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QUADRATURE FORMULAS FOR RATIONAL FUNCTIONS*

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Abstract. Let ω be an L_1-integrable function on $[-1,1]$ and let us denote

$$I_\omega(f) = \int_{-1}^{1} f(x)\omega(x)dx,$$

where f is any bounded integrable function with respect to the weight function ω. We consider rational interpolatory quadrature formulas (RIQFs) where all the poles are preassigned and the interpolation is carried out along a table of points contained in $\overline{\mathbb{C}} \setminus [-1,1]$.

The main purpose of this paper is the study of the convergence of the RIQFs to $I_\omega(f)$.

Key words. weight functions, interpolatory quadrature formulas, orthogonal polynomials, multipoint Padé–type approximants.

AMS subject classifications. 41A21, 42C05, 30E10.

1. Introduction.

This work is mainly concerned with the estimation of the integral

$$I_\omega(f) = \int_{-1}^{1} f(x)\omega(x)dx, \tag{1.1}$$

where $\omega(x)$ is an L_1–integrable function (possibly complex) on $[-1,1]$ and f is a bounded complex valued function. The existence of the integral $I_\omega(f)$ should be understood in the sense that the real and imaginary parts of $f(x)\omega(x)$ are Riemann integrable functions on $[-1,1]$, either properly or improperly. We propose approximations of the form

$$I_n(f) = \sum_{j=1}^{n} A_{j,n} f(x_{j,n}) \tag{1.2}$$

which we will refer to as an n–point quadrature formula with coefficients or weights $\{A_{j,n}\}$ and nodes $\{x_{j,n}\}$. As it is well known, the key question in this context is how to choose the nodes and weights so that $I_n(f)$ turns out to be a "good" estimation of $I_\omega(f)$.

Classical theory is based on the fact of the density of the space Π of all polynomials in the class $C([-1,1])$ of the continuous functions. Assuming that the integrals

$$c_k = \int_{-1}^{1} x^k \omega(x)dx, \qquad k = 0, 1, ...,$$

exist and are easily computable, when replacing $f(x)$ in (1.1) by a certain polynomial $P(x)$, $I_\omega(P)$ will provide us with an approximation for $I_\omega(f)$.

Concerning the choice of the polynomial $P(x)$, many techniques have been developed in the last decades making use of interpolating polynomials. More precisely, given n-distinct

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nodes $\{x_{j,n}\}$ on $[-1,1]$, let $P_{n-1}(f;x)$ denote the interpolating polynomial of degree at most $n-1$ to the function f at these nodes, i.e.,

$$P_{n-1}(f;x) = \sum_{j=1}^{n} l_{j,n}(x) f(x_{j,n}) \qquad \text{(Lagrange Formula)} \tag{1.3}$$

where $l_{j,n} \in \Pi_{n-1}$ (space of polynomials of degree at most $n-1$), satisfies

$$l_{j,n}(x_{k,n}) = \delta_{j,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

We see that $I_\omega(P_{n-1}(f,.))$ provides us with a quadrature formula of the form (1.2) with $A_{j,n} = I_\omega(l_{j,n}),\ j = 1,2,...,n$. $I_n(f)$ is called an n-point interpolatory quadrature formula and clearly integrates exactly any polynomial P in Π_{n-1}, i.e.,

$$I_\omega(P) = I_n(P), \qquad \forall P \in \Pi_{n-1}. \tag{1.4}$$

An important aspect in this framework is the problem of the convergence. That is, how to choose the nodes $\{x_{j,n}\}_{j=1}^{n},\ n = 1,2,...$ so that the resulting interpolatory quadrature formula sequence $\{I_n(f)\}$ converges to $I_\omega(f)$, with f belonging to a class of functions "as large as possible".

Many contributions have been given in the last two decades. For the sake of completeness we shall state a result by Sloan and Smith (see [12]), culminating a series of previous works of these authors (see e.g. [13] and [14]).

THEOREM 1.1. *Let $\beta(x)$ be a real and L_1-integrable function on $[-1,1]$ and $\omega(x)$ be a weight function on $[-1,1]$ $(\omega(x) \geq 0)$ such that*

$$\int_{-1}^{1} \frac{|\beta(x)|^2}{\omega(x)} dx < +\infty.$$

Let $\{x_{j,n}\},\ j = 1,2,\ldots,n,\ \ n \in \mathbb{N}$ be the zeros of the n^{th}-monic orthogonal polynomial with respect to $\omega(x)$ on $[-1,1]$, and

$$I_n(f) = \sum_{j=1}^{n} A_{j,n} f(x_{j,n}),$$

the n-point interpolatory quadrature formula at the nodes $\{x_{j,n}\}$. Then

$$\lim_{n\to\infty} I_n(f) = I_\beta(f),$$

for all real-valued bounded function $f(x)$ on $[-1,1]$ such that the integral

$$I_\beta(f) = \int_{-1}^{1} f(x)\beta(x)dx$$

exists.

In this work, we propose to make use of quadrature formula (1.2), integrating exactly rational functions with prescribed poles outside $[-1,1]$. Observe that polynomials can be considered as rational functions with all the poles at infinity.

The main aim will be to prove a similar result to Theorem 1.1 for rational interpolatory quadratures formulas (RIQFs), where orthogonal polynomials with respect to a varying

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weight function will play a fundamental role. In this respect this work can be considered as a continuation of the paper by González-Vera, et al.(see [8]), where the convergence of this type of quadrature exactly integrating rational functions with prescribed poles, was proved for the class of continuous functions satisfying a certain Lipschitz condition, and it can be also considered as a continuation of the paper written by Cala Rodríguez and López Lagomasino (see [3]) where they proved convergence (exact rate of convergence) of this type of interpolating quadrature formulas approximating Markov-type analytic functions.

