(s))x(s)\mathrm{d}s, \qquad t \in [a,b], x \in X,$$

and

$$\begin{aligned}(\mathcal{K'}_N u)x(t) = \\ & h|t-a|^\lambda \sum_{j=-N}^{N} \frac{k\Big(t,\varphi_{a,t_i}(jh)\Big)}{(1+e^{jh})^\lambda(1+e^{-jh})}\frac{\partial \psi}{\partial u}\Big(\varphi_{a,t_i}(jh),u(\varphi_{a,t_i}(jh))\Big)x(jh) \\ & + h|b-t|^\lambda \sum_{j=-N}^{N} \frac{k\Big(t,\varphi_{t_i,b}(jh)\Big)}{(1+e^{jh})^\lambda(1+e^{-jh})}\frac{\partial \psi}{\partial u}\Big(\varphi_{t_i,b}(jh),u(\varphi_{t_i,b}(jh))\Big)x(jh).\end{aligned}$$

THEOREM 3.4. *Assume that $u(t)$ is the true solution of equation (1.1) such that $I - \mathcal{K}'u$ is a non-singular operator. Additionally, suppose that the term $\frac{\partial^2 \psi}{\partial u^2}(t,s,u)$ is well defined and continuous on its domain. Furthermore, assume that $g \in \mathcal{M}_\lambda(\varphi_{a,b}(\mathcal{D}_d))$ and $\mathcal{K}u \in \mathcal{M}_\lambda(\varphi_{a,b}(\mathcal{D}_d))$ for all $u \in \mathfrak{B}(u,r)$. Then, there is a constant C independent of N such that*

$$\|u - u_N\| \le C\xi_N\sqrt{N}\log(N+1)\exp(-\sqrt{\pi d\lambda N}),$$

where $\xi_N = \|(I - \mathcal{P}_N(\mathcal{K}_N)'(u))^{-1}\|$.

Proof. To find an upper error bound, we subtract (1.3) from (3.5) and obtain

$$u - u_N = \mathcal{K}u - \mathcal{P}_N\mathcal{K}_N u_N + g - \mathcal{P}_N g.$$

This relation is rewritten as

$$\begin{aligned} u - u_N = (I - \mathcal{P}_N(\mathcal{K}'_N)(u))^{-1}\big\{ & (g - \mathcal{P}_N g) \\ & + (\mathcal{K}u - \mathcal{P}_N\mathcal{K}u) + \mathcal{P}_N(\mathcal{K}u - \mathcal{K}_N u) \\ & + \mathcal{P}_N(\mathcal{K}_N u - \mathcal{K}_N u_N - (\mathcal{K'}_N)(u)(u-u_N))\big\}.\end{aligned}$$

Finally, the following estimate is obtained

$$\begin{aligned}\|u - u_N\| \le \|(I - \mathcal{P}_N(\mathcal{K}'_N)(u))^{-1}\|\big\{\|g - \mathcal{P}_N g\| \\ + \|\mathcal{K}u - \mathcal{P}_N\mathcal{K}u\| + \|\mathcal{P}_N\|\|\mathcal{K}u - \mathcal{K}_N u\|\big\} + \|\mathcal{P}_N\|\mathcal{O}(\|u-u_N\|^2).\end{aligned}$$

Because of $g, \mathcal{K}u \in \mathcal{M}_\lambda(\varphi_{a,b}(\mathcal{D}_d))$ and Theorem 2.2, we find

$$\begin{aligned}\|g - \mathcal{P}_N g\| &\le C_1\sqrt{N}\exp(-\sqrt{\pi d\lambda N}), \\ \|\mathcal{K}u - \mathcal{P}_N\mathcal{K}u\| &\le C_2\sqrt{N}\exp(-\sqrt{\pi d\lambda N}).\end{aligned}$$

By using Theorem 2.4, we conclude that

$$\|\mathcal{K}u - \mathcal{K}_N u\| \le C_3\exp(-\sqrt{\pi d\lambda N}),$$

and, finally, we find an upper bound for $\|\mathcal{P}_N\|$ by Lemma 3.1. Hence,

$$\|u - u_N\| \le C\xi_N\log(N+1)\sqrt{N}\exp(-\sqrt{\pi d\lambda N}). \qquad \square$$

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3.2. Sinc convolution. Let $f(t)$ be a function with a singularity at the origin and $g(t)$ be a function with singularities at both endpoints. The method of sinc convolution is based on an accurate approximation of the following integrals

$$
\begin{aligned}
p(s) &= \int_a^s f(s-t)g(t)\mathrm{d}t, \quad s \in (a,b), \\
q(s) &= \int_s^b f(t-s)g(t)\mathrm{d}t, \quad s \in (a,b),
\end{aligned} \tag{3.10}
$$

which can then be used to approximate the definite convolution integral

$$
\int_a^b f(|s-t|)g(t)\mathrm{d}t.
$$

In the sequel, the following notation is used.