The common contribution in those papers ([3] and [8]) was to display the connection between Multipoint Padé–type Approximants and Interpolating Quadrature Formulas. Here, we start from a "purely" numerical integration point of view and, as an immediate consequence of this approach, a known result about uniform convergence for Multipoint Padé–type Approximants will be easily deduced.

2. Preliminary results. Let $\hat{\alpha} = \{\alpha_{j,n} : \ j = 1, 2, ..., n, \ n = 1, 2, ...\}$ be compactly contained in $\overline{\mathbb{C}} \setminus [-1, 1]$, i.e., such that

$$d(\hat{\alpha}; [-1, 1]) = \min \operatorname{dist}[\alpha_{j,n}; [-1, +1]] = \delta > 0. \tag{2.1}$$

In the sequel, we shall refer to this property as the "δ-condition"for $\hat{\alpha}$. Set

$$\pi_n(x) = \prod_{j=1}^{n} (x - \alpha_{j,n}).$$

In what follows, we need to introduce the following spaces of rational functions. For each $n \in \mathbb{N}$, define

$$\mathcal{L}_{2n} = \left\{ \frac{P(x)}{|\pi_n(x)|^2} : \ P \in \Pi_{2n-1} \right\} \qquad \text{and} \qquad \mathcal{R}_n = \left\{ \frac{P(x)}{\pi_n(x)} : \ P \in \Pi_{n-1} \right\}$$

Let $\omega(x)$ be a given weight function on $[-1, 1]$ and consider the function

$$\omega_n(x) = \frac{\omega(x)}{|\pi_n(x)|^2} \geq 0, \qquad \forall x \in [-1, 1]. \tag{2.2}$$

Let $Q_n(x)$ be the n^{th}-orthogonal polynomial with respect to $\omega_n(x)$ on $[-1, 1]$ and let $\{x_{j,n}\}_{j=1}^{n}$ be the n zeros of $Q_n(x)$. Then, positive numbers $\tilde{\lambda}_{1,n}, \tilde{\lambda}_{2,n}, \dots, \tilde{\lambda}_{n,n}$ exist such that

$$\int_{-1}^{1} f(x)\omega_n(x)dx = \sum_{j=1}^{n} \tilde{\lambda}_{j,n} f(x_{j,n}), \qquad \forall f \in \Pi_{2n-1} \tag{2.3}$$

Take $R \in \mathcal{L}_{2n}$. Then $R(x) = \dfrac{P(x)}{|\pi_n(x)|^2}$ $P \in \Pi_{2n-1}$ and one has

$$\int_{-1}^{1} R(x)\omega(x)dx = \int_{-1}^{1} P(x)\omega_n(x)dx = \sum_{j=1}^{n} \tilde{\lambda}_{j,n} P(x_{j,n}) = \sum_{j=1}^{n} \lambda_{j,n} R(x_{j,n}) = \tilde{I}_n(R),$$

where $\lambda_{j,n} = \tilde{\lambda}_{j,n} |\pi_n(x_{j,n})|^2 > 0$. Thus, an n-point quadrature formula, with positive coefficients or weights, which is exact in $\mathcal{L}_{2n}$, has been defined. We will refer to it as the n-point Gauss formula for $\mathcal{L}_{2n}$. Now, we give the following result of uniform boundness for the coefficients of this quadrature formulas.

LEMMA 2.1. *Under the conditions above, a positive constant M exists such that*

$$\sum_{j=1}^{n} \lambda_{j,n} \leq M, \qquad n \geq 1.$$

REMARK 1. *In case that $\hat{\alpha}$ is a Newtonian table, i.e.,*

$$\hat{\alpha} = \{\alpha_{j,n} = \alpha_{j,k}, \quad 1 \leq j \leq k, \ k = 1, \ldots, n, \quad \textit{and} \quad n \in \mathbb{Z}_+\}$$

contained in $\overline{\mathbb{C}} \setminus [-1, 1]$, condition (2.1) can be omitted in order to prove Lemma 2.1.

Proceeding as in [10, Theorem 1, p. 101], and making use of Theorem 1.1 in [7], we can now give a characterization theorem for these quadrature formulas.

THEOREM 2.2. *A quadrature formula of the type*

$$\tilde{I}_n(f) = \sum_{j=1}^{n} \lambda_{j,n} f(x_{j,n})$$

is exact in $\mathcal{L}_{2n}$, if and only if,

(i) *$I_n(f)$ is exact in $\mathcal{R}_n$, and*

(ii) *for each $n \in \mathbb{N}$, $\{x_{j,n}\}_{j=1}^{n}$ are the zeros of the n^{th}-orthogonal polynomial $Q_n(x)$ with respect to function $\omega_n(x)$ given by* (2.2).

These Gauss formulas can be obtained in the same way as Markov's for the polynomial case (see e.g. [11] and references found therein), integrating the rational interpolation function $R_{2n} \in \mathcal{L}_{2n}$, which is the solution of the Hermite interpolation problem:

$$\left.\begin{array}{rcl} R_{2n}(f; x_{j,n}) & = & f(x_{j,n}) \\ R'_{2n}(f; x_{j,n}) & = & f'(x_{j,n}) \end{array}\right\} \qquad j = 1, 2, ..., n$$

where $\{x_{j,n}\}_{j=1}^{n}$ are n distinct nodes in $[-1, 1]$ and f is a differentiable function on $[-1, 1]$. Following this procedure, error formulas can be derived by integrating the interpolation error (see [11] and [7]).

Assuming that $\hat{\alpha}$ satisfies the δ–condition, the class of rational functions $\mathcal{R} = \cup_{n \in \mathbb{N}} \mathcal{R}_n$ is dense in $C[-1, 1]$ ([8, Theorem 4]. Thus, a theorem on convergence of Gauss quadrature formulas in $\mathcal{L}_{2n}$ can be proved in an analogous way to the polynomial case. We give only a sketch of its proof.

THEOREM 2.3. *The sequence $\{\tilde{I}_n(f)\}$ of Gauss quadrature formulas for $\mathcal{L}_{2n}$, $n = 1, 2, \ldots$, converges to*

$$I_\omega(f) = \int_{-1}^{1} f(x)\omega(x)dx,$$

for any bounded Riemann integrable function on $[-1, 1]$.

Proof. Take $f \in C([-1, 1])$. Now, since a positive constant K exists such that

$$\sum_{j=1}^{n} |\lambda_{j,n}| = \sum_{j=1}^{n} \lambda_{j,n} \leq K,$$

and by the density of the class $\mathcal{R}$ in $C([-1, 1])$, it follows that

$$\lim_{n \to \infty} I_n(f) = I_\omega(f).$$

Convergence in the class of the bounded Riemann integrable functions is a consequence of the fact that $\lambda_{j,n} > 0, \quad j = 1, 2, \dots, n, \quad n \in \mathbb{N}$ ([6, pp. 127–129]). $\square$

We state two lemmas that will be useful in the next section. The former can be found in [15, Theorem 1.5.4].

LEMMA 2.4. *Let ω be a weight function on $[-1, 1]$ with*

$$\int_{-1}^{1} \omega(x)dx < +\infty,$$

and let f be a real-valued and bounded function on $[-1, 1]$ such that the Riemann integral

$$\int_{-1}^{1} f(x)\omega(x)dx$$

exists. Then, for any $\varepsilon > 0$, there exist polynomials p and P such that

$$\int_{-1}^{1} [P(x) - p(x)]\omega(x)dx < \varepsilon,$$

and $-M - \varepsilon \le p(x) \le f(x) \le P(x) \le M + \varepsilon, \quad \forall x \in [-1, 1]$ with

$$M = \max\left\{ \left| \inf_{x\in[-1,1]} f(x) \right|, \left| \sup_{x\in[-1,1]} f(x) \right| \right\}.$$

We will state a similar result for rational functions. Let $\hat{\alpha} = \cup_{n\in\mathbb{N}} \hat{\alpha}_n \subset \overline{\mathbb{C}} \setminus [-1, 1]$, with $\hat{\alpha}_n = \{\alpha_{j,n} \in \overline{\mathbb{C}} \setminus [-1, 1],\ j = 1, 2, \dots, n\}$, satisfy the δ–condition (2.1) and furthermore, assume that for each $n \in \mathbb{N}$, there exists $m = m(n)$, with $1 \le m \le n$, such that $\alpha_{m,n} \in \hat{\alpha}_n$ satisfies $|\Re(\alpha_{m,n})| > 1$.

LEMMA 2.5. *Let ω be a given weight function on $[-1, 1]$ with*

$$\int_{-1}^{1} \omega(x)dx < +\infty,$$

and let f be a complex bounded function on $[-1, 1]$ such that the integral

$$\int_{-1}^{1} f(x)\omega(x)dx$$

exists. Then, for any $\varepsilon > 0$, there exists $R \in \mathcal{R}$ satisfying

$$\int_{-1}^{1} |f(x) - R(x)|\omega(x)dx < \varepsilon,$$

and

$$|f(x) - R(x)| \le 2(M + \varepsilon),$$

where M is a positive constant depending on f.

Proof. We can write $f(x) = f_1(x) + i f_2(x)$ where f_j $(j = 1, 2)$ are bounded real–valued functions on $[-1, 1]$ such that $\int_{-1}^{1} f_j(x)\omega(x)dx$ exists for $j = 1, 2$. By using Lemma 2.4, polynomials p_1, p_2, P_1 and P_2 exist such that for $j = 1, 2$ and any $\varepsilon' > 0$, we have

$$-M_j - \varepsilon' \le p_j(x) \le f_j(x) \le P_j(x) \le M_j + \varepsilon', \quad \forall x \in [-1, 1], \tag{2.4}$$

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with

$$M_j = \max\left\{ \left| \inf_{x\in[-1,1]} f_j(x) \right|, \left| \sup_{x\in[-1,1]} f_j(x) \right| \right\},$$

and

$$\int_{-1}^{1} [P_j(x) - p_j(x)]\omega(x)dx < \varepsilon'. \tag{2.5}$$

The functions $F(x) = p_1(x) + ip_2(x)$ and $G(x) = P_1(x) + iP_2(x)$ are continuous and complex–valued on $[-1,1]$. So, by the δ-condition, there exist sequences $\{r_n\}$ and $\{R_n\}$ in $\mathcal{R}$ such that

$$\lim_{n\to\infty} r_n(x) = F(x) \quad \text{and} \quad \lim_{n\to\infty} R_n(x) = G(x), \tag{2.6}$$

uniformly on $[-1,1]$.