DEFINITION 3.5. *For a given positive integer N, let D_N and V_N denote the linear operators acting on a function u by*

$$
\begin{aligned}
D_N u =& \operatorname{diag}[u(t_{-N}), \ldots, u(t_N)], \\
V_N u =& (u(t_{-N}), \ldots, u(t_N))^T,
\end{aligned}
$$

where the superscript T specifies the transpose and diag *denotes the diagonal matrix. Set the basis functions as*

$$
\begin{aligned}
\gamma_j(t) &= S(j,h)(\varphi_{a,b}(t)), && j = -N, \ldots, N, \\
\omega_j(t) &= \gamma_j(t), && j = -N, \ldots, N, \\
\omega_{-N}(t) &= \frac{b-t}{b-a} - \sum_{j=-N+1}^{N} \frac{1}{1+e^{jh}} \gamma_j(t), \\
\omega_N(t) &= \frac{t-a}{b-a} - \sum_{j=-N}^{N-1} \frac{e^{jh}}{1+e^{jh}} \gamma_j(t).
\end{aligned}
$$

With the aid of these basis functions, for a given vector $\mathbf{c} = (c_{-N}, \ldots, c_N)^T$, we consider a linear combination denoted as Π_N as follows:

$$
(\Pi_N \mathbf{c})(t) = \sum_{j=-N}^{N} c_j \omega_j(t).
$$

Let us define the interpolation operator $\mathcal{P}_N^c : \mathcal{M}_\lambda(\mathcal{D}) \to X_N = \operatorname{span}\{\omega_j(t)\}_{j=-N}^{N}$ as

$$
\mathcal{P}_N^c f(t) = \sum_{j=-N}^{N} f(t_j)\omega_j(t),
$$

where the points t_j are defined in (3.1). The numbers σ_k and e_k are determined by

$$
\begin{aligned}
\sigma_k &= \int_0^k \operatorname{sinc}(t)\mathrm{d}t, \quad k \in \mathbb{Z}, \\
e_k &= \frac{1}{2} + \sigma_k.
\end{aligned}
$$

We now define an $(2N+1)\times(2N+1)$ (Toeplitz) matrix $I^{(-1)} = [e_{i-j}]$, where e_{i-j} represents the (i,j)-th element of $I^{(-1)}$. In addition, the operators $\mathcal{I}^+$ and $\mathcal{I}^-$ are given as

$$(\mathcal{I}^+ g)(t) = \int_a^t g(s)\mathrm{d}s, \qquad (\mathcal{I}^- g)(t) = \int_t^b g(s)\mathrm{d}s.$$

The following discrete operators $\mathcal{I}_N^+$ and $\mathcal{I}_N^-$ approximate the operators $\mathcal{I}^+$ and $\mathcal{I}^-$ as

$$\begin{aligned} (\mathcal{I}_N^+ g)(t) &= \Pi_N \mathbf{A}^{(1)} V_N g(t), \qquad & \mathbf{A}^{(1)} &= h I^{(-1)} D_N(\frac{1}{\varphi'_{a,b}}), \\ (\mathcal{I}_N^- f)(t) &= \Pi_N \mathbf{A}^{(2)} V_N g(t), \qquad & \mathbf{A}^{(2)} &= h (I^{(-1)})^T D_N(\frac{1}{\varphi'_{a,b}}). \end{aligned} \tag{3.11}$$

For a function f, the operator $\mathcal{F}[f](s)$ is defined by

$$\mathcal{F}[f](s) = \int_0^c e^{\frac{-t}{s}} f(t)\mathrm{d}t, \tag{3.12}$$

and it is assumed that equation (3.12) is well defined for some $c \in [b-a, \infty]$ and for all s in the right half of the complex plane, $\Omega^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$.

Sinc convolution methods provide formulae of high accuracy and allow $f(s)$ to have an integrable singularity at $s = b - a$ and g to have singularities at both endpoints of (a, b) [22]. This property of sinc convolution makes this method suitable for approximating weakly singular integral equations.