Take real and imaginary parts and set $r_n(x) = r_{n,1}(x) + ir_{n,2}(x)$ and $R_n(x) = R_{n,1}(x) + iR_{n,2}(x)$. From (2.6) it clearly follows that

$$\begin{aligned} \lim_{n\to\infty} r_{n,1}(x) = p_1(x), &\qquad \lim_{n\to\infty} R_{n,1}(x) = P_1(x), \\ \lim_{n\to\infty} r_{n,2}(x) = p_2(x), &\qquad \lim_{n\to\infty} R_{n,2}(x) = P_2(x), \end{aligned}$$

uniformly on [-1,1]. Therefore, for $\varepsilon'' > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n > n_0$

$$\left.\begin{array}{rcl} r_{n,1}(x) - \varepsilon'' < & p_1(x) & < r_{n,1}(x) + \varepsilon'' \\ R_{n,1}(x) - \varepsilon'' < & P_1(x) & < R_{n,1}(x) + \varepsilon'' \end{array}\right\} \quad \forall x \in [-1,1]. \tag{2.7}$$

Without loss of generality we can assume that $\alpha_m = a + ib$ with $a < -1$ and $b > 0$ (α_m such that $|\Re(\alpha_m)| > 1$). Let $\bar{\alpha}_m$ denotes the complex conjugate of α_m. On the other hand, the function $(x - \alpha_m)^{-1}$ is obviously in $\mathcal{R}$. Write

$$\frac{1}{x - \alpha_m} = \frac{x - \bar{\alpha}_m}{|x - \alpha_m|^2} = \frac{x - a}{|x - \alpha_m|^2} + i\frac{b}{|x - \alpha_m|^2} := h_1(x) + ih_2(x),$$

where $x - a > 0$ (since $a < -1$). Set

$$\gamma_1 = \min_{x\in[-1,1]}\{h_1(x)\} > 0, \qquad \text{and} \qquad \gamma_2 = \max_{x\in[-1,1]}\{h_1(x)\} > 0.$$

Take $\varepsilon'' = \tilde{\varepsilon}\gamma_1$ with $\tilde{\varepsilon} > 0$ arbitrary. Then, by (2.7), for all $x \in [-1,1]$,

$$r_{n,1}(x) - \tilde{\varepsilon}h_1(x) < p_1(x) < r_{n,1}(x) + \tilde{\varepsilon}h_1(x) \tag{2.8}$$

and

$$R_{n,1}(x) - \tilde{\varepsilon}h_1(x) < P_1(x) < R_{n,1}(x) + \tilde{\varepsilon}h_1(x). \tag{2.9}$$

Define now

$$S_1(x) = r_{n,1}(x) - \tilde{\varepsilon}h_1(x), \quad R_1(x) = R_{n,1}(x) + \tilde{\varepsilon}h_1(x), \quad x \in [-1,1].$$

Then, by (2.4), (2.8) and (2.9), we have $S_1(x) \le f_1(x) \le R_1(x)$, $\forall x \in [-1,1]$. On the other hand, by (2.7),

$$S_1(x) = r_{n,1}(x) - \tilde{\varepsilon}h_1(x) > p_1(x) - \varepsilon'' - \tilde{\varepsilon}h_1(x) \ge p_1(x) - \varepsilon'' - \tilde{\varepsilon}\gamma_2,$$

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(recall that $\gamma_2 = \max_{x\in[-1,1]}\{h_1(x)\} > 0$). Now, by (2.4), $S_1(x) \geq -M_1 - \varepsilon' - \varepsilon'' - \tilde{\varepsilon}\gamma_2$ ($\varepsilon'' = \gamma_1\tilde{\varepsilon}$). Then,

$$S_1(x) \geq -M_1 - \varepsilon' - (\gamma_1 + \gamma_2)\tilde{\varepsilon}.$$

By (2.4) and (2.7),

$$R_1(x) < P_1(x) + \varepsilon'' + \tilde{\varepsilon}h_1(x) < M_1 + \varepsilon' + \varepsilon'' + \tilde{\varepsilon}\gamma_2 = M_1 + \varepsilon' + (\gamma_1 + \gamma_2)\tilde{\varepsilon}.$$

In short, the functions $S_1(x)$ and $R_1(x)$ defined above satisfy

$$-M_1 - \varepsilon' - (\gamma_1 + \gamma_2)\tilde{\varepsilon} \leq S_1(x) \leq f_1(x) \leq R_1(x) < M_1 + \varepsilon' + (\gamma_1 + \gamma_2)\tilde{\varepsilon}. \tag{2.10}$$

Similarly, considering $h_2(x)$, it can be deduced for the function $f_2(x)$ that

$$-M_2 - \varepsilon' - (\delta_1 + \delta_2)\tilde{\varepsilon} \leq S_2(x) \leq f_2(x) \leq R_2(x) < M_2 + \varepsilon' + (\delta_1 + \delta_2)\tilde{\varepsilon}, \tag{2.11}$$

where

$$\begin{aligned} S_2(x) &= r_{n,2}(x) - \tilde{\varepsilon}h_2(x), & \delta_1 &= \min_{x\in[-1,1]}\{h_2(x)\} > 0,\\ R_2(x) &= R_{n,2}(x) + \tilde{\varepsilon}h_2(x), & \delta_2 &= \max_{x\in[-1,1]}\{h_2(x)\} > 0. \end{aligned}$$

Define

$$\begin{aligned} S(x) &= S_1(x) + iS_2(x) = [r_{n,1}(x) - \tilde{\varepsilon}h_1(x)] + i[r_{n,2}(x) - \tilde{\varepsilon}h_2(x)]\\ &= r_n(x) - \tilde{\varepsilon}[h_1(x) + ih_2(x)] = r_n(x) - \frac{\tilde{\varepsilon}}{x - \alpha_m} \in \mathcal{R}. \end{aligned}$$

Similarly

$$\begin{aligned} R(x) &= R_1(x) + iR_2(x) = [R_{n,1}(x) + \tilde{\varepsilon}h_1(x)] + i[R_{n,2}(x) + \tilde{\varepsilon}h_2(x)]\\ &= R_n(x) + \tilde{\varepsilon}[h_1(x) + ih_2(x)] = R_n(x) + \frac{\tilde{\varepsilon}}{x - \alpha_m} \in \mathcal{R}. \end{aligned}$$

Now, by (2.10) and (2.11), it follows

$$\begin{aligned} |f(x) - R(x)| &\leq |f_1(x) - R_1(x)| + |f_2(x) - R_2(x)|\\ &= (R_1(x) - f_1(x)) + (R_2(x) - f_2(x))\\ &< 2[M_1 + \varepsilon' + (\gamma_1 + \gamma_2)\tilde{\varepsilon}] + 2[M_2 + \varepsilon' + (\delta_1 + \delta_2)\tilde{\varepsilon}]. \end{aligned} \tag{2.12}$$

On the other hand, by the uniform convergence, we have, for $j = 1, 2$

$$\lim_{n\to\infty}\int_{-1}^{+1} r_{n,j}(x)\omega(x)dx = \int_{-1}^{+1} p_j(x)\omega(x)dx, \quad \text{and}$$
$$\lim_{n\to\infty}\int_{-1}^{+1} R_{n,j}(x)\omega(x)dx = \int_{-1}^{+1} P_j(x)\omega(x)dx.$$

Recalling the notation $I_\omega(f) = \int_{-1}^{+1} f(x)\omega(x)dx$, for $\varepsilon''' > 0$, there exists $n_1 \in \mathbb{N}$, such that for any $n > n_1$,

$$\begin{aligned} -\varepsilon''' + I_\omega(p_1) < I_\omega(r_{n,1}) < \varepsilon''' + I_\omega(p_1)\\ -\varepsilon''' + I_\omega(P_1) < I_\omega(R_{n,1}) < \varepsilon''' + I_\omega(P_1). \end{aligned}$$

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We have now

$$I_\omega(R_1 - S_1) = I_\omega(R_{n,1} + \tilde{\varepsilon}h_1 - r_{n,1} + \tilde{\varepsilon}h_1) = I_\omega(R_{n,1}) - I_\omega(r_{n,1}) + 2\tilde{\varepsilon}I_\omega(h_1),$$

and

$$I_\omega(h_1) = \int_{-1}^{+1} h_1(x)\omega(x)dx \leq \gamma_2 c_0,$$

with $c_0 = \int_{-1}^{+1} \omega(x)dx$, which can be taken as 1. Thus, from (2.5),

$$I_\omega(R_1 - S_1) < I_\omega(P_1) - I_\omega(p_1) + 2(\varepsilon''' + \gamma_2\tilde{\varepsilon}) < \varepsilon' + 2(\varepsilon''' + \gamma_2\tilde{\varepsilon}).$$

Similarly, it can be deduced that

$$I_\omega(R_2 - S_2) < \varepsilon' + 2(\varepsilon''' + \delta_2\tilde{\varepsilon}).$$

This yields

$$\begin{aligned}\int_{-1}^{+1} |f(x) - R(x)|\omega(x)dx &\leq \int_{-1}^{+1} |f_1(x) - R_1(x)|\omega(x)dx + \int_{-1}^{+1} |f_2(x) - R_2(x)|\omega(x)dx \\ &= \int_{-1}^{+1} (R_1(x) - f_1(x))\omega(x)dx + \int_{-1}^{+1} (R_2(x) - f_2(x))\omega(x)dx \\ &\leq I_\omega(R_1 - S_1) + I_\omega(R_2 - S_2) \\ &< 2\varepsilon' + 2[\varepsilon''' + (\gamma_2 + \delta_2)\tilde{\varepsilon}]. \qquad (2.13)\end{aligned}$$

Taking $M = M_1 + M_2$, from (2.12) and (2.13), the proof follows. □

3. Convergence of interpolatory quadrature formulas. In this section we will be concerned with the estimation of the integral

$$I_\beta(f) = \int_{-1}^{1} f(x)\beta(x)dx,$$

where $\beta(x)$ is an L_1-integrable function (possibly complex) in $[-1, 1]$, i.e.

$$\int_{-1}^{1} |\beta(x)|dx < +\infty.$$

For given n distinct nodes $x_{1,n}, x_{2,n}, x_{3,n}, \ldots, x_{n,n}$ in $[-1, 1]$, there exist n coefficients $A_{1,n}, A_{2,n}, \ldots, A_{n,n}$ such that

$$I_\beta(f) = \sum_{j=1}^{n} A_{j,n} f(x_{j,n}) := I_n(f), \quad \forall f \in \mathcal{R}_n.$$

Let $R_{n-1}(f, x)$ be the unique interpolant to f in $\mathcal{R}_n$,

$$R_{n-1}(f, x_{j,n}) = f(x_{j,n}), \qquad j = 1, 2, \ldots, n, \;\; n \in \mathbb{N}.$$

Setting $\pi_k(x) = \prod_{j=1}^{k}(x - \alpha_{j,k}),\; k = 1, 2, \ldots$, and since $\{\pi_k^{-1}\}_{k=1}^{n}$ is a Chebyshev system in $(-1, 1)$, existence and uniqueness of such interpolant is guaranteed (see e.g. [5, p. 32]).

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Then, as in the polynomial case ($\alpha_{j,n} = \infty, \ \ j = 1, 2, \dots, n$), it is easily proved (see [10, p. 80]), that

$$I_n(f) = \sum_{j=1}^{n} A_{j,n} f(x_{j,n}) = I_\beta(R_{n-1}(f, \cdot)).$$

Hence, we will sometimes refer to $I_n(f)$ as an n-point interpolatory quadrature formula for $\mathcal{R}_n$.