Now for convenience, some useful theorems related to the sinc convolution method are introduced. The following theorem predicts their convergence rate.

THEOREM 3.6 ([22]). *(a) Suppose that the integrals $p(t)$ and $q(t)$ in (3.10) exist and are uniformly bounded on (a, b), and let $\mathcal{F}$ be defined by (3.12). Then the following operator identities hold*

$$p = \mathcal{F}(\mathcal{I}^+)g, \qquad q = \mathcal{F}(\mathcal{I}^-)g. \tag{3.13}$$

(b) Assume that $\dfrac{g}{\varphi_{a,b}} \in \mathcal{L}_\lambda(\mathcal{D})$. If for some positive C' independent of N, the inequality $|\mathcal{F}'(s)| \le C'$ holds for all $\Re(s) \ge 0$, then there is a constant C, which is independent of N, such that

$$\begin{aligned} \| \, p - \mathcal{F}(\mathcal{I}_N^+)g \, \| \; &\le C\sqrt{N}\exp(-\sqrt{\pi\lambda d N}), \\ \| \, q - \mathcal{F}(\mathcal{I}_N^-)g \, \| \; &\le C\sqrt{N}\exp(-\sqrt{\pi\lambda d N}). \end{aligned}$$

3.2.1. Sinc convolution scheme. In order to make practical use of the convolution method, it is assumed that the dimension of the matrices, $2N+1$, is such that the matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are diagonalizable [22]:

$$\mathbf{A}^{(j)} = X^{(j)} S (X^j)^{-1}, \qquad j = 1, 2, \tag{3.14}$$

where

$$\begin{aligned} S &= \mathrm{diag}(s_{-N}, \ldots, s_N), \\ X^{(1)} &= [x_{k,l}], \qquad (X^{(1)})^{-1} = [x^{k,l}], \\ X^{(2)} &= [\xi_{k,l}], \qquad (X^{(2)})^{-1} = [\xi^{k,l}]. \end{aligned}$$

The integral in (1.1) is split in the following way:

$$\int_a^b |t-s|^{-\lambda} k(t,s,u(s))\mathrm{d}s = \int_a^t |t-s|^{-\lambda} k(t,s,u(s))\mathrm{d}s + \int_t^b |t-s|^{-\lambda} k(t,s,u(s))\mathrm{d}s. \tag{3.15}$$

Based on formulae (3.11), two discrete nonlinear operators are defined as

$$(\mathcal{K}_N^1 u)(t) = \Pi_N \mathbf{A}^{(1)} V_N k(t,s,u(s)), \qquad (\mathcal{K}_N^2 u)(t) = \Pi_N \mathbf{A}^{(2)} V_N k(t,s,u(s)).$$

The approximate solution takes the form

$$u_N^c(t) = \sum_{j=-N}^{N} c_j \omega_j(t),$$

where the c_j are unknown coefficients to be determined. The integrals on the right-hand side of (3.15) are approximated by formulae (3.11), (3.13), and (3.14). We substitute these approximations to (1.1), and then the obtained equation is collocated at the sinc points. This process reduces the solution of (1.1) to solving the following finite-dimensional system of equations

$$c_j - \sum_{k=-N}^{N} x_{j,k} \sum_{l=-N}^{N} x^{k,l} \mathcal{F}(s_k) k(z_j, z_l, c_l) - \sum_{k=-N}^{N} \xi_{j,k} \sum_{l=-N}^{N} \xi^{k,l} \mathcal{F}(s_k) k(z_j, z_l, c_l) = y(z_j), \tag{3.16}$$

for $j = -N, \ldots, N$. Equation (3.16) can be expressed in operator notation as

$$u_N^c - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c - \mathcal{P}_N^c \mathcal{K}_N^2 u_N^c = \mathcal{P}_N^c y. \tag{3.17}$$

3.2.2. Convergence analysis. The convergence analysis of the sinc convolution method is discussed in this section. The main result is formulated in the following theorem.