Let $\omega(x)$ be a given weight function on $[-1, 1]$, (i.e., $\omega(x) > 0$, a.e. on $[-1, 1]$), satisfying

$$\int_{-1}^{1} \frac{|\beta(x)|^2}{\omega(x)} dx = K_1^2 < +\infty.$$

We can establish the following

THEOREM 3.1. *Let f be a bounded function in* $\mathrm{L}_{2,\omega} = \{f : [-1, 1] \to \mathbb{C} : \int_{-1}^{1} |f(x)|^2 \omega(x) dx < \infty\}$. *Then,*

$$|I_\beta(f) - I_n(f)| \leq K_1 . \|f - R_{n-1}\|_{2,\omega},$$

where $\|f\|_{2,\omega}$ denotes the weighted L_2-norm, i.e.,

$$\|f\|_{2,\omega} = \left[\int_{-1}^{1} |f(x)|^2 \omega(x) dx \right]^{\frac{1}{2}}.$$

Proof. Making use of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|I_\beta(f) - I_n(f)| = |I_\beta(f) - I_\beta(R_{n-1}(f, .))| &= \left| \int_{-1}^{1} (f(x) - R_{n-1}(f, x)) \beta(x) dx \right| \\
&= \left| \int_{-1}^{1} (f(x) - R_{n-1}(f, x)) \sqrt{\omega(x)} \frac{\beta(x)}{\sqrt{\omega(x)}} dx \right| \\
&\leq \left(\int_{-1}^{1} (f(x) - R_{n-1}(f, x))^2 \omega(x) dx \right)^{\frac{1}{2}} \left(\int_{-1}^{1} \frac{|\beta(x)|^2}{\omega(x)} dx \right)^{\frac{1}{2}} \\
&\leq K_1 \|f - R_{n-1}(f, .)\|_{2,\omega}. \qquad \square
\end{aligned}$$

Thus, we see that the $L_{2,\omega}$ convergence of the interpolants at the nodes of the quadrature implies convergence of the sequence of quadrature formulas. Now, the questions are: How to find nodes $\{x_{j,n}\}$ in $[-1, 1]$ such that

$$\lim_{n \to \infty} \|f - R_{n-1}(f, .)\|_{2,\omega} = 0,$$

and in which class of functions (as large as possible) does it hold?

As a first answer, we have

THEOREM 3.2. *Let f be a complex continuous function on $[-1, 1]$ and $\omega(x) > 0$ a.e. on $[-1, 1]$. Let $\{x_{j,n}\}_{j=1}^{n}, \ \ n = 1, 2, \dots,$ denote the zeros of $Q_n(x)$, the n^{th} monic orthogonal polynomial with respect to*

$$\frac{\omega(x)}{|\pi_n(x)|^2}$$

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on $[-1,1]$. *Then,*

$$\lim_{n\to\infty} \|f - R_{n-1}(f,\cdot)\|_{2,\omega} = 0.$$

Proof. Let $T_{n-1}(x) \in \mathcal{R}_n$ denote the best minimax rational approximant to $f(x)$, i.e.,

$$\rho_{n-1}(f) := \|f - T_{n-1}\|_{[-1,1]} = \max_{x\in[-1,1]} |f(x) - T_{n-1}(x)| \le \|f - R\|_{[-1,1]}, \quad \forall R \in \mathcal{R}_n.$$

Then, we have

$$\begin{aligned} \|f - R_{n-1}(f,\cdot)\|_{2,\omega} &= \|f - T_{n-1} + T_{n-1} - R_{n-1}(f,\cdot)\|_{2,\omega} \\ &\le \|f - T_{n-1}\|_{2,\omega} + \|T_{n-1} - R_{n-1}(f,\cdot)\|_{2,\omega} \\ &= \left\{\int_{-1}^{1} |f(x) - T_{n-1}(x)|^2 \omega(x) dx\right\}^{\frac{1}{2}} \\ &+ \left\{\int_{-1}^{1} |T_{n-1}(x) - R_{n-1}(f,x)|^2 \omega(x) dx\right\}^{\frac{1}{2}}. \end{aligned}$$

But $|T_{n-1}(x) - R_{n-1}(f,x)|^2 \in \mathcal{L}_{2n}$, since x is real. Then,

$$\begin{aligned} \|f - R_{n-1}(f,\cdot)\|_{2,\omega} &\le \rho_{n-1}(f).\sqrt{c_0} + \left\{\sum_{j=1}^{n} \lambda_{j,n} |T_{n-1}(x_{j,n}) - R_{n-1}(f,x_{j,n})|^2\right\}^{\frac{1}{2}} \\ &\le \rho_{n-1}(f).\sqrt{c_0} + \rho_{n-1}(f)\left\{\sum_{j=1}^{n} \lambda_{j,n}\right\}^{\frac{1}{2}}, \end{aligned}$$

with $c_0 = \int_{-1}^{1} \omega(x) dx$, that is,

$$\|f - R_{n-1}(f,.)\|_{2,\omega} \le \rho_{n-1}(f)\left\{\sqrt{c_0} + \left(\sum_{j=1}^{n} \lambda_{j,n}\right)^{\frac{1}{2}}\right\}. \tag{3.1}$$

By Lemma 2.1, there exists a constant M such that

$$\sum_{j=1}^{n} \lambda_{j,n} \le M, \qquad \forall n \ge 1.$$

Since, (see [8])

$$\lim_{n\to\infty} \rho_n(f) = 0,$$

from (3.1) the proof of the theorem follows. □

We have immediately the following

COROLLARY 3.3. *Let* β *be an* L_1*-integrable complex function on* $[-1,1]$ *such that*

$$\int_{-1}^{1} \frac{|\beta(x)|^2}{\omega(x)} dx < +\infty.$$

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Let $I_n(f) = \sum_{j=1}^{n} A_{j,n} f(x_{j,n})$ be the n-point interpolatory quadrature formula in $\mathcal{R}_n$ with nodes $\{x_{j,n}\}_{j=1}^{n}$ at the zeros of Q_n, the n^{th} monic orthogonal polynomial with respect to

$$\omega_n(x) = \frac{\omega(x)}{|\pi_n(x)|^2}, \quad x \in [-1, 1].$$

Then

$$\lim_{n\to\infty} I_n(f) = I_\beta(f),$$

for any complex function f continuous on $[-1, 1]$

Now, making use of Banach-Steinhaus Theorem (see e.g. [10, p. 264]), we have,

COROLLARY 3.4. *Under the same conditions as in Corollary 3.3, there exists a positive constant M such that*

$$\sum_{j=1}^{n} |A_{j,n}| \leq M, \qquad n = 1, 2, \ldots$$

EXAMPLE 1. *(Multipoint Padé-type Approximants)*

For $z \in \mathbb{C} \setminus [-1, 1]$, consider the function $f(x, z) = (z - x)^{-1}$ (in the variable x, and z as a parameter), so that

$$I_\beta(f(\cdot, z)) = \int_{-1}^{1} \frac{\beta(x)}{z - x} dx = F_\beta(z).$$

We have

$$I_n(f(\cdot, z)) = \sum_{j=1}^{n} \frac{A_{j,n}}{z - x_{j,n}} = \frac{P_{n-1}(z)}{Q_n(z)},$$

with $Q_n(z) = \prod_{j=1}^{n}(z - x_{j,n})$ and $P_{n-1}(z) \in \Pi_{n-1}$. In order to characterize such rational functions, it should be recalled that

$$I_n(f) = I_\beta(R_{n-1}(f, \cdot)),$$

$R_{n-1}(f, \cdot)$ being the interpolant in $\mathcal{R}_n$ at the nodes $\{x_{j,n}\}_{j=1}^{n}$ to $f(x)$.

Write $R_{n-1}(z, x) = R_{n-1}((z - x)^{-1}, x)$. We have that

$$R_{n-1}(z, x) = \frac{1}{z - x}\left[1 - \frac{Q_n(x)}{Q_n(z)} \frac{\pi_n(z)}{\pi_n(x)}\right],$$

which can be easily checked that belongs to $\mathcal{R}_n$, and since $Q_n(x_{j,n}) = 0$, then

$$R_{n-1}(z, x_{j,n}) = \frac{1}{z - x_{j,n}}, \quad j = 1, 2, \ldots, n.$$

Thus

$$\begin{aligned} \frac{P_{n-1}(z)}{Q_n(z)} &= I_\beta\left[\frac{1}{z - x}\left(1 - \frac{Q_n(x)\pi_n(z)}{Q_n(z)\pi_n(x)}\right)\right] \\ &= F_\beta(z) - I_\beta\left[\frac{Q_n(x)\pi_n(z)}{(z - x)Q_n(z)\pi_n(x)}\right] \end{aligned} \tag{3.2}$$

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Hence,

$$F_\beta(z) - \frac{P_{n-1}(z)}{Q_n(z)} = I_\beta\left[\frac{Q_n(x)\pi_n(z)}{(z-x)Q_n(z)\pi_n(x)}\right] = \frac{\pi_n(z)}{Q_n(z)}\int_{-1}^{1}\frac{Q_n(x)}{\pi_n(x)}\frac{\beta(x)}{z-x}dx. \tag{3.3}$$

We see that the rational function $\frac{P_{n-1}(z)}{Q_n(z)}$ *(with a prescribed denominator) interpolates* $F_\beta(z)$ *at the nodes* $\{\alpha_{j,n}\}_{j=1}^n$*. Following* [3]*, we will refer to this rational function as a Multipoint Padé-type Approximant (MPTA) to* $F_\beta(z)$.

REMARK 2. *The same expression as in* (3.3) *for the error was also obtained in* [8]*, which is basically inspired from* [16, p. 186].

By using Corollary 3.4 and the Stieltjes-Vitali Theorem (see e.g. [9, Theorem 15.3.1]), the following can be proved:

COROLLARY 3.5. *The sequence of MPTA*

$$\left\{\frac{P_{n-1}(z)}{Q_n(z)}\right\}_{n\in\mathbb{N}},$$

defined in (3.2)*, converges to* $F_\beta(z)$*, uniformly on compact subsets of* $\overline{\mathbb{C}}\setminus[-1,1]$.

Now, we are in a position to prove the following

THEOREM 3.6. *Let* $L_{n-1}^f(x)$ *denote the interpolant in* $\mathcal{R}_n$ *to the function* $f(x)$ *at the nodes* $\{x_{j,n}\}_{j=1}^n$ *which are the zeros of the* n^{th} *orthogonal polynomial with respect to* $\omega_n(x)$ *on* $[-1,1]$*. Then,*

$$\lim_{n\to\infty}\|L_{n-1}^f - f\|_{2,\omega}^2 = \lim_{n\to\infty}\int_{-1}^{+1}|L_{n-1}^f(x) - f(x)|^2\omega(x)dx = 0,$$

for any complex-valued and bounded function on $[-1,1]$*, such that the integral* $\int_{-1}^{+1} f(x)\omega(x)dx$*, exists.*

Proof. We have

$$\|L_{n-1}^f - f\|_{2,\omega}^2 = \int_{-1}^{+1}|L_{n-1}^f(x) - f(x)|^2\omega(x)dx = \|f\|_{2,\omega}^2 + \|L_{n-1}^f\|_{2,\omega}^2 - 2\int_{-1}^{+1}\Re(L_{n-1}^f(x)\overline{f(x)})\omega(x)dx.$$

Hence

$$\|L_{n-1}^f - f\|_{2,\omega}^2 \le \|f\|_{2,\omega}^2 + \|L_{n-1}^f\|_{2,\omega}^2 + 2I_\omega(|L_{n-1}^f||f|). \tag{3.4}$$