THEOREM 3.7. *Suppose that $u(t)$ is an exact solution of equation (1.1) and that the kernel k satisfies a Lipschitz condition with respect to the third variable. Also, let the assumptions of Theorem 3.3 be fulfilled. Then there is a constant C independent of N such that*

$$\|u - u_N^c\| \leq C\sqrt{N} \log(N) \exp(-\sqrt{\pi d \lambda N}).$$

Proof. By subtracting equation (1.1) from (3.17), the following bound can be derived:

$$\|u - u_N^c\| \leq \|\mathcal{K}^1 u - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c\| + \|\mathcal{K}^2 u - \mathcal{P}_N^c \mathcal{K}_N^2 u_N^c\| + \|y - \mathcal{P}_N^c y\|.$$

The derivation of upper bounds for the first and second terms is almost identical. For this aim, the first term is rewritten as

$$\mathcal{K}^1 u - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c = \mathcal{K}_1 u - \mathcal{P}_N^c \mathcal{K}_1 u_N^c + \mathcal{P}_N^c \mathcal{K}_1 u_N^c - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c,$$

so we have

$$\|\mathcal{K}^1 u - \mathcal{P}_N \mathcal{K}_N^1 u_N^c\| \leq \|\mathcal{K}^1 u - \mathcal{K}_1 u_N^c\| + \|\mathcal{K}^1 u_N^c - \mathcal{P}_N \mathcal{K}^1 u_N^c\| + \|\mathcal{P}_N^c\| \|\mathcal{K}^1 u_N^c - \mathcal{K}_N^1 u_N^c\|,$$

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where the second term is bounded by Theorem 2.2. Due to the Lipschitz condition, the first term is bounded by

$$\|\mathcal{K}^1 u - \mathcal{K}^1 u_N^c\| \le C_1 \|u - u_N^c\|,$$

where C_1 is a suitable constant. In addition, Lemma 3.1 and Theorem 3.6 help us to find the upper bound

$$\|\mathcal{P}_N^c\| \|\mathcal{K}_1 u_N^c - \mathcal{K}_N^1 u_N^c\| \le C_2 \sqrt{N} \log(N) \exp(-\sqrt{\pi d \lambda N}).$$

Finally, we get

$$\|u - u_N^c\| \le C \sqrt{N} \log(N) \exp(-\sqrt{\pi d \lambda N}). \qquad \square$$

4. Numerical experiments. This section is devoted to numerical experiments concerning the accuracy and the rate of convergence of the presented methods. The proposed algorithms are implemented in Mathematica. To solve the nonlinear systems which arise in the formulation of the methods, we have utilized Newton's iteration. In order to find an initial guess for the Newton procedure, the steepest descent method is employed, which is less sensitive to the initial guess [4]. The convergence rate of the sinc convolution and sinc convolution methods depends on two parameters α and d. Specifically, the parameter d specifies the size of the holomorphic domain of u. In all examples, the parameter α is determined by Theorem 3.3 and d is set to 3.14. Furthermore the parameter c in formula (3.12) is taken as infinity.

EXAMPLE 4.1 ([15, 19]). Let us examine the integral equation

$$u(t) - \int_0^1 |t-s|^{\frac{-1}{2}} u^2(s) \mathrm{d}s = g(t), \qquad t \in (0,1),$$

where

$$g(t) = [t(1-t)]^{\frac{1}{2}} + \frac{16}{15} t^{\frac{5}{2}} + 2t^2 (1-t)^{\frac{1}{2}} + \frac{4}{3} t (1-t)^{\frac{3}{2}} + \frac{2}{5} (1-t)^{\frac{5}{2}} - \frac{4}{3} t^{\frac{3}{2}} - 2t(1-t)^{\frac{1}{2}} - \frac{2}{3} (1-t)^{\frac{3}{2}},$$

with the exact solution $u(t) = \sqrt{t(1-t)}$. The exact solution has a singularity near zero. The numerical results are given in Figure 4.1. As reported in [19], the maximum of the absolute errors at the collocation points for a piecewise polynomial collocation method is around 10^{-7} due to the super-convergence property of the piecewise collocation method. Furthermore, in [15] the authors have applied the multi-Galerkin method for weakly singular integral equations of Hammerstein type. A comparison between the reported results reveal better findings for the sinc approach.

EXAMPLE 4.2 ([23]). In this example, we consider the following integral equation

$$u(t) - \int_0^1 |t-s|^{\frac{-1}{4}} u^2(s) \mathrm{d}s = g(t), \qquad t \in (0,1).$$

The function $g(t)$ is chosen such that $u(t) = t^{\frac{3}{2}}$ is the exact solution. The first derivative of the exact solution has a singularity near zero. Figure 4.2 illustrates the error results achieved for the sinc convolution and the sinc collocation methods, which are competitive with the results reported in [23].