Now, $L_{n-1}^f \in \mathcal{R}_n$ implies that $|L_{n-1}^f|^2 \in \mathcal{L}_{2n} = \{\frac{P(x)}{|\pi_n(x)|^2}, P \in \Pi_{2n-1}\}$, when restricted to the real line. So ,

$$\|L_{n-1}^f\|_{2,\omega}^2 = I_\omega(|L_{n-1}^f|^2) = \sum_{j=1}^n \lambda_{j,n}|L_{n-1}^f(x_{j,n})|^2 = \sum_{j=1}^n \lambda_{j,n}|f(x_{j,n})|^2.$$

Setting $f(x) = f_1(x) + if_2(x)$, we have

$$\|L_{n-1}^f\|_{2,\omega}^2 = \sum_{j=1}^n \lambda_{j,n}[f_1^2(x_{j,n}) + f_2^2(x_{j,n})].$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\|L_{n-1}^f\|_{2,\omega}^2 &= \lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{j,n}f_1^2(x_{j,n}) + \lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{j,n}f_2^2(x_{j,n})\\ &= \int_{-1}^{+1} f_1^2(x)\omega(x)dx + \int_{-1}^{+1} f_2^2(x)\omega(x)dx\\ &= \int_{-1}^{+1}|f(x)|^2\omega(x)dx\\ &= \|f\|_{2,\omega}^2.\end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\left\{I_\omega(|f||L_{n-1}^f|)\right\}^2 &= \left(\int_{-1}^{+1}|f(x)||L_{n-1}^f(x)|\omega(x)dx\right)^2\\ &\le \left(\int_{-1}^{+1}|f(x)|^2\omega(x)dx\right)\left(\int_{-1}^{+1}|L_{n-1}^f(x)|^2\omega(x)dx\right)\\ &= \|f\|_{2,\omega}^2\cdot\|L_{n-1}^f\|_{2,\omega}^2.\end{aligned}$$

Therefore, $\limsup_{n\to\infty} I_\omega(|f||L_{n-1}^f|) \le \|f\|_{2,\omega}^2$, and by (3.4) it follows that

$$\limsup_{n\to\infty} I_\omega(|f-L_{n-1}^f|^2) \le 4\|f\|_{2,\omega}^2. \tag{3.5}$$

Now, given $\varepsilon > 0$, by Lemma 2.5, there exists $R \in \mathcal{R}$, such that

$$|f(x)-R(x)| \le 2(M+\varepsilon), \qquad \forall x \in [-1,1],$$

and

$$\int_{-1}^{+1}|f(x)-R(x)|\omega(x)dx < \varepsilon.$$

Hence,

$$\begin{aligned}\|f-R\|_{2,\omega}^2 &= \int_{-1}^{+1}|f(x)-R(x)|^2\omega(x)dx\\ &= \int_{-1}^{+1}|f(x)-R(x)||f(x)-R(x)|\omega(x)dx\\ &\le 2(M+\varepsilon)\int_{-1}^{+1}|f(x)-R(x)|\omega(x)dx < 2\varepsilon(M+\varepsilon). \qquad (3.6)\end{aligned}$$

For sufficiently large n, we have $L_{n-1}^R = R$, and we get $f - L_{n-1}^f = f - R + R - L_{n-1}^f = f - R - (L_{n-1}^f - L_{n-1}^R) = f - R - L_{n-1}^{f-R}$. Hence, $\|f - L_{n-1}^f\|_{2,\omega}^2 = \|(f-R) - L_{n-1}^{f-R}\|_{2,\omega}^2$ and by (3.5–3.6), it holds that

$$\begin{aligned}\limsup_{n\to\infty}\|f-L_{n-1}^f\|_{2,\omega}^2 &= \limsup_{n\to\infty}\|(f-R)-L_{n-1}^{f-R}\|_{2,\omega}^2\\ &\le 4\|f-R\|_{2,\omega}^2 \le 8(M+\varepsilon)\varepsilon. \qquad (3.7)\end{aligned}$$

Clearly, from (3.7) the proof follows. □

REMARK 3. *The theorem above can be considered as an extension to the rational case of the famous Erdös-Turán result for polynomial interpolation (see* [4, pp. 137–138]*). Actually, an earlier rational extension was carried out by Walter Van Assche et al. in* [2]*, under the restriction that the points $\alpha_{j,n}$ are real, distinct and $\hat{\alpha}$ is a Newtonian table, and only considering continuous functions on* $[-1, 1]$.

Finally, making use of Theorem 3.1 and Theorem 3.6, we can state the main result we referred to in the beginning, (compare with Theorem 1.1).

THEOREM 3.7. *Let β be an L_1-integrable function on $[-1, 1]$ and $\omega(x) > 0$, a.e. on $[-1, 1]$ be such that*

$$\int_{-1}^{+1} \frac{|\beta(x)|^2}{\omega(x)} dx < +\infty.$$

Let $I_n(f) = \sum_{j=1}^{n} A_{j,n} f(x_{j,n})$ be the n-point interpolatory quadrature formula in $\mathcal{R}_n$, whose nodes $\{x_{j,n}\}_{j=1}^{n}$, are the zeros of $Q_n(x)$, the n^{th} monic orthogonal polynomial with respect to $\frac{\omega(x)}{|\pi_n(x)|^2}$, $x \in [-1, 1]$. Assume that the table $\hat{\alpha}$ satisfies the same conditions as those in Lemma 2.5*. Then,*

$$\lim_{n\to\infty} I_n(f) = I_\beta(f) = \int_{-1}^{+1} f(x)\beta(x)dx,$$

for all bounded complex–valued function f on $[-1, 1]$ such that the integral $\int_{-1}^{+1} f(x)\beta(x)dx$ exists.

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