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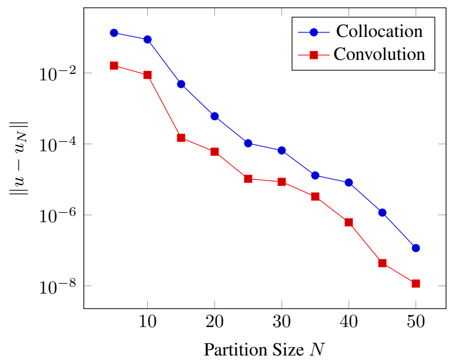

FIG. 4.1. *Plots of the absolute error for the sinc convolution and sinc collocation method for Example 4.1.*

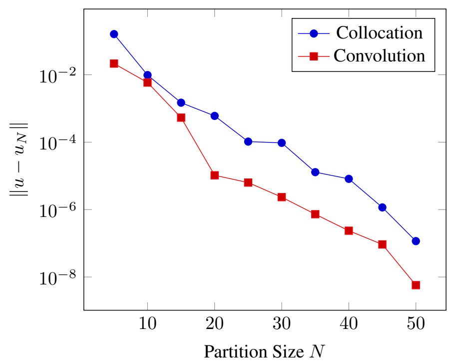

FIG. 4.2. *Plots of the absolute error for the sinc convolution and sinc collocation method for Example 4.2.*

EXAMPLE 4.3. Consider the integral equation

$$u(t) - \int_0^1 |t-s|^{\frac{-1}{2}} \cos(s + u(s))\mathrm{d}s = g(t),$$

where $g(t)$ is selected so that $u(t) = \cos(t)$. This example with an infinitely smooth solution is discussed in [12]. Here we compare the solutions of sinc collocation and sinc convolution. Figure 4.3 displays better results for the sinc convolution approach in comparison with the sinc collocation method.

EXAMPLE 4.4. In this experiment, we explore the sensitivity of the methods to the parameter $\lambda \in (0, 1)$ in the weakly singular integral equation. We consider the equation

$$u(t) - \int_0^1 \frac{1}{|t-s|^{1-\lambda}} u^2(s)\mathrm{d}s = g(t),$$

with the exact solution $u_\lambda(t) = t^{2-\lambda}$. We choose $\lambda = \frac{k}{10}$, for $k \in \{1, 2, \ldots, 9\}$, and the errors for the sinc convolution method are displayed in Figure 4.4.

Conclusion. In this paper, sinc collocation and sinc convolution methods are considered for nonlinear weakly singular Fredholm integral equations, and rigorous proofs of the exponential convergence of the schemes are obtained. The theoretical arguments show that directly applying the collocation method with sinc basis functions leads to a parameter ξ_N in the

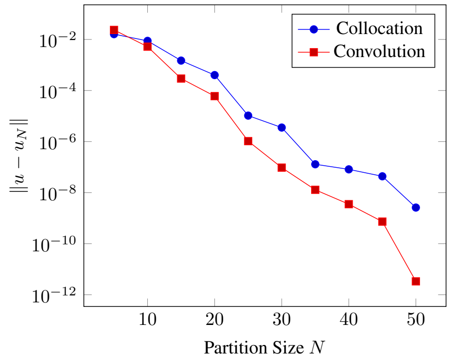

FIG. 4.3. *Plots of the absolute error for the sinc convolution and sinc collocation method for Example 4.3.*

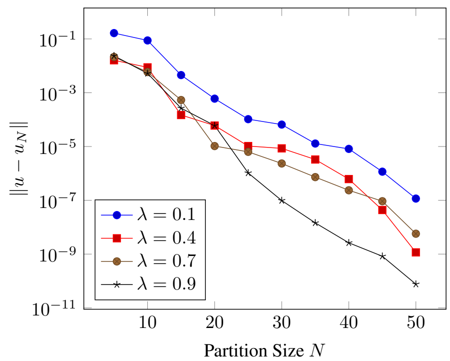

FIG. 4.4. *Plots of the absolute error for the sinc convolution method for different values of* λ.

error bound. This parameter is unavoidable due to the non-uniform boundedness of the sinc interpolation operator. Hence, a numerical method based on the sinc convolution is proposed. It is shown both in theory and by numerical experiments that convolution methods are more accurate and achieve exponential convergence with respect to N. The main advantage of the sinc methods for the weakly singular kernels is the fact that they allow for singularities at the boundaries. The method is capable of handling discrete sinc convolution operators and extendable to the case to fully implicit integral equations by utilizing double exponential sinc methods.

